Commands, Capacities, and What Remains to Be Done

The triadic structure of Sweedler measuring

28 October 2026; revised 5 November 2026

Abstract

An agent need not carry out an instruction from beginning to end. It can respond to one command language by generating a command in another: a cab driver takes a traveller to a station, a software handler turns a request into a database query, and a human delegate transforms an ambitious instruction into more manageable tasks. This suggests reading Sweedler measuring as a three-way relation between a theory of demands T, a comonadic form of responding capacity D, and a residual theory R. The three associated universal constructions describe, respectively, the commands a capacity can interpret, the capacities adequate for a given translation, and the residual theory universally generated by the interaction. The account becomes particularly suggestive for dependent command languages, where responding requires more than executing a fixed alphabet: it requires coherent command formation, substitution, context extension, and delegation. We distinguish competence for one command theory from a uniform capacity across theories in a doctrine, and identify the duoidal, constructive, and contextual representation problems that remain open. Finally, we distinguish lawful interaction from adequate interaction. Assessing a response requires standards, which are naturally supplied by algebras of the residual theory. The resulting adequacy relation is genuinely triadic among particular demands, capacities and standards, rather than among the theories themselves.

1 A command is not always an instruction to finish the job

Imagine a thermostat whose setting is changed, a sheepdog hearing a whistle, a computer receiving a store or lookup instruction, and a person asked to organise a complicated journey. All are responding to commands, but the languages in which the commands are expressed have very different structures. The thermostat may need only a few transitions. The sheepdog must interpret its whistle in the light of an unfolding situation. A store must maintain locations and satisfy equations governing reads and writes. A human may have to interpret references, introduce new objects, organise subtasks, or ask someone else for help. It is tempting to arrange these responders on a single scale of competence: the more sophisticated the language, the more sophisticated the agent must be. But that ignores *delegation*. Asked to arrange travel to a remote village, a cab driver may take us only as far as the railway station. That is nevertheless a perfectly useful response if the next stage of the journey can be handled elsewhere. In the same way, an agent may carry out part of a complicated command and pass the remaining work on as a residual computation. This is the central insight of the note: *competence is relative both to the demands and to the residual operations permitted*. There is no useful absolute measure of D's command-following capacity without mentioning both T and R.

2 The three participants in an interaction

In the familiar Set-based presentation, a measuring interaction has components

$$\theta_{X,Y} : TX \times DY \longrightarrow R(X \times Y), \tag{1}$$

natural in the parameters (and dinatural in formulations using an associated pairing). The mathematical data have three different jobs:

- T is the *command theory*: it supplies commands, their composition, returns, and equations.
- D is a *comonadic form of responding capacity*: it supplies observations or structured ways of continuing from a present situation. An individual responder is typically a D-coalgebra, not the comonad itself.
- R is the *residual theory*: it specifies the operations in which the response is allowed to leave its work expressed.

The map θ is not merely an arbitrary interpreter. Its measuring laws connect the multiplication and unit of T, the comultiplication and counit of D, and the multiplication and unit of R. In plain terms, a composite instruction must be handled compatibly with the way its pieces are handled, and the response must remain usable for continuation. The residual R is not a specific unfinished task: the element of $R(X \times Y)$ *is* the residual computation. The monad R describes the repertoire in which that task may be expressed.

The cab-driver reading

Let T express an ambitious demand, such as arriving at a village; let D represent a kind of transport service capable of handling some part of the demand; and let R allow the production of onward transport tasks. A driver who reaches the station and leaves the rail leg to others may support a valid T-to-R interaction, even though the same driver cannot supply a T-to-Id interaction. This is an analogy, not a literal identification of journeys with monads: an actual model would have to specify the types, commands, effects, and laws.

3 Three universal ways to view the same interaction

Under the standard existence and duoidal hypotheses, the Sweedler adjunctions relate three sets of data:

$$\boxed{\begin{aligned}\mathrm{Mon}(D \triangleright T, R) &\cong \mathrm{Mon}(T, [D, R]) \\ &\cong \mathrm{Comon}(D, \{T, R\}).\end{aligned}} \tag{2}$$

These are descriptions of the same measuring laws, not three unrelated correspondences. Here $D \triangleright T$ is the Sweedler copower, $[D, R]$ the convolution monad, and $\{T, R\}$ the Sweedler hom or measuring comonad.

Fix D and R: what commands can be supported?

The convolution monad $[D, R]$ is an *effective command theory* formed from a capacity and the residual operations permitted to it. A monad morphism

$$T \longrightarrow [D, R]$$

interprets the given command language compositionally in that effective theory. In the ordinary endofunctor presentation the underlying convolution expression is, when defined,

$$[D, R](X) \cong \mathrm{Nat}_Y\big(DY, R(X \times Y)\big).$$

This describes operations that can be run uniformly against any Y-labelled responding environment. Naturality prevents an implementation from relying on accidental details of the label representation.

Fix T and R: which capacities can support it?

The measuring comonad $\{T, R\}$ classifies the ways of responding to T-commands through R. A comonad morphism

$$D \longrightarrow \{T, R\}$$

exhibits a measuring interaction of the candidate capacity D with T into R. It does *not* say that every comonad is already competent: the morphism is the necessary additional structure. Coalgebras of the universal measuring comonad provide particular stateful implementations under the established runner–coalgebra correspondence.

Fix T and D: what residual is universal?

The Sweedler copower $D \triangleright T$ is a monad with the universal property

$$\mathrm{Meas}(T, D; R) \cong \mathrm{Mon}(D \triangleright T, R).$$

It is the universal *residual theory generated by this kind of interaction.* Choosing an actual residual theory R amounts to interpreting those generated operations by a monad morphism $D \triangleright T \to R$. Universal does not mean the minimal work remaining in an optimisation sense. It means initial among algebraic destinations for such measuring laws.

Held fixed	Universal object	Everyday question
D, R	$[D, R]$	Which instructions can this service interpret using these resources?
T, R	$\{T, R\}$	What kind of responding capacity would make this delegation possible?
T, D	$D \triangleright T$	What is the universal language for handing on what this service leaves to be done?

4 Why varying the residual changes the account of competence

Suppose $R \to R'$ is a monad morphism. Postcomposing a measuring interaction with this map gives

$$TX \times DY \xrightarrow{\theta} R(X \times Y) \longrightarrow R'(X \times Y).$$

Thus every R-mediated interpretation gives an R'-mediated interpretation whenever such a translation is available. This is a precise sense in which residual choice changes what a responder can accomplish. It is *not* a claim that all residual theories lie in one linear order of expressive strength. This is clearest when T is demanding and D is modest. A

sophisticated source language can be translated through a simple responder into a slightly less demanding residual language. The responder may do little more than identify the first executable step, select an appropriate tool, or re-express the remainder of the task. Even so, its measuring law must respect the units, substitutions, and equations of the source theory. Merely forwarding arbitrary strings does not automatically produce a lawful runner. One can also compose delegations. If a first runner takes T-demands to residual R, and a second handles R-demands through Q, their composition, where the relevant runner structure exists, gives a T-to-Q runner. This is already part of the composition of loose arrows in the double category of runners. It supplies an algebra of handoffs, distinct from any optimisation problem about selecting the cheapest or fastest sequence of services.

5 Where do the theories live?

The examples above range from commands drawn from a fixed alphabet to instructions involving dependent contexts. This is not merely an increase in vocabulary. It can change the *kind of theory* being interpreted and the category in which that theory is represented. Before passing to dependent instructions, we should locate the Sweedler constructions in their categorical home.

A doctrine, a theory, and its presentation

A *doctrine* specifies the forms of judgement and structure available for making theories: for example, finite-product algebraic logic or a dependent doctrine supporting contexts, substitution, and comprehension. A *particular theory* specifies operations and equations within that doctrine. Its *presentation* is one syntactic description of those operations and equations; a semantic description might instead be a Lawvere theory, a structured category, or a suitable monad. These distinctions are emphasized in Shulman's discussion of levels of theory and syntactic versus semantic theories [2]. They prevent us from treating a change of doctrine as simply adding more generators to a fixed command monad. For ordinary accessible endofunctors, our standard ambient example is

$$\mathcal{D} = [\mathbf{Set}, \mathbf{Set}]_{\mathrm{acc}}.$$

There are two compatible monoidal products, composition $\circ$ and a second product $\star$, with duoidal interchange. Both T and R are monoids for $\circ$ (hence monads), and D is a comonoid for $\circ$ (hence a comonad). The second product, and its compatibility with composition, supply the convolution and measuring machinery. The three adjunctions in (2) require still further closure and existence hypotheses. A collection of monads and comonads in an arbitrary category is not enough.

One tool, or a toolkit?

A knife is suited to cutting; a corkscrew to opening a bottle. A multifunction penknife can respond to several kinds of demand by selecting the appropriate implement. It need not be as good at each task as a dedicated tool. Nor can it necessarily complete a task alone: a tool may prepare the work for another service, leaving a residual obligation. This distinguishes *competence for one theory* from *competence uniform across a class of theories*. For a fixed T and R, a measuring map $D \to \{T, R\}$ witnesses a particular interpretation. A "penknife" capacity instead invites a coherent family of measuring laws

$$\theta_{T,R} : (T, D; R),$$

indexed by the admissible theories and residuals of a doctrine. Such a family should respect translations of theories, and possibly changes of residual. A single comonad D does not automatically supply this family: its existence and coherence are extra mathematical structure. In particular, varying R may let even a modest capacity respond to sophisticated commands by passing most of the work onward.

Moving to dependent syntax

A dependent doctrine can be presented by contextual categories, categories with attributes, categories with families, or categories with representable maps, under their respective hypotheses. Such structures organize contexts, extensions, and substitutions; they do not in general directly produce the operational monad of commands we wish to execute. One may model suitable dependent theories by monads on a presheaf category, or instead by indexed, fibred, or relative monads over a comprehension structure. These are distinct semantic choices. In particular, a monad *on* a presheaf topos is not automatically a monad *internal to* that topos. To extend the Sweedler triad, we must identify an appropriate ambient category of operational theories, construct or verify its duoidal interaction, and establish the relevant adjoints. Only then can the three transposes be read literally in the new setting. The problem of transporting them under a change of doctrine remains open; even a functor between ambient categories need not preserve measuring objects or copowers. The resulting question is two-tiered: can this responder interpret *this* dependent command theory through *this* residual, and can one responder interpret *arbitrary* theories of a chosen doctrine in a way coherent under changes of presentation and substitution? The latter is a uniformity requirement, not merely a claim that D has more states.

6 What changes as the command language grows richer?

For a fixed alphabet A of sequential actions, the free command monad has the simple form

$$T_A(X) = A^* \times X.$$

A deterministic responder may have transitions $u_a : S \to S$ for each $a \in A$. Its possible futures are organised by words in A^*; one familiar behaviour comonad is $DY = Y^{A^*}$. The structure records all action histories, and its comultiplication shifts attention from a history to the histories continuing after it. A store-and-retrieve language brings equations: two reads without intervening writes must agree, and a read after a write must return the written value. A responder is not competent merely because it can process the individual instructions. Their *compositions* must satisfy the equations, at least in the residual interpretation. A dependent command language brings another layer. Consider a sequence of commands that introduces an $x : A$, subsequently constructs a $y : B(x)$, and then performs an action depending on both. The later command is not available in the original context:

$$\Gamma \longrightarrow \Gamma, x : A \longrightarrow \Gamma, x : A, y : B(x).$$

To interpret the language, a responder may need to track environments, context extension, typing dependencies, and substitution. But again this is relative to R. A simple D might delegate type checking, proof search, or allocation to the residual, provided the resulting map respects the dependent syntax and its equations. A model of a dependent type theory and a runner of commands expressed in that theory are therefore not automatically the same thing. A model interprets the theory's operations; a runner may translate them into

permitted residual operations. Nor does lawful interpretation mean that the responder can solve arbitrary proof-search or inhabitation problems. At the natural-language end, one encounters an additional phenomenon: commands can establish or revise the meaning of subsequent commands ("from now on, by the active region I mean ..."). Categories of contexts and theories, together with translations between them, offer possible ingredients for describing this. The required measuring representation has not yet been established.

7 The dependent and constructive research questions

Our recent investigations of categories with attributes, contextual categories, and categories with representable maps explain where *context and substitution* come from. They do not themselves give an operational monad of composable commands. The distinction is essential: reindexing a command along $f : \Delta \to \Gamma$ is not the same operation as executing a command that changes Γ into a new context. The task for fibred Sweedler theory is to place a command monad, a behaviour comonad, and a residual monad over a suitable comprehension structure, and require the measuring law to be stable under substitution and context extension. The earlier presheaf–copresheaf coend pairing is a useful arena for this interaction, but the pairing by itself neither supplies command composition nor proves representability. The constructive programme should ultimately address *all three* vertices of the adjunction triangle (2), not solely the measuring comonad: construct $[D, R]$, $\{T, R\}$, and $D \triangleright T$ under explicit internal hypotheses, and establish that their universal properties persist under reindexing. The polynomial and coequational route from Paper A provides a candidate method for some of these constructions; Paper B isolates the contextual transport problem. General internal existence and stability are still open.

8 Algebras, coalgebras, and universal models of competence

The three transposes of a measuring law also act on *models*. Fix

$$\bar{\theta} : D \triangleright T \to R, \qquad \widehat{\theta} : T \to [D, R], \qquad \widetilde{\theta} : D \to \{T, R\}.$$

The directions of these maps matter. A monad morphism restricts algebras backwards, whereas a comonad morphism transports coalgebras forwards. Hence we obtain three functors:

$$\begin{aligned} \mathrm{Coalg}(D) &\longrightarrow \mathrm{Coalg}(\{T, R\}), \\ \mathrm{Alg}([D, R]) &\longrightarrow \mathrm{Alg}(T), \\ \mathrm{Alg}(R) &\longrightarrow \mathrm{Alg}(D \triangleright T). \end{aligned}$$

The first turns a system with a specified D-capacity into a T-to-R runner. Explicitly, a D-coalgebra $\gamma : S \to DS$ becomes

$$S \xrightarrow{\gamma} DS \xrightarrow{\widetilde{\theta}_S} \{T, R\}S.$$

The second regards an algebra for the effective command theory $[D, R]$ as an algebra for the particular commands selected by T. The third regards an R-handler as a handler for the universally generated residual demands $D \triangleright T$. These are distinct comparisons: neither their source categories nor their forgetful functors should be identified.

Free extensions and what additional competence costs

Under the usual extension-of-scalars hypotheses, the restriction along $\widehat{\theta}$ has a left adjoint. It freely extends a T-algebra to an $[D, R]$-algebra; similarly restriction along $\bar{\theta}$ may have a left adjoint that freely equips a $D \triangleright T$-algebra with R-operations. Such free extensions do not assert that the original carrier already supports the additional operations, or that the extension is computationally feasible. They express an algebraic universal property. For a monad M on **Set**, the free algebra MX on a set X is more informative here than the initial algebra $M\varnothing$: the latter can be empty for theories with no constants. Whenever $M\varnothing$ exists as usual, it is initial in $\mathrm{Alg}(M)$; the singleton is terminal. For example, $[D, R]\varnothing$ and $(D \triangleright T)\varnothing$ record closed terms of their respective theories, which may be scarce or nonexistent.

Terminal behaviours and relative expressiveness

Dually, the cofree D-coalgebra on 1 is terminal in $\mathrm{Coalg}(D)$, with carrier $D1$; the empty coalgebra is initial for a Set-comonad. The measuring morphism yields a canonical comparison of terminal behaviours,

$$\widetilde{\theta}_1 : D1 \longrightarrow \{T, R\}1.$$

Indeed, the image of the terminal D-coalgebra under the induced runner functor has a unique coalgebra map to the terminal $\{T, R\}$-coalgebra, and its underlying map is precisely $\widetilde{\theta}_1$. This gives a possible measure of *relative behavioural expressiveness*: which universal T, R-behaviours are realized by the D-capacity? Its image need be neither all of $\{T, R\}1$ nor a faithful encoding of $D1$. The last caveat matters. For $DY = Y^{A^*}$, one has $D1 = 1$, so passing to terminal coalgebras forgets the action dynamics unless the comonad itself includes nontrivial observations. For Moore-style behaviours with observation labels, terminal behaviour can be substantially richer. Moreover, the image of $\widetilde{\theta}_1$ need not itself carry an inherited subcomonad structure. Such a claim needs extra stability hypotheses.

An agenda suggested by the three model categories

These comparisons propose three linked questions. Which D-behaviours can be distinguished after translation into T, R-runner behaviour? Which T-algebras admit an extension to the effective theory $[D, R]$? Which universal residual obligations can be handled by a chosen R-algebra? Their conjunction connects the triadic algebra of delegation to reachability, observability, and the Chu-style dualities proposed for a later study. It does not yet supply an adjunction directly between the algebra and coalgebra categories.

9 Pattern runs on matter: the copower compared with a one-step residual

Libkind and Spivak's *Pattern Runs on Matter* [5] exhibits a natural "runs-on" map of polynomial functors

$$\Xi_{p,q} : m_p \otimes c_q \longrightarrow m_{p \otimes q},$$

where m_p is the free substitution monad on a polynomial p, c_q the cofree comonad on q, and $\otimes$ the polynomial (Dirichlet) tensor. The map puts the distinction between a complete responding capacity and the residual exposed at each step into focus.

A universal comparison. Assume the polynomial Sweedler copower and the relevant tensor–hom adjunctions exist. For every monad R, freeness and the Sweedler adjunction yield natural bijections

$$\begin{aligned}\mathrm{Mon}(c_q \triangleright m_p, R) &\cong \mathrm{Mon}(m_p, [c_q, R]) \\ &\cong \mathrm{Poly}(p, [c_q, R]) \\ &\cong \mathrm{Poly}(p \otimes c_q, R) \\ &\cong \mathrm{Mon}(m_{p \otimes c_q}, R).\end{aligned}$$

Here the tensor–hom step is interpreted in the polynomial ambient category, with its usual coherence conventions. Yoneda therefore gives

$$c_q \triangleright m_p \cong m_{p \otimes c_q}. \tag{3}$$

The cofree counit (the map to the generating endofunctor) $\varepsilon_q : c_q \to q$ then induces a canonical monad morphism

$$m_{p \otimes c_q} \xrightarrow{m_{1_p \otimes \varepsilon_q}} m_{p \otimes q}. \tag{4}$$

The runs-on map factors through the universal copower interaction:

$$m_p \otimes c_q \longrightarrow c_q \triangleright m_p \cong m_{p \otimes c_q} \longrightarrow m_{p \otimes q}.$$

This factorization identifies the comparison of residual theories; it does not assert that the runs-on module action is itself an isomorphism of monads or a universal copower.

An elementary failure of invertibility. Let $p = q = y$, the identity polynomial, which is the unit for $\otimes$. Then $m_y(X) = \mathbb{N} \times X$ and $c_y(X) = X^{\mathbb{N}}$. The comparison becomes

$$m_{y^{\mathbb{N}}} \longrightarrow m_y,$$

induced by evaluation at 0, $X^{\mathbb{N}} \to X$. For $X = \{0, 1\}$, the one-node terms whose families of return labels are $f = (0, 0, \ldots)$ and $g = (0, 1, 0, \ldots)$ are distinct in the source, yet both map to the same unary term returning 0. Hence the comparison is not injective, and is not generally an isomorphism.

Interpretation. The universal residual $c_q \triangleright m_p$ retains the possible continuations encoded in the entire cofree behaviour c_q. The runs-on target $m_{p \otimes q}$ retains only the one-step q-interface at each generated operation. This supplies a concrete instance of our triadic distinction: a capacity D may generate a universal residual $D \triangleright T$, while a particular operational account chooses a smaller presentation of residual demands through a monad map out of it.

10 The three transposes as squares in the double category of runners

The same measuring law also explains how the tight and loose directions of $\mathrm{Run}(\mathcal{D})$ fit together. A tight arrow is a monad morphism; a loose arrow $T \rightsquigarrow R$ is a $\{T, R\}$-coalgebra. Squares express compatibility between a change of source or residual theory and a runner. Fix a measuring law $\theta : (T, D; R)$, with transposes

$$\bar{\theta} : D \triangleright T \to R, \qquad \widehat{\theta} : T \to [D, R], \qquad \widetilde{\theta} : D \to \{T, R\},$$

and a D-coalgebra $\gamma : S \to DS$. The induced runner h_θ has carrier S and structure map

$$S \xrightarrow{\gamma} DS \xrightarrow{\widetilde{\theta}_S} \{T, R\}S.$$

There are two canonical ways to display this runner in a double-category square.

Restriction of the source (convolution). The universal measuring interaction of $([D, R], D; R)$ turns γ into a runner $h_{\mathrm{conv}} : [D, R] \rightsquigarrow R$. Precomposing its commands with the tight map $\widehat{\theta}$ gives h_θ. Thus the following is a genuine square in $\mathrm{Run}(\mathcal{D})$, whose map of carriers is 1_S:

$$\begin{array}{ccc} T & \xrightarrow{\;h_\theta\;} & R \\ {\scriptstyle \widehat{\theta}}\downarrow & & \downarrow{\scriptstyle 1_R} \\ [D, R] & \xrightarrow{\;h_{\mathrm{conv}}\;} & R. \end{array}$$

This square says that the responder is first able to execute the effective command theory $[D, R]$, and a T-command is interpreted in that theory before execution.

Extension of the residual (copower). The universal measuring interaction $(T, D; D \triangleright T)$ turns γ into a runner $h_{\mathrm{cop}} : T \rightsquigarrow D \triangleright T$. Postcomposing its residual with the tight map $\bar{\theta}$ gives h_θ. We obtain a second genuine square, again over 1_S:

$$\begin{array}{ccc} T & \xrightarrow{\;h_{\mathrm{cop}}\;} & D \triangleright T \\ {\scriptstyle 1_T}\downarrow & & \downarrow{\scriptstyle \bar{\theta}} \\ T & \xrightarrow{\;h_\theta\;} & R. \end{array}$$

This square says that the responder first generates universal residual obligations in $D \triangleright T$, which are then interpreted in the chosen residual theory R.

Why both squares describe the same runner. Under the Sweedler adjunctions, precomposition by $\widehat{\theta}$ and postcomposition by $\bar{\theta}$ are precisely the two transposes of the original measuring law θ. Applying either universal interaction to the same coalgebra γ therefore yields the same $\{T, R\}$-coalgebra h_θ. This is an instance of the naturality of the transposition bijections, not a new adjunction. The two displayed squares are distinct 2-cells, with different tight boundaries; they should not be confused with a single square whose horizontal sides are the two factorizations.

A limitation worth retaining. These squares require the relevant monads, comonads, and Sweedler transposes to exist, and use the change-of-theory squares of the chosen runner double category. They do not imply that an arbitrary T-to-R runner factors through one fixed D; the factorization applies to runners *induced* by a specified measuring law and D-coalgebra. They show, however, how the tight arrows of $\mathrm{Run}(\mathcal{D})$ and its loose implementations arise from the same triadic interaction.

11 Triadic mediation, Biedermann and Guitart's trijunctions

The same construction also suggests a connection with Peirce's idea of *Thirdness*: mediation by a rule or general relation, rather than a merely juxtaposed pair of things. A command

is not simply matched with an agent. An agent's response takes place *through* a residual language, so its competence is relative to what may legitimately be handed on. The cab driver who reaches the railway station is competent for one kind of onward-journey residual, but not necessarily for a residual demanding that the traveller already be at the destination. This is a philosophical analogy, not a mathematical consequence of Peirce's categories.

Lehmann and Wille introduced a triadic form of formal concept analysis inspired by Peirce, in which objects have attributes under conditions [8]; Biedermann developed its diagrammatic and order-theoretic side [7]. This was taken up by Guitart as form of 2-variable adjunction that he called a *trijunction* [1]. His definition (Section 2.1) uses three categories and three contravariant bifunctors, with a compatible cycle of hom-set equivalences. Our Sweedler data instead comprise a mixed-variance two-variable adjunction:

$$\begin{aligned}\mathrm{Mon}(D \triangleright T, R) &\cong \mathrm{Mon}(T, [D, R]) \\ &\cong \mathrm{Comon}(D, \{T, R\}).\end{aligned}$$

The common point is that fixing two arguments produces a universal description of the third, subject to existence hypotheses. The resemblance should not be pressed further at the level of theories. The Sweedler data form a two-variable adjunction, whose three transposes describe one and the same measuring; they are not a cyclic triality of interchangeable objects [10]. The closest analogue of a triadic context appears one level down, among *instances*: particular demands, states of realised capacities, and standards of assessment, related by whether the capacity meets the demand under the standard. Section 12 returns to this.

In everyday terms, we may ask three questions about one episode of lawful delegation. Given a demand and a capacity, what kinds of residual obligations can it generate? Given a capacity and permitted residuals, which commands can be interpreted? Given a demand and a residual theory, which responding capacities can support the task? The copower, convolution hom, and measuring comonad answer these questions universally; none amounts to ranking agents by competence without specifying the other two terms. Nor does any of them say whether a particular responder has done well; that question needs standards of assessment as well (Section 12).

12 From possible to adequate interaction

A measuring interaction says that demands can be *coherently interpreted* through a capacity into a residual. It does not say that they have been *met well*. An order in a restaurant can be served correctly, but slowly, and the food may be poor. Every step of the service may be lawful, and the diner may still be disappointed.

A very small example makes the point mathematically. Take the demand theory $T = \mathbb{N} \times -$ of counting ticks, with capacity $D = \mathrm{Id}$ and residual $R = \mathrm{Id}$. There is exactly one measuring interaction, since $\mathrm{Meas}(T, \mathrm{Id}; \mathrm{Id}) \cong \mathrm{Mon}(T, \mathrm{Id})$ is a singleton. The runner it induces does nothing at all when a tick is demanded. It satisfies every law, because nothing in the theory of ticks asks a tick to accomplish anything.

We should therefore separate three questions that the triadic language tends to run together:

- *Can the demand be interpreted?* This is answered by a measuring $D \to \{T, R\}$.
- *What happens when it is executed?* This is answered by a D-coalgebra and the runner it induces.
- *How well does the execution meet the demand?* This needs a further participant: a *standard of assessment*.

The first two belong to the triad of this note. The third does not follow from them.

Where standards come from. The natural source of standards is the residual itself. An algebra for R evaluates residual computations: the expected value of a probabilistic residual, the total of a cost residual, the truth of a predicate over nondeterministic outcomes. Running a command from a state of a realised capacity and then evaluating the residual with such an algebra yields a verdict. This is the familiar pattern of weakest-precondition semantics [9], and it is already present in our companion study as the *delegated algebra* of a runner [10]. Three consequences matter for the present account.

- *The residual limits what can be assessed.* With $R = \mathrm{Id}$ there is nothing that could evaluate a tick, so the verdict on the idle runner above is not "inadequate" but "not assessable". With $R = \mathbb{N} \times -$ the possible measurings are indexed by $k \in \mathbb{N}$ (ticks performed per tick demanded), and suitable standards separate the idle ($k = 0$), honest ($k = 1$) and over-eager ($k \geq 2$) responders. What a responder may hand on is also what can be checked.
- *Standards see only what the residual records.* Slowness and poor quality stay invisible to every standard unless the command or residual language is enriched to record time, cost or quality, for instance by graded or temporal theories. Assessment does not repair an underspecified demand; it shows which specification is missing.
- *Assessment composes backwards along delegation.* A standard for the final residual of a chain of delegations pulls back to a standard for an intermediate residual [10]. The cab driver's leg is judged adequate if arrival at the station is the kind of arrival from which the rail leg will meet the traveller's standards. This is an assume–guarantee principle, and it is the evaluative counterpart of the algebra of handoffs described in Section 4.

Adequacy is three-place among instances. The resulting relation holds between a particular demand, a state of a realised capacity, and a standard, according to whether the capacity meets the demand under the standard. This has exactly the form of a triadic context in the sense of Lehmann and Wille [8]: objects, attributes and conditions. Its triadic concepts are maximal triples consisting of a set of demands, a set of capacities and a set of standards such that every capacity meets every demand under every standard. Fixing any two components determines a closure of the third. The genuinely triadic structure of adequacy therefore lives among *instances*, while the theories T, D, R are related by the two-variable adjunction of Section 3. When the standards take values in an ordered or quantitative domain, the incidence itself becomes graded. That is where enrichment belongs: in the measure of how well the instances fit, not in comparing measuring laws. The companion study develops these points [10].

13 Perspective: a theory of lawful delegation

The most useful shift in viewpoint is from a binary question ("Can this agent follow these instructions?") to a ternary one:

> Given a language of demands, a specified form of responding capacity, and a permitted language of residual effects, is there a coherent way to transform the demands into those effects?

The three Sweedler universal constructions answer three differently directed versions of this question. This explains why increasing sophistication in the command language does not simply demand an ever-more-complex responder. Sophistication can be distributed across a chain of interpretations and delegations. What must remain invariant is the lawful relationship between the original commands, the responder's continuations, and the residual computation it produces. Lawfulness is not yet success. Whether a particular delegation served its purpose is a further, three-place question among demands, realised capacities and standards of assessment, and the residual language determines which standards can be applied at all.

References

[1] R. Guitart, *Trijunctions and Triadic Galois Connections*, Cahiers de Topologie et Géométrie Différentielle Catégoriques 54 (2013), no. 1, 13–27. https://cahierstgdc.com/wp-content/uploads/2017/04/Guitart_54-1.pdf.

[2] M. Shulman, *What is an n-theory?*, The n-Category Café (2018), https://golem.ph.utexas.edu/category/2018/04/what_is_an_ntheory.html.

[3] D. McDermott, E. Rivas and T. Uustalu, *Sweedler Theory of Monads*, FoSSaCS (2022).

[4] M. Hyland, G. Plotkin and J. Power, *Combining Effects: Sum and Tensor*, Theoretical Computer Science 357 (2006).

[5] S. Libkind and D. I. Spivak, *Pattern Runs on Matter: The Free Monad Monad as a Module over the Cofree Comonad Comonad*, EPTCS 429 (2025), 1–28.

[6] R. Garner, work on the behaviour of algebraic effects and comodels, Mathematical Structures in Computer Science (2021–2022).

[7] K. Biedermann, *How triadic diagrams represent conceptual structures.* In: Lukose, D., Delugach, H., Keeler, M., Searle, L., Sowa, J. (eds) Conceptual Structures: Fulfilling Peirce's Dream. ICCS 1997. Lecture Notes in Computer Science, vol 1257. Springer, Berlin, Heidelberg.

[8] F. Lehmann and R. Wille, *A triadic approach to formal concept analysis*, in: Conceptual Structures: Applications, Implementation and Theory (ICCS 1995), LNAI 954, Springer (1995), 32–43.

[9] I. Hasuo, *Generic weakest precondition semantics from monads enriched with order*, Theoretical Computer Science 604 (2015), 2–29.

[10] *Measuring Interactions, Realisations, and Interaction Profiles*, companion research study (November 2026).