Clocks and Sweedler Runners

A bridge from trajectories to operational theories, tests, and composition

October 2026 — self-contained bridge chapter

The question of this chapter. In the DOTS approach, a behaviour is a map *out of a clock*. In the Sweedler approach, a theory specifies operations, a runner implements one theory using another, and comodels carry observable dynamics. How do these descriptions meet? We start with the simplest clock map, discover why interfaces require *charts*, and show that those charts assemble the representable clocks of Garner's comodel semantics. This gives a translation, not an identification of the two approaches. We then ask what changes in the presence of nondeterminism or probability, how potentials evaluate the resulting executions, and which clock constructions respect system composition.

Scope. The elementary results are proved below. A general double category of charted runners and a compositional behaviour-profunctor comparison are proposed, not assumed. The discussion uses public sources only.

1 Why translate clocks into runners?

1.1 The common question, asked from opposite directions

A systems theory should say not just which states and transitions exist, but what it means to *observe an execution*, to *implement* a prescribed operation, and to combine machines without losing the structure on which their specifications depend. Two categorical approaches put different parts of this task first.

For a reader coming from *DOTS*, a clock is itself a particularly simple system. A map from a clock into a machine is a run of that machine; allowing a clock to communicate through an interface produces input signals and observed traces alongside the evolving state. Allowing the clock to vary over finite intervals, with translations and inclusions as morphisms, organizes its trajectories into a presheaf on the category of temporal intervals. This is precisely the construction encountered in our earlier study of temporal sheaves, following Schultz, Spivak and Vasilakopoulou.[1] When compatible local trajectories glue uniquely, the presheaf satisfies the temporal sheaf condition. Thus the DOTS principle that behaviours are maps out of clocks leads directly to the temporal-topos viewpoint, while Garner's construction offers a complementary organization through complete operational behaviours. This is a geometric, morphism-based starting point: *what are the probes from which system behaviour can be seen?*

For a reader coming from *Sweedler theory*, one starts instead with a monad T presenting an operational language and, optionally, a residual monad R. A stateful runner implements operations of T using operations of R; taking $R = y = \mathrm{Id}$ gives comodels. Garner's behaviour category then organizes comodels through their complete observable behaviour. The starting question is

[1] Cf. 'Temporal machines'

algebraic: *how can implementations and observations of a theory themselves be organized by universal constructions?*

These starting points pull in different directions. DOTS takes maps *from* clocks into systems; Garner's final-comodel semantics takes maps *from* systems into a universal object of behaviours. We should not force these into the same universal property. The translation is useful precisely because it explains how they coexist.

1.2 The smallest clock and the first meeting point

For the free unary tick theory $T = \mathbb{N}y$ (one operation tick, with no extra equations), a residual-free comodel is an endomap $u : S \to S$. The clock is the pointed endomap

$$(\mathbb{N}, \mathrm{succ}, 0), \qquad n \longmapsto n + 1.$$

A pointed map of dynamical systems from this clock to (S, u) is determined by its starting state s:

$$\gamma_s : \mathbb{N} \longrightarrow S, \qquad \gamma_s(n) = u^n(s).$$

Thus the clock represents the *choice of an execution starting point*. This is an initial pointed dynamical-system property—a reachability universal property. When observations $o : S \to A$ are introduced, the same starting point has a complete observable future $(o(u^n s))_{n\geq 0}$. Garner's cosemantics records this future, now in the opposite, observability direction. The Hankel pairing from Part X.A offers another way to see this passage from histories to future tests; we return to it after the charted-clock comparison.

The hinge. A map of clock states alone does not specify which input a controlled machine receives at time n, or which target observation corresponds to the clock's reading. A *chart of interfaces* supplies this missing information. Once we include it, the single DOTS clock can be compared with Garner's whole *family* of representable observable clocks.

Our first explicit destination is the following identity, proved in Section 3 for an autonomous observable system X:

$$\underbrace{\mathrm{Sys}(\mathfrak{c}, X)}_{\substack{\text{one DOTS clock}\\ \text{with charts}}} \;\cong\; \coprod_{\alpha \in A^{\mathbb{N}}} \underbrace{\mathrm{Comod}(T_A)(C_\alpha, X)}_{\substack{\text{Garner's}\\ \text{representable clocks}}}.$$

The left side stores the chart as part of the *map*; the right side stores the resulting observation history in the *domain clock*. This is a change of organization, not a choice between rival descriptions.

Our route is therefore deliberate. Section 2 introduces charts and connects them with syntactic wiring; Section 3 proves the Garner comparison. Section 4 shows why the easy clock universality cannot simply survive effects. Section 5 explains how models of the residual effect theory give outgoing quantitative tests. Sections 6 and 7 ask what happens when we compose executions and when we vary their durations. The final section isolates what each approach contributes and the genuinely missing theorem.

The smooth case is an important instance of the clock-based approach. A smooth vector field $F : X \to TX$ defines a differential system, and a solution on an open interval I is precisely a morphism of differential systems from the unit-speed clock $(I, \mathbf{1}_I)$ into (X, F). For every smooth potential $V : X \to \mathbb{R}$, tangent functoriality gives the exact identity

$$\frac{d}{dt}(V \circ \gamma) = \big(dV(F)\big) \circ \gamma.$$

Thus incoming trajectories and outgoing scalar tests are related by tangent–cotangent evaluation, naturally with respect to morphisms of differential systems. This provides a concrete smooth instance of the trajectory–potential pattern that we also study for effectful runners. The remaining question is not whether this pairing exists, but how its tangent-geometric realization relates to the algebra–comodel evaluation and measuring constructions of Sweedler theory.

The concrete clock constructions for stochastic and nondeterministic systems that we later discuss are treated in the paper of Lynch–Myers–Rischel–Staton [1].

2 Interfaces first: charts and simulation

A deterministic clock answers the question "where does the state go next?" but a controlled run must also answer "which action was supplied?" For a Moore machine, an observation can govern that choice. This is why a bare map of state spaces cannot be the whole notion of a trajectory or simulation.

Fix a category $\mathcal{E}$ with finite products and an endofunctor $M : \mathcal{E} \to \mathcal{E}$. An M-Moore system is a tuple (X, I, O, o, k), with state X, inputs I, outputs O, observation $o : X \to O$, and transition $k : X \times I \to MX$. A charted morphism to (Y, J, P, p, ℓ) is a triple

$$h : X \to Y, \qquad a : O \to P, \qquad \psi : O \times I \to J,$$

satisfying

$$ph = ao, \qquad Mh\,k(x, i) = \ell\big(h(x), \psi(o(x), i)\big). \tag{1}$$

The input translator depends on the *visible* output of X, not an arbitrary hidden state. It is the elementary constant-input-bundle version of a Moore interface lens. In DOTS terminology, a chart transports both the observed quantities and the admissible inputs between interfaces. In the syntactic Moore theory of Part III of our tutorial, the same dependence appears when one implements a command by first reading an output and then choosing an update.

From a Sweedler standpoint the important change is that the interface translation is *not merely a map of fixed monads*: it may inspect a current observation. From a DOTS standpoint the extra datum is h: it witnesses that the chosen translation respects the implemented state dynamics. We first check that these data compose.

Theorem 2.1 (Charted systems form a category)**.** *For every such $\mathcal{E}, M$, charted Moore systems and the indicated triples form a category. If (h, a, ψ) is followed by (g, b, χ), their composite is*

$$(gh, ba, \chi \star \psi), \qquad (\chi \star \psi)(o, i) = \chi\big(a(o), \psi(o, i)\big).$$

Proof. The observation equation is $qgh = bph = bao$. Functoriality of M and the transition equations give

$$\begin{aligned} M(gh)k(x, i) &= Mg\,\ell(hx, \psi(ox, i)) \\ &= m(ghx, \chi(p(hx), \psi(ox, i))) \\ &= m\big(ghx, \chi(a(ox), \psi(ox, i))\big). \end{aligned}$$

The identity chart is $(o, i) \mapsto i$. For three composable charts, both parenthesizations evaluate to $\theta(ba(o), \chi(a(o), \psi(o, i)))$, proving associativity. The symbol θ denotes the input chart of the *third* morphism. No monad structure on M is needed. □

The category theorem is a coherence check: translating twice is the same as composing translations, so charts can be used in larger diagrams without ambiguity. But what is their *physical* meaning? A chart ψ determines the following command in the joint Moore theory: read o, update the first system with i, then update the second with $\psi(o, i)$. The read occurs before updating, and the updates are understood in a noninterfering product implementation. A state map h witnesses a property of this wiring. This turns the familiar Part III syntax into a semantic statement about invariant simulations.

Proposition 2.2 (Wiring and invariant graphs)**.** *For deterministic systems $M = \mathrm{Id}$ in* **Set***, let $u : X \times I \to X$ and $v : Y \times J \to Y$, and wire them by*

$$H((x, y), i) = \big(u(x, i), v(y, \psi(o(x), i))\big). \tag{2}$$

Then the charted transition equation for h is equivalent to forward invariance, under every input i, of the graph $\Gamma_h = \{(x, h(x))\} \subseteq X \times Y$. With $ph = ao$, the graph describes an output-compatible simulation.

Proof. The transition from $(x, h(x))$ lands at $\big(u(x, i), v(h(x), \psi(o(x), i))\big)$. It lies in Γ_h precisely when its second coordinate is $h(u(x, i))$, which is (1) for $M = \mathrm{Id}$. □

Proposition 2.3 (Graph-supported coupling)**.** *For any endofunctor M, suppose a clock $c : C \to MC$, a system $k : X \times I \to MX$, a state map $h : C \to X$ and a chart $\iota : C \to I$ satisfy*

$$Mh\, c = k\langle h, \iota\rangle. \tag{3}$$

Then

$$\widehat{c} = M\langle \mathrm{Id}_C, h\rangle\, c : C \to M(C \times X)$$

has marginals $M\pi_C\, \widehat{c} = c$ and $M\pi_X\, \widehat{c} = k\langle h, \iota\rangle$.

Proof. Apply M to the two projection identities, using (3) for the second. □

This joint effect is available *along a given simulation graph.* In general two independently specified marginal effects need not determine a joint effect. Thus these vertical charted morphisms do not yet provide all the squares of a runner double category.

We have answered our first question: a clock map with a chart is a genuine morphism of systems, and a simulation can be checked as an invariant graph. Now we can ask the question of direct interest to a Sweedler reader. Where does this *single charted clock* appear in the universal representation of comodels?

3 From a single charted clock to Garner's representables

To see the answer without concealing it in general machinery, temporarily omit external inputs and retain one update and one observation. Let T_A be the free observation theory whose comodels are $X = (X, u, o)$, with $u : X \to X$ and $o : X \to A$. Its observation operation has output in A; after the operation a new state remains. The final observable future of x is

$$\beta_X(x) = (o(x), o(ux), o(u^2x), \ldots) \in A^{\mathbb{N}}. \tag{4}$$

Garner's behaviour category in this example is the action category

$$\mathcal{B}_A = A^{\mathbb{N}} \rtimes \mathbb{N}, \qquad (\alpha, n) : \alpha \longrightarrow \sigma^n \alpha,$$

where σ is left shift. Under the Sweedler/comodel interpretation the cosemantics comonad has the polynomial form

$$\{T_A, y\}(Z) \cong (A \times Z)^{\mathbb{N}} \cong A^{\mathbb{N}} \times Z^{\mathbb{N}}. \tag{5}$$

An element of $C(Z) \cong A^{\mathbb{N}} \times Z^{\mathbb{N}}$ consists of an observation stream α and a stream z of Z-labels. The counit returns the initial label z_0. The comultiplication retains α and replaces the label stream by the sequence of pairs of tails $(\mathrm{shift}^n \alpha, \mathrm{shift}^n z)$, one for each $n \in \mathbb{N}$. Thus the polynomial positions are complete observation streams and the directions at each position are $\mathbb{N}$. The statement of the polynomial/categorical correspondence is the relevant case of Garner's cosemantics theory [2]; its identification with the Sweedler hom assumes the representing comonad has been constructed.

For each $\alpha \in A^{\mathbb{N}}$ let C_α be the comodel with states $\mathbb{N}$, update $n \mapsto n+1$, and observation $o(n) = \alpha_n$. Under Garner's equivalence $\mathrm{Comod}(T_A) \simeq [\mathcal{B}_A, \mathbf{Set}]$, it is the covariant representable $\mathcal{B}_A(\alpha, -)$.

There is an apparent discrepancy. Garner needs a clock C_α for *every possible observation stream*; DOTS needed only one self-observing clock $\mathfrak{c}$, whose states are time instants. The discrepancy disappears once one remembers that a DOTS morphism includes an observation chart. That chart can supply the entire stream α, so the two constructions are organized differently, not in competition.

Let $\mathfrak{c}$ be the self-observing clock on $\mathbb{N}$, with observation $n \mapsto n$ and one command go. A charted map into X contains a state map $h : \mathbb{N} \to X$ and an observation chart $a : \mathbb{N} \to A$, with

$$h(n+1) = u(h(n)), \qquad a(n) = o(h(n)).$$

Theorem 3.1 (Chart-indexed Garner decomposition)**.** *For every autonomous observable deterministic system X there are natural bijections*

$$\mathrm{Sys}(\mathfrak{c}, X) \cong \coprod_{\alpha \in A^{\mathbb{N}}} \mathrm{Comod}(T_A)(C_\alpha, X) \cong X. \tag{6}$$

The arrow $(\alpha, m) : \alpha \to \sigma^m \alpha$ in $\mathcal{B}_A$ induces the comodel map $C_{\sigma^m \alpha} \to C_\alpha$, $n \mapsto n+m$.

Proof. The charted morphism equations say exactly that $h : C_a \to X$ preserves update and observation. Conversely each such map gives a clock trajectory with chart $a = \alpha$. Evaluation at 0 gives $h \mapsto h(0)$, with inverse $x \mapsto (n \mapsto u^n x)$; the chart is then $\beta_X(x)$. The shift map preserves update and observation because $(\sigma^m \alpha)_n = \alpha_{m+n}$, and compositions add offsets. □

For example, take a two-state toggle $0 \leftrightarrow 1$ whose output is its current state. Choosing initial state 0 gives the chart $\alpha = 0101\cdots$ and a map $C_\alpha \to X$; choosing 1 gives $\alpha = 1010\cdots$ and a map from a different representable. The *underlying clock* $\mathbb{N}$ has not changed: the observable chart has.

The chart labels a *fibre* of trajectories by its observation future. This is suggestive of a presentation or gauge choice, but two different observable streams need not describe the same physical behaviour; no gauge equivalence is asserted. There is also a stronger density statement: writing F_X for the copresheaf corresponding to X, co-Yoneda gives

$$F_X \cong \int^{\alpha \in \mathcal{B}_A} F_X(\alpha) \times \mathcal{B}_A(\alpha, -). \tag{7}$$

This coend glues the reachability probes along their common tails. It is not merely a disjoint decomposition of states.

Why this is a translation. To a DOTS reader, the theorem explains why a single fully observable time clock is powerful enough: its charts determine the target behaviour fibre. To a Sweedler reader, it explains how the covariant representables of $\mathcal{B}_A$ acquire an operational meaning as individual charted probes. The shift arrows tell us how advancing time relates one probe to another. The theorem does *not* say that different observable streams are gauge-equivalent: the chart is a presentation choice, but changing it may change the actual behaviour.

3.1 Histories, future tests, and the Hankel pairing

The familiar Hankel construction, discussed already in Part X.A, sharpens the reachability–observability distinction without requiring a second account of minimisation here. To allow genuine branching of experiments, consider a *controlled* deterministic Moore machine (S, δ, o), with input alphabet I, output set O, and a chosen starting state s_0. Write $\delta_u(s)$ for the state reached after executing the letters of $u \in I^*$ in order; thus $\delta_{uv} = \delta_v \delta_u$. Define

$$r_{s_0} : I^* \to S, \quad u \mapsto \delta_u(s_0), \qquad \mathrm{beh} : S \to O^{I^*}, \quad \mathrm{beh}(s)(v) = o(\delta_v(s)).$$

Here r_{s_0} *executes a history*, whereas beh assigns to every state the outcomes of *all future experiments*. Their composite is the curried Hankel pairing

$$H_{s_0}(u, v) := o(\delta_{uv}(s_0)) = \mathrm{beh}(r_{s_0}(u))(v), \qquad \widehat{H}_{s_0} = \mathrm{beh}\ r_{s_0} : I^* \to O^{I^*}. \tag{8}$$

An operation can be moved from the already-executed history to the beginning of the future test without changing the observation:

$$H_{s_0}(uw, v) = H_{s_0}(u, wv). \tag{9}$$

Thus a Hankel *row* is the complete observable future of a reached state: $\widehat{H}_{s_0}(u) = u^{-1}\,\mathrm{beh}(s_0)$, with $(u^{-1}b)(v) = b(uv)$. The distinct reachable rows give the usual reachable–observable realisation, as recalled in Part X.A. In this sense, Garner's outgoing semantics collects all continuation tests into a single behaviour, while clock maps supply the incoming, history-executing side. Garner's behaviour category also retains the *arrows* induced by operations, not only the resulting set of futures.

For the autonomous one-tick system of Theorem 3.1, the input alphabet is a singleton, so $I^* \cong \mathbb{N}$. If $\alpha = \beta_X(s_0)$, the Hankel matrix becomes

$$H_{s_0}(m, n) = o(u^{m+n} s_0) = \alpha_{m+n}, \qquad \widehat{H}_{s_0}(m) = \sigma^m \alpha.$$

Its rows are exactly the successive tails organised by $\mathcal{B}_A = A^{\mathbb{N}} \rtimes \mathbb{N}$; the arrows remember the number of ticks even when two tails agree. This is the unary instance of the history/future balancing law, *not* a claim that our one linear clock, with a single chosen input chart, already probes every word of a multi-input machine. For the full $I^* \times I^*$ pairing one must vary the charts or use a branching input clock. The distinction will matter once clocks become effectful.

There is a limitation, however. Our simple clock is initial because a deterministic tick has a unique successor. What remains when an operation can branch or sample randomly?

4 Effects change what a universal clock can mean

Here a first payoff of the Sweedler translation appears. Rather than inventing a new notion of clock for each kind of dynamics, we can retain the unary operational theory and change the *residual* effect theory. For a strong monad M on **Set**, a runner implementing the free unary tick theory $\mathbb{N}y$ with residual effects in M and state C is determined by

$$c : C \longrightarrow MC. \tag{10}$$

Indeed, the runner implements the single generating operation by c and extends it to words of ticks using the monad laws. Strict runner maps with fixed theory boundaries are precisely M-coalgebra maps. This object-level calculation does not require the general measuring comonad $\{\mathbb{N}y, M\}$ to have been constructed.

For nondeterminism let $M = \mathcal{P}$ and take a nonempty branching alphabet Ω. The finite-word clock is

$$c : \Omega^* \to \mathcal{P}(\Omega^*), \qquad c(w) = \{w\omega \mid \omega \in \Omega\}.$$

Its charted maps into $k : X \times I \to \mathcal{P}X$ satisfy

$$k(h(w), \iota(w)) = \{h(w\omega) \mid \omega \in \Omega\}. \tag{11}$$

Thus every visited target state has at most $|\Omega|$ distinct successors; if $\Omega \neq \varnothing$ there can be no visited deadlock. This is a restriction imposed by the choice of clock, not a claim of initiality. The construction and graph-decorated variants are developed in [1].

For stochastic systems, the public construction of [1, Definition 4.7 and Theorem 4.10] uses a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$ and the clock $\widetilde{\Omega} = \coprod_n (\Omega, \mathcal{F}_n)$, with transitions defined by conditional-distribution kernels. Maps out of it recover adapted processes satisfying the system transition law. Strict equality of kernels depends on chosen versions; passing to almost-sure equivalence is essential in the probability-sheaf result [1, Theorem 6.7]. The alternative independent-sampling word clock already appears in [1, Remark 4.11].

Proposition 4.1 (No general initial pointed effectful clock)**.** *There is no initial object among pointed strict coalgebras for either the finite-powerset monad $\mathcal{P}_{\mathrm{fin}}$ or the finite-distribution monad $\mathsf{D}_{\mathrm{fin}}$.*

Proof. For finite powersets, a pointed map to the one-state deadlock forces the initial state's successor set to be empty, while a map to the one-state self-loop forces it to be nonempty. For distributions, take $Q = \{0, 1, 2\}$, pointed at 0, with $q(0) = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$ and $q(j) = \delta_j$ for $j = 1, 2$. The automorphism interchanging 1 and 2 and fixing 0 forces the unique pointed map from a hypothetical initial coalgebra to have image in $\{0\}$. At its distinguished state the coalgebra law would then equate δ_0 with $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$, a contradiction. □

The obstruction is not a failure of the clock method. It distinguishes two tasks. A particular stochastic or branching clock can *represent a specified class of runs* through charted maps; it need not be initial among all pointed effectful systems. The choice of filtration, source of randomness, or branching alphabet now becomes meaningful data rather than a hidden implementation detail.

So far we have asked how to *produce* runs. To test safety or stability we need the complementary operation: evaluating quantities *along* those runs. This is where the Sweedler reader recognizes a second familiar ingredient, namely an algebra for the residual effect monad.

5 Testing runs: effect models and smooth potentials

A behaviour theory should support both execution and specification. DOTS expresses the first through a map from the clock, and the second through a map from the system to a space of quantities or a specification template. Sweedler's implementation picture suggests how to evaluate the second against the first: supply a *model of the residual effects* in which to interpret the observable values.

Let $a : M\mathbb{R} \to \mathbb{R}$ be an algebra for a monad M. Examples are expectation for finite distributions and maximum for the *nonempty* finite-powerset monad. It evaluates an M-effect on a real-valued observable. Given a coalgebra $c : C \to MC$, a potential $W : C \to \mathbb{R}$ has one-step drift

$$\mathfrak{D}_c W = a\, MW\, c - W. \tag{12}$$

The expression $a\,MW\,c$ is a model–coalgebra interaction: the coalgebra supplies the effect, W translates its result into scalars, and the M-model a interprets the effect. This is already Sweedler-shaped on the operational side, although it is not by itself an identification with the measuring pairing in an arbitrary duoidal category.

Theorem 5.1 (Naturality of model evaluation along a charted trajectory)**.** *Let $c : C \to MC$, $k : X \times I \to MX$, and (h, ι) satisfy $Mh\,c = k\langle h, \iota\rangle$. For every potential $V : X \to \mathbb{R}$,*

$$\mathfrak{D}_c(Vh)(z) = a\,MV\,k(h(z), \iota(z)) - V(h(z)). \tag{13}$$

Proof. The effect part of the left side is $a\,M(Vh)c = a\,MV\,Mh\,c = a\,MV\,k\langle h, \iota\rangle$; subtract Vh. In particular, an inequality on the target drift pulls back to the clock. □

For a stochastic kernel this is expected drift; for nonempty finite branching it is worst-case (maximum) drift. No integrability for arbitrary Giry kernels or convention for the empty maximum is asserted.

In the smooth case the vector field is a section $F : X \to TX$, not an ordinary coalgebra for the preceding monad M. A clock is an interval I with unit vector field ∂_t, and $\gamma : I \to X$ is a trajectory if $T\gamma\,\partial_t = F\gamma$. A smooth potential $V : X \to \mathbb{R}$ satisfies

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}(V\gamma)(t) = \langle \mathrm{d}V_{\gamma(t)}, F(\gamma(t))\rangle = \mathrm{pr}_2\big(TV(F(\gamma(t)))\big).} \tag{14}$$

Here $T\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ and pr_2 extracts the infinitesimal scalar component. This is the tangent-bundle analogue of evaluating an effect against a potential, *not* a claim that $\mathrm{pr}_2 : T\mathbb{R} \to \mathbb{R}$ is an algebra for the same monad M. It is the physically familiar version of the same *incoming trajectory/outgoing test* pattern: a potential produces a scalar history, and its infinitesimal change is governed by the dynamics.

If $\langle \mathrm{d}V, F\rangle \le 0$ along a solution, $V\gamma$ is nonincreasing. More quantitatively, suppose $\rho : X \to \mathbb{R}_{\ge 0}$ is a goal function and increasing comparison functions $\alpha_\pm$ satisfy

$$\alpha_-(\rho(x)) \le V(x) \le \alpha_+(\rho(x)), \qquad \langle \mathrm{d}V_x, F(x)\rangle \le 0. \tag{15}$$

Then every solution on an interval I obeys the two-time estimate

$$\rho(\gamma(t)) \le \alpha_-^{-1}(\alpha_+(\rho(\gamma(s)))) \quad (s \le t \text{ in } I). \tag{16}$$

The proof is the chain of inequalities $\alpha_-(\rho\gamma(t)) \le V\gamma(t) \le V\gamma(s) \le \alpha_+(\rho\gamma(s))$. The same estimate holds after any translation-embedding of intervals; hence it is natural in the clock's duration. Stronger differential decay and properness of ρ are substantive *additional* hypotheses for asymptotic-stability conclusions.

The two-time form of the estimate matters: it holds on every subinterval and survives translating the temporal origin. In other words, the specification is compatible with the way clock trajectories restrict. But systems are not only observed separately; DOTS also insists on *modular composition*. Does a clock-based account of behaviour preserve it? The answer is yes in the simplest deterministic case and already more subtle for stochastic systems.

6 Composing trajectories: copying time versus effects

A clock map turns the composition of systems into a question about the structure of the *clock*. Given separate trajectories $C \to X$ and $C \to Y$, we would like a trajectory into a synchronous product $X \otimes Y$. To construct it, the time source must split coherently so that the same clock drives both components. This is not an extra detail of a product state space; it is a property of the representable behaviour functor.

An object C in a symmetric monoidal category $\mathscr{S}$ represents runs by $\mathscr{S}(C, -)$. A comonoid $C \to C \otimes C$, $C \to \mathbb{I}$ then makes this representable functor lax monoidal, since pairs of clock maps combine by precomposing with the diagonal. In an indexed theory one instead expects colax clock comparison maps

$$d_{p,q} : C_{p \otimes q} \longrightarrow C_p \otimes C_q, \tag{17}$$

coherent with the tensor. Such maps are *additional data*, not the Sweedler measuring-coalgebra structure.

Theorem 6.1 (Synchronous decomposition of Garner clocks)**.** *For autonomous deterministic observation systems with output alphabets A, B, the clocks have canonical maps*

$$d_{\alpha,\beta} : C_{\langle \alpha,\beta \rangle} \longrightarrow C_\alpha \times C_\beta, \qquad n \longmapsto (n, n).$$

They induce natural bijections

$$\mathrm{Comod}(T_{A \times B})(C_{\langle \alpha,\beta \rangle}, X \times Y) \cong \mathrm{Comod}(T_A)(C_\alpha, X) \times \mathrm{Comod}(T_B)(C_\beta, Y),$$

coherently for finite synchronous products.

Proof. The diagonal preserves successor and pairs the correct observations. A map to a product machine is a pair of maps satisfying the individual equations; pointwise pairing gives its inverse. Associativity, symmetry and units follow from the corresponding identities of iterated diagonals $\mathbb{N} \to \mathbb{N}^n$. □

The deterministic theorem shows what the two perspectives gain from one another. DOTS asks for decompositions of its representing clocks; Garner's indexed clocks provide them explicitly by the diagonal on $\mathbb{N}$, coherently across behaviour labels. It is tempting to copy the same proof for probability, but that would confuse synchronizing the *time index* with synchronizing the *random draw*.

For stochastic systems equipped with *independent product* transitions, a similar bijection generally fails. Write D for the finite-distribution monad; the same argument applies to measurable kernels wherever the products and pushforwards exist.

Theorem 6.2 (Stochastic product comparison)**.** *Fix any D-coalgebra $c : C \to \mathsf{D}C$ and stochastic systems $k : X \to \mathsf{D}X$, $\ell : Y \to \mathsf{D}Y$. Give $X \times Y$ the independent product kernel $(x, y) \mapsto k(x) \otimes \ell(y)$. Projections induce an injective map*

$$\Phi : \mathrm{Coalg}_{\mathsf{D}}(c, k \otimes \ell) \hookrightarrow \mathrm{Coalg}_{\mathsf{D}}(c, k) \times \mathrm{Coalg}_{\mathsf{D}}(c, \ell), \tag{18}$$

where $\mathrm{Coalg}_{\mathsf{D}}(c, k)$ denotes the set of strict coalgebra maps from (C, c) to (X, k). Its image consists exactly of pairs (f, g) satisfying

$$\mathsf{D}\langle f, g \rangle(c(z)) = k(f(z)) \otimes \ell(g(z)) \qquad (z \in C). \tag{19}$$

This comparison need not be surjective. The condition is conditional independence of the next two observed states *under the clock's one-step kernel, not merely that both components individually satisfy their Markov laws. The corresponding statement for an adapted stochastic clock uses conditional distributions given the current filtration.*

Proof. The marginals of an independent product distribution are $k(x)$ and $\ell(y)$, so projections are coalgebra maps and Φ is well-defined. A function into $X \times Y$ is determined by its components, hence Φ is injective. The pair $\langle f, g\rangle$ is a coalgebra morphism precisely when (19) holds. To see failure of surjectivity, let $C = \{0, 1\}$ with $c(z) = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$, and let $X = Y = \{0, 1\}$ with constant fair-coin kernels $k = \ell = c$. Then both identity maps $f = g = \mathrm{Id}$ are coalgebra morphisms, but the joint successor under $\langle f, g\rangle$ is supported on $(0, 0), (1, 1)$, whereas the product kernel gives probability $1/4$ to each of the four pairs. □

The failure persists for adapted processes: on the canonical filtration of independent fair coin tosses, take both state processes to equal the most recently observed toss (and give them an arbitrary fixed state at time zero). Both component conditional laws are fair coin at each step, while the joint next-state law is concentrated on the diagonal. Thus stochastic clock maps admit a canonical *comparison from joint runs to pairs of marginal runs*, rather than the isomorphism of Theorem 6.1. In categorical language this is a colax/oplax direction; its compatibility with a full 2-algebra of interfaces remains to be proved. The image condition (19) is an obvious candidate for a sheaf-theoretic interpretation when clocks vary over filtrations.

There is an elementary obstruction to the opposite comparison. A strict diagonal $C \to C \times C$ would require $\mathsf{D}(\Delta_C)c(z) = c(z) \otimes c(z)$; for finite distributions it holds exactly when $c(z)$ is a point mass. Likewise for powersets with product transitions it holds exactly when the successor set is empty or a singleton. *Copying a shared time coordinate does not mean copying the same random choice.*

For two smooth systems, however, the interval-clock diagonal $I \to I \times I$, $t \mapsto (t, t)$, is a vector-field morphism from 1 to $(1, 1)$; hence $\mathrm{Sol}_{F\times G}(I) \cong \mathrm{Sol}_F(I) \times \mathrm{Sol}_G(I)$, naturally under interval translations.

This difference gives a concrete target for a future compositional semantics: identify the colax structure on clocks that permits a joint run, and specify what coupling information is being retained or discarded. The existence of a runner coalgebra by itself does not supply this structure. We now turn to a different dimension of the same problem: how varying the *amount of time observed* organizes runs into presheaves.

7 From complete futures to finite durations

Garner's behaviour objects record complete observable futures. Much of practical systems work, however, concerns finite tests and observations on bounded time windows. DOTS accommodates this by replacing one clock with a *theory of clocks*: a functor assigning a clock system to each duration, with embeddings of shorter durations into longer ones. Mapping out of those clocks gives a time-varying set of runs.

A duration-indexed clock functor $C : \mathsf{Time} \to \mathrm{Sys}$ gives the presheaf nerve

$$N_C(S)(t) = \mathrm{Sys}(C_t, S), \qquad N_C(S) : \mathsf{Time}^{\mathrm{op}} \to \mathbf{Set}. \tag{20}$$

This differs from Garner's copresheaf presentation of complete observable behaviours: the indexing category and the variance express different universal properties. The distinction is worth keeping even in the unary example, where the one-object monoid category $\mathbf{B}\mathbb{N}$ is accidentally self-opposite

because addition is commutative. There is nevertheless a striking, exact construction linking it to finite intervals.

Let $\mathsf{Int}_{\mathbb{N}}$ have objects $n \in \mathbb{N}$ and morphisms $n \to m$ given by offsets $h \geq 0$ with $h + n \leq m$. The offset selects a length-n window inside a length-m interval. Let $\mathbf{B}\mathbb{N}$ be the one-object category with morphism monoid $(\mathbb{N}, +)$.

Proposition 7.1 (Interval time as twisted arrows)**.** *With the convention that an arrow $f \to g$ in $\mathrm{Tw}(\mathcal{B})$ is a factorization $g = vfu$, there is an isomorphism*

$$\mathsf{Int}_{\mathbb{N}} \cong \mathrm{Tw}(\mathbf{B}\mathbb{N}).$$

A morphism $n \to m$ with offset h corresponds to the factorization $m = h + n + (m - n - h)$.

Proof. The objects of $\mathrm{Tw}(\mathbf{B}\mathbb{N})$ are elements $n \in \mathbb{N}$. A morphism $n \to m$ is exactly a pair (h, q) with $m = h + n + q$, equivalently an allowable offset h. Composition adds the leading and trailing offsets, which is the composition rule of interval-window embeddings. □

For $\mathcal{B}_A = A^{\mathbb{N}} \rtimes \mathbb{N}$, an object of $\mathrm{Tw}(\mathcal{B}_A)$ is an arrow $(\alpha, n) : \alpha \to \sigma^n \alpha$. A factorization of it into a longer arrow remembers not only the window's offset but the ambient *complete* observation stream. The forgetful functor

$$\mathrm{Tw}(\mathcal{B}_A) \longrightarrow \mathsf{Int}_{\mathbb{N}}, \qquad (\alpha, n) \longmapsto n \tag{21}$$

forgets that stream. On morphisms it sends a factorization to its leading offset. It is faithful in this elementary example, but generally not full: an offset from n to m lifts between specified (α, n) and (β, m) only if $\alpha = \sigma^h \beta$. Thus $\mathrm{Tw}(\mathcal{B}_A)$ is a well-defined candidate for *indexed windows of complete behaviours*, not automatically the correct category of finite observable histories. This sharpens the translation question: how does one *forget unobserved future data* while preserving the operations on windows? An arbitrary twisted-arrow construction does not perform that forgetting for us.

Finite charted clocks also differ from total comodels: a length-n clock has states $0, \ldots, n$, with no required update at the terminal state. Its variable interface permits precisely n selected inputs and $n + 1$ observations. It is not generally a comodel for the *fixed total* unary-operation theory. Nevertheless, for relational systems the finite-clock nerve is fully faithful. This tells us something different from Theorem 3.1: the first theorem packages initial-state runs by their *complete observation streams*; the next theorem shows how a system can be recovered from its *finite labelled paths*. The two density results operate in different categories, and neither makes the other redundant.

Theorem 7.2 (Fixed-interface nerve–realisation)**.** *Fix an observation set O and input family $I(o)$. Let $\mathscr{R}_{I,O}$ be the category of sets S with observations $o : S \to O$ and relations $x \xrightarrow{i} x'$ $(i \in I(o(x)))$, with output- and transition-preserving maps. Let $W_{I,O}$ be the presheaf of finite interface words. Then the finite-path nerve*

$$N : \mathscr{R}_{I,O} \to [\mathsf{Int}_{\mathbb{N}}^{\mathrm{op}}, \mathbf{Set}]/W_{I,O}$$

is fully faithful and has a left adjoint R. The state set of RP is P_0; there is a transition $x \xrightarrow{i} x'$ when an element of P_1 has these endpoints and the input label i. The counit $RN \to \mathrm{Id}$ is invertible.

Proof. A relational path of length n is uniquely determined by its sequence of n labelled one-step transitions. Thus $N(S)_n$ is the iterated fibre product of $N(S)_1$ over $N(S)_0$. A map $RP \to S$ assigns to each P_0-state an S-state, respecting observations, and sends the endpoints and label of every element of P_1 to an existing transition in S. It extends uniquely to a displayed natural transformation $P \to NS$ by mapping each higher cell to the sequence of its one-step windows. Conversely, a displayed transformation restricts to such a map of relations. This proves $R \dashv N$. Since relational transitions have no extra parallel-edge witnesses, realizing the paths of S recovers its original relation, hence $RN \cong S$. □

The unit $P \to NRP$ need *not* be surjective in higher degrees: take one state, one loop in P_1, and $P_n = \varnothing$ for $n \geq 2$. The realization has a length-two loop but P_2 is empty. Thus fully faithfulness of the nerve is not a claim that arbitrary temporal presheaves already satisfy the one-dimensional Segal condition. Also, the above adjunction realizes relations; it is not a general minimal-realisation right adjoint.

Finally, for the *one-sided prefix* subcategory $(\mathbb{N}, \leq)$, compatible finite-prefix trajectories are exactly infinite runs. Inverse limits of such prefixes do not include deadlocked finite trajectories, since those cannot be extended to all lengths. A deadlock-inclusive completion would need an explicit stopping or partial-path convention. This distinction is relevant when comparing finite histories with Garner's completed futures.

We now have two answers to "what do clocks see?": complete executions organized by Garner's behaviour category, and local finite paths organized by duration. A successful general comparison must respect both the information *in* a run and the ways the run can be restricted and combined. This leads to the last question: what have we actually gained on each side, and what remains an open theorem?

8 What each approach learns—and the theorem still missing

8.1 A translation with two useful directions

The results above should be read as a bridge rather than as an equivalence of two frameworks.

For a DOTS reader, the Sweedler description explains why the ordinary clock is related to a universal operational object: the tick is a unary theory, an M-effectful tick is a runner with residual theory M, and complete observable histories are indexed by Garner's category $\mathcal{B}_A$. The single charted clock does not compete with Garner's representables; Theorem 3.1 identifies its maps with a disjoint union of their maps, and co-Yoneda shows how these probes generate comodels. The Hankel factorisation (8) makes the complementary roles of executed histories and future tests explicit. Moreover the residual-effect algebra evaluates tests on such trajectories through the naturality equation (13).

For a Sweedler reader, the clock perspective gives the representables an immediate operational interpretation: they are probes used to generate runs, and the chart tells us how their inputs and observations attach to a machine. It insists that an input chart can be state-of-clock dependent even when no strict map between fixed operational theories expresses it. It also exposes structures absent from the bare comodel equivalence: durations and restriction maps, synchronization of clocks, and the difference between a common time coordinate and a joint stochastic effect.

The achievements are modest but tangible. We have a category of charted maps (Theorem 2.1); simulations as invariant graphs (Proposition 2.2); an exact deterministic chart-to-Garner comparison (Theorem 3.1); an obstruction to universal pointed effectful clocks (Proposition 4.1); naturality of effect-model evaluation (Theorem 5.1); and a precise difference between deterministic and stochastic clock composition (Theorems 6.1 and 6.2). The twisted-arrow result and finite-path nerve–realisation theorem then place the comparison inside duration-indexed semantics.

8.2 What is not yet established

We have *not* constructed a double category in which every charted interface map is a loose Sweedler runner square; nor have we proved a double functor from such runners to behavioural profunctors preserving all the relevant wiring operations. Having a vertical category of charted maps does not

supply horizontal runner composition and coherent squares by itself. And a comonoid structure for synchronous clocks is additional to the measuring coalgebra used to express a runner.

The next tests are therefore specific. One is to build charted runner squares whose state-dependent operation choices reduce to (1), and prove their compatibility with the syntactic wiring of Moore theories. Another is to test $\mathrm{Tw}(\mathcal{B}_T)$ as a time category, starting with T_A, and identify a sound quotient or restriction that forgets the ambient infinite tail. A third is to characterize the joint-run image condition (19) over varying stochastic filtrations, including almost-sure equivalence.

There is also an analytic test suggested by continuous-time dynamics. For a Markov semigroup (P_t) on a compact state space K, positive unital operators $P_t : C(K) \to C(K)$ correspond to Radon kernels $K \to \mathrm{Rad}(K)$, giving an effectful-clock description at each time. But the infinitesimal generator L is generally unbounded and has a proper domain $D(L)$; a trajectory map intertwining generators must satisfy both

$$\nabla_C(Vh) = (LV)h \qquad (V \in D(L)) \tag{22}$$

and the requisite domain preservation. For diffusion generators the Leibniz defect

$$\Gamma_L(V, W) = L(VW) - VLW - WLV$$

measures infinitesimal covariance; for $L = b \cdot \nabla + \frac{1}{2}a : \nabla^2$ it is $\nabla V^{\mathsf{T}} a \nabla W$. The model-evaluation viewpoint should eventually account for both drift and second-order covariance. The analytic domains are not supplied automatically by ordinary Set-based measuring theory.

Published work provides relevant tools but not the missing comparison theorem: the clock constructions in [1] give effectful examples, while Capucci–Myers [3] study lifting discrete-opfibration classifiers through suitable 2-algebraic structure. Invoking that technology for Sweedler runners would require a specific enhanced 2-category and verification of its hypotheses.

The proposed synthesis. DOTS supplies clock-shaped *incoming probes*, charts, and a calculus of system interconnection. Sweedler theory supplies operational *implementation* with residual effects, Garner's organization of observable futures, and models of effects for evaluating *outgoing tests*. Our calculations show these structures meet in deterministic examples and reveal exactly where effectful and compositional versions need more structure. The goal is not to replace either approach, but to prove that the translation respects wiring, observation and specification.

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