The modality of physical law in modal homotopy type theory

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What is intriguing about HoTT is the convergence of:

- Constructive type theory (Martin-Löf,...)
- Categorical logic (Lawvere,...)

From the perspective of the latter:

HoTT is the internal logic of $(\infty, 1)$-toposes.

- 1-toposes beautifully blend logic and space.
- $(\infty, 1)$-toposes are even better, and are used by our leading geometers (Lurie, Toën, ...)

Some tasks for philosophy

We can look again at any place philosophy has been tempted to use untyped logic, and then consider whether or not type theory (especially HoTT) might fare better.

- Consider all types evenly – propositions, sets, and higher groupoids.
- Notice how the line between mathematics and logic is blurred.
- Observe how the seeds of deep mathematical ideas are already present in everyday thought.

The A

Form 'the A' when A is contractible.

$$(a, p) : IsContr(A) \vdash the\ A(a, p) : A.$$

- Do say 'the product' of types B and C.
- Don't say 'the algebraic closure' of a field.
- Don't say 'the end of the row' under conditions of symmetry.

Preprint

Lawvere on quantifiers

For $\mathbf{H}$ a topos (or ∞-topos) and $f : X \to Y$ an arrow in $\mathbf{H}$, then base change induces between over-toposes:

$$(\sum_f \dashv f^* \dashv \prod_f) : \mathbf{H}/X \underset{f_*}{\overset{\overset{f_!}{\to}}{\overset{f^*}{\leftarrow}}} \mathbf{H}/Y$$

Lawvere on quantifiers

Take a mapping

$$Owner : Dog \to Person,$$

then any property of people can be transported over to a property of dogs, e.g.,

$$Being\ French \mapsto Being\ owned\ by\ a\ French\ person.$$

We shouldn't expect every property of dogs will occur in this fashion.

In other words, we can't necessarily invert this mapping to send, say, 'Pug' to a property of People.

Lawvere on quantifiers

We can try...

Pug $\mapsto$ *Owning some pug* $\mapsto$???

Lawvere on quantifiers

But then

Pug $\mapsto$ *Owning some pug* $\mapsto$ *Owned by someone who owns a pug*.

However, people may own more than one breed of dog.

Lawvere on quantifiers

How about

$$Pug \mapsto Owning\ only\ pugs \mapsto ???$$

Lawvere on quantifiers

But this leads to

$$Pug \mapsto Owning\ only\ pugs \mapsto Owned\ by\ someone\ owning\ only\ pugs$$

But again, not all pugs are owned by single breed owners.

Lawvere on quantifiers

In some sense, these are the best approximations to an inverse (left and right adjoints). They correspond to the type theorist's dependent sum and dependent product.

Were we to take the terminal map so as to group all dogs together ($Dog \to \mathbf{1}$), then the attempts at inverses would send a property such as 'Pug' to familiar things:

'Some dog is a pug' and 'All dogs are pugs'.

Modal logic

What if we take a map *Worlds* $\to \mathbf{1}$?

We begin to see the modal logician's *possibly* (in some world) and *necessarily* (in all worlds) appear.

Modal logic

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Things work out well if we form the (co)monad of dependent sum (product) followed by base change, so that possibly P and necessarily P are dependent on the type *Worlds*.

This resembles this dog-owner case better if we consider just an equivalence relation on Worlds, represented by a surjection, $W \to V$. Necessarily P holds at a world if P holds at all related worlds.

These constructions applied to our pug case are:

$$Pug \mapsto Owning\ some\ pug \mapsto Owned\ by\ someone\ who\ owns\ a\ pug.$$

$$Pug \mapsto Owning\ only\ pugs \mapsto Owned\ by\ someone\ owning\ only\ pugs$$

$\bigcirc_{owner} Pug(d)$ means of a dog, d, that some co-owned dog is a pug.
$\Box_{owner} Pug(d)$ means of a dog, d, that all co-owned dogs are pugs.

We have equivalents of

- $P \to \bigcirc P$ and $\bigcirc \bigcirc P \to \bigcirc P$
- $\Box P \to P$ and $\Box P \to \Box\Box P$

Such composites will be adjoint to each other, expressing their 'opposition'.

$$\bigcirc A(w) \to B(w) \Leftrightarrow A(w) \to \Box B(w)$$

[Note: for types dependent on a type which is a delooped group, we see appear constructions such as fixed points and orbits of a group action, and, in general, representation theory.]

Temporal logic

Moving away from S5, we might want to represent time as a type.

Say, Time_1 is a type of temporal intervals, and Time_0 a type of instants.

Then we have maps:

- $b, e : \mathit{Time}_1 \to \mathit{Time}_0$, *beginning* and *end*

Each arrow, b and e, generates an adjoint triple, e.g.,

$$\sum_b \dashv b^* \dashv \prod_b,$$

formed of dependent sum, base change, dependent product, going between the slices $\mathcal{C}/\mathit{Time}_0$ and $\mathcal{C}/\mathit{Time}_1$.

Temporal logic

Then we find two adjunctions $\sum_b e^* \dashv \prod_e b^*$ and $\sum_e b^* \dashv \prod_b e^*$.

Now consider for the moment that C and D are propositions. Then

- $\sum_b e^* C$ means "there is some interval beginning now and such that C is true at its end", i.e. FC.
- $\prod_e b^* D$ means "for all intervals ending now, D is true at their beginning", i.e. HD
- Hence our adjunction is $F \dashv H$.
- Similarly, interchanging b and e, we find $P \dashv G$.
- Note that we don't have to assume the classical $G\phi = \neg F \neg \phi$ and $H\phi = \neg P \neg \phi$.

[$F, H.P, G$ are the standard temporal modalities.]

The various units and counits

- $\phi \to GP\phi$ "What is, will always have been"
- $PG\phi \to \phi$ "What came to be always so, is"
- $\phi \to HF\phi$ "What is, has always been to come"
- $FH\phi \to \phi$ "What always will have been, is"

With maps $p, q, c : Time_1 \times_{Time_0} Time_1 \to Time_1$, we can be more expressive, e.g., to capture *since* and *until*.

Such a map of predicates on $Time_0$ as $P := \sum_e b^*$, is a form of integral transform. This is just like the transforms Urs uses in his 'Quantization via Linear homotopy types' (arXiv:1402.7041).

Towards physics

- All very well, but we need to recreate differential topology and geometry. We have something like the 'total space' and 'space of sections' constructions, but we need spatial cohesion and smoothness.
- To try to do this in plain HoTT would commit the same mistake as to adopt set theoretic 'in principle' foundations.
- We need a tailored way to express spatial cohesion and smoothness.
- Fortunately, Urs developed Lawvere's ideas on cohesion to do just this, see dcct.

8.2 11d Supergravity with M2/M5-branes

↑ unravel bouquet of Whitehead towers

8.1 Lorentzian supergeometry

↑ represent rheonomy and form universal cover

6 Higher supergeometry

↑ faithful model

4.3 Solid homotopy theory

↑ resolve finite/infinitesimal

4.2 Elastic homotopy theory

↑ resolve continuous/discrete

4.1 Cohesive homotopy theory

↑ resolve initial opposition

3 Homotopy topos theory

↑ internal self-reflection (univalent universe)

2 Homotopy type theory

↑ first law of thought: $A = A$

Logical substance
type theory

$$
\begin{array}{rcccl}
 & \mathrm{id} & \dashv & \mathrm{id} & \\
 & \vee & & \vee & \\
\text{fermionic} & \rightrightarrows & \dashv & \rightsquigarrow & \text{bosonic} \\
 & \bot & & \bot & \\
\text{bosonic} & \rightsquigarrow & \dashv & \mathrm{Rh} & \text{rheonomic} \\
 & \vee & & \vee & \\
\text{reduced} & \Re & \dashv & \Im & \text{infinitesimal} \\
 & \bot & & \bot & \\
\text{infinitesimal} & \Im & \dashv & \& & \text{étale} \\
 & \vee & & \vee & \\
\text{cohesive} & \int & \dashv & \flat & \text{discrete} \\
 & \bot & & \bot & \\
\text{discrete} & \flat & \dashv & \sharp & \text{continuous} \\
 & \vee & & \vee & \\
 & \emptyset & \dashv & * &
\end{array}
$$

Lawvere's cohesion

Consider a chain of adjunctions between a category of spaces and the category of sets. If we take the former to be topological spaces, then

- One basic mapping takes such a space and gives its underlying set of points. All the cohesive 'glue' has been removed.
- There are two ways to generate a space from a set: one is to form the space with the *discrete* topology, where no point sticks to another.
- The other is to form the space with the *codiscrete* topology, where the points are all glued together into a single blob so that no part is separable,in the sense that any map into it is continuous.
- Finally, we need a second map from spaces to sets, one which 'reinforces' the glue by reducing each connected part to an element of a set, the connected components functor, π_0.

We have

$$(\pi_0 \dashv \mathit{Disc} \dashv U \dashv \mathit{coDisc}) : \mathit{Top} \to \mathit{Set}$$

These four functors form an adjoint chain, where any of the three compositions of two adjacent functors ($U \circ \mathit{coDisc}, U \circ \mathit{Disc}, \pi_0 \circ \mathit{Disc}$) from the category of sets to itself is the identity, whereas, in the other direction, composing adjacent functors to produce endofunctors on *Top* ($\mathit{coDisc} \circ U, \mathit{Disc} \circ U, \mathit{Disc} \circ \pi_0$) yields two idempotent monads and one idempotent comonad.

These correspond to the three adjoint modalities of the diagram:

$$\int \vdash \flat \vdash \sharp$$

To participate in such adjoint strings is demanding. By the time we find another (correctly related) layer, smoothness, or differential cohesion is expressed.

$$\Re \vdash \Im \vdash \&$$

- $*$ behaves very similarly to $\Im$
- The unit $W \to *(W) = \mathbf{1}$ is the morphism we used for modal logic.
- What if we use the unit $\Sigma \to \Im(\Sigma)$ for some domain of variation, identifying infinitesimally close points?

Let's return to an equivalence relation on a set, this time animals mapped to their species.

A property on animals is mapped by dependent product to a property on species which holds iff all conspecifics satisfy that property.

Pulling this back, an animal has a property necessarily iff all its conspecifics have that property.

You might say it's an essential characteristic. (Possibility corresponds to an accidental property?)

Now we should pass to general types to imitate bundles.

Animal legs sit above legged animals. This type is sent by dependent product to the type which above a species is the set of maps from each animal of that species to one of their legs.

Pulled back, for an animal, it is the set of maps from conspecifics to one of their legs. E.g., the last leg of a conspecific to have left the ground.

In general, there won't be a map from a type to the $\Box$ version. Think of the dependent type Offspring(animal).

So which types do allow a map to the $\Box$ version? Which are 'necessary'?

Which types do allow a map to the $\Box$ version?

Those types pulled back from ones over species. A 'standard' is provided to allow comparison across conspecifics.

E.g., we might have a subbundle of the original bundle of choices of legs of conspecifics which allows only those choices which make sense in terms of the skeletal structure of the species, e.g., the right foremost leg.

Try to do the same for differential cohesion, taking a bundle over some base, it might be time, $\mathbb{R}$.

That bundle is sent to a bundle of jets within the fibres now over $\Im(\mathbb{R})$.

Then this is sent to the bundle of jets within the fibres over $\mathbb{R}$, where horizontal infinitesimal paths are now allowed. The joint operation forms the jet comonad.

In a context $\mathbf{H}$ of differential cohesion with $\Im$ the infinitesimal shape modality, then for any object $X \in \mathbf{H}$ the comonad

$$Jet_X := i^* i_*$$

for base change along the X-component of the unit of $\Im$

$$\mathbf{H}_{/X} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathbf{H}_{/\Im(X)},$$

may be interpreted as sending any bundle over X to its jet bundle.

For which bundles is there a map from E to $Jet(E)$?

Those E which themselves have been pulled back from $\Im(\Sigma)$.

Marvan showed that these 'coalgebras' are solutions sets of PDEs.

Pulling back from $\Im(\Sigma)$ gives a way of comparing infinitesimally close fibres, just as a species defined characteristic gave a way to compare across conspecifics.

The differential equation expresses a rule for infinitesimal change depending solely on the value of the ordinary point.

E.g., in the first order, one dimensional case, $\frac{dx}{dt} = f(t)$

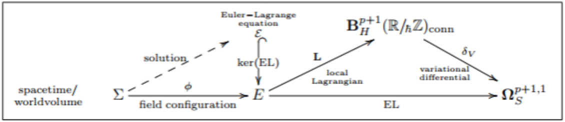
So a solution $\Sigma \to E$ is one such that at each point x in Σ it lifts any infinitesimal path in Σ starting at x to a jet that exists in the solution bundle.

Over $\Im(\Sigma)$ we already have all these admissible jets, but we don't have the infinitesimal paths that are in Σ that we can lift along.

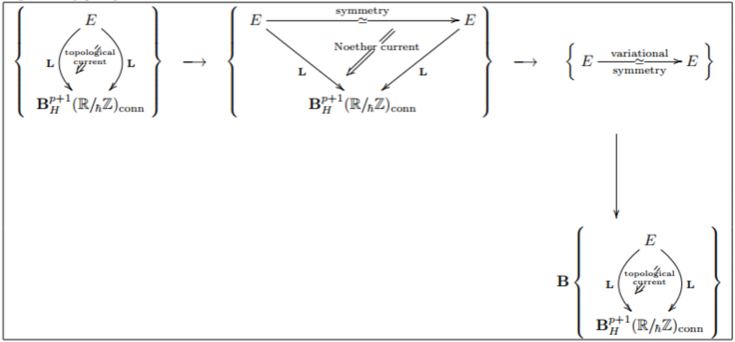
A PDE and its solutions are very much like a type of rigid designators or of modal counterparts.

Perhaps this analogy doesn't take us so far, but differential type theory possesses the resources to do astonishing things...

For $E \in \mathbf{H}_{/\Sigma} \xrightarrow{F} \mathrm{PDE}_\Sigma(\mathbf{H})$ this formalizes the principle of extremal action in field theory/variational calculus:



Theorem 7.1 (Noether's theorem [Fiorenza-Rogers-S 13a, Sati-S 15, Khavkine-S]). *There is homotopy fiber sequence of group stacks like so:*



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$$

There are plenty more modalities to consider from the table. E.g., $\flat B \to B$ is a counit from the discrete version of a space to that space.

What would it be to form modalities from this map?

> *An ionad is a set X together with a finite limit-preserving comonad Int_X on the category Set^X. (Garner)*
>
> *In the original expression, we consider the geometric morphism from the topos $Sets/|X|$ of sets indexed over a set $|X|$ to the topos $Sh(X)$ of sheaves over a topological space induced by the (continuous) identity map $id : |X| \to X$. The modal operator $\Box$ is interpreted by the interior operation int that the comonad $id^* \circ id_*$ induces on the Boolean algebra $Sub_{Sets/|X|}(id^* F) \cong \mathcal{P}(F)$ of subsets of F. (Awodey and Kishida)*

Perhaps the appearance of $\flat$ in the modality diagram explains why these models for modal logic appear.

I began by pointing out these two sources for HoTT:

- Constructive type theory (Martin-Löf,...)
- Categorical logic (Lawvere,...)

In the course of this discussion of modal type theory, I have followed the categorical logical line.

I have also been speaking 'externally'. The adjoint logic of Shulman and Licata provides an 'internal' formalism.

There is already a developed field of modal type theory in a more Martin-Löfian style, with its own formalism. This would be well worth integrating.

In the judgmental approach to modal type theory, we view modal types $\Diamond A$ as internalizations of categories of judgment.

That is, in Martin-Löf's judgmental methodology, we take the assertion "P is true", and then introduce a judgement of "P is true", which explains what constitutes evidence for P (the introduction rules), and how to use a P (the elimination rules). We can extend this to modalities by introducing new judgements to represent new categories of assertion. So in addition to "P is true", we might also have categories of judgment such as "P is known to X", "P will eventually be true", "P is possible", and so on. Then, a modal type like $\Diamond A$ is an internalization of a judgement. That is, we can say that the introduction rule for the judgement "$\Diamond A$ is true", is actually evidence for the judgement "A is possible". (Neel Krishnaswami)

Thank you.