

Getting to Grips with Sweedler Runners for Systems

Conclusion: what the architecture is beginning to show
Theories, implementations, changing ambient worlds, and a programme for categorical systems

October 2026

The shortest summary. The tutorial began from one question: what is a stateful system *between two theories*? For a locally presentable base and accessible monads T, R , the endofunctor Sweedler theory of McDermott–Rivas–Uustalu supplies a measuring comonoid $\mathcal{M}(T, R)$, whose coalgebras are residual stateful runners [8]

$$\theta_X : TX \otimes S \longrightarrow R(X \otimes S).$$

The proposal developed through Parts I–XII and the appendices is to take this seriously as systems architecture. Theories are objects, direct interpretations are tight maps, stateful implementations are loose maps, and the ambient duoidal category determines what kinds of state, resource, geometry, probability, analysis, symmetry, or certification the systems may carry.

1 What changed in the point of view

We did not end with a new universal definition of “system”. That would have been the wrong goal. Moore and Mealy machines, handlers, Petri nets, higher-dimensional automata, probabilistic processes, smooth dynamical systems, analytic evolution equations, temporal specifications, and resource-sensitive programs remain genuinely different constructions.

What changed is the way they can be organised. Instead of beginning only with a state space or an interface, we began with a pair of theories

$$T, \quad R$$

and asked for a stateful realisation

$$T \rightsquigarrow R.$$

The state is then not merely a place where dynamics happens. It is the residual mechanism by which one operational language is implemented inside another.

This separates two ideas which are often conflated:

$T \rightarrow R$: stateless interpretation of theories,
$T \rightsquigarrow R$: stateful implementation of one theory in another.

The first is the tight direction of **Run**($\mathcal{D}$); the second is the loose direction. Much of the tutorial can be read as exploring the consequences of keeping those directions distinct while allowing them to interact.

2 The architecture in one view

The recurring dictionary is now compact enough to state in one place.

Role	Categorical object	Systems reading
Object	theory / monoid T	an operational language or interface
Tight map	$T \rightarrow R$	compilation, interpretation, restriction, wiring of operations
Loose map	$T \rightsquigarrow R$	stateful implementation with residual effects
2-cell	compatible transformation	simulation or comparison of implementations
Handler	R -algebra	downstream semantics for residual effects
Sweedler hom	$\mathcal{M}(T, R)$	universal object of ways of running T in R

The resulting picture is deliberately plural. A theory says *what operations are available and what laws they satisfy*. A runner says *how those operations are realised through state*. A handler says *how the residual effects are interpreted*. Tight maps reorganise the vocabulary; loose maps execute it. The double-categorical proposal is that these should compose with one another in a controlled way.

The Sweedler and interaction-law ingredients here are inherited from the existing theory [6, 8, 7, 10]. The specific proposal of this tutorial is to organise theories, tight interpretations, stateful runners and their cells into the double category $\mathbf{Run}(\mathcal{D})$, and then to study how that architecture varies with the ambient world.

3 What Parts I–XII added

The early parts established the algebraic core. Parts I–III developed comodels, residual runners, induction and delegation, tight theory maps, coproduct and tensor constructions, parameter change, and wiring. Following Katsumata–Rivas–Uustalu and McDermott–Rivas–Uustalu [6, 8], Part IV then exposed the structural reason the machinery works: the interaction between the two monoidal products in a duoidal category is not background bookkeeping but the source of the monad–comonad interaction underlying Sweedler runners.

From Part V onward the series increasingly varied the ambient world. Valued predicates turned runners into carriers of safety, guarantees and quantitative certification. C -sets and graded structure turned hidden state into stored resource and made amortised reasoning compositional. Precubical sets replaced mere transition graphs by higher-dimensional execution geometry and suggested a Sweedler calculus of concurrency. Species and digraphical species brought exchange symmetries and graph-shaped processes into the same architecture.

Part IX shifted attention from implementation to behaviour. Final comodels, reachability and observability made it possible to ask what a runner exposes to an observer and how different internal mechanisms can realise the same external behaviour. The comparison with filtering/control duality showed that this is not merely automata theory in new notation: the same reachability–observability polarity appears in stochastic and control-theoretic settings.

Part X moved to smooth worlds and separated smooth dependence from infinitesimal change. Part XA then asked what changes when probability belongs to the ambient category itself rather than appearing only as a residual monad. Part XB pushed further into condensed and solid settings, where the intended gain is analytic rather than merely smooth: infinite-dimensional spaces, topological completion, and well-posed evolution can become part of the ambient semantics rather than external side conditions.

Part XI studied traces, wheels and closed execution. The emerging lesson was that one need not force all feedback structure directly into $\mathbf{Run}(\mathcal{D})$: a behaviour construction landing in a profunctorial or cofunctorial equipment can inherit a shadow there, so that loop behaviour is computed semantically. Part XII finally turned back toward syntax and internal logic, asking what kind of linear or dependent type theory could present theories, runners, handlers, resources and composition from inside the framework.

The appendices widened the picture in complementary directions. $\mathbf{IntRun}(\mathcal{D})$ treats paired interfaces and prism-like wiring; the psychological examples test genuinely closed-loop multi-agent readings; actegories and Tambara modules suggest a route from monoid actions to richer interface geometry. Further proposed extensions—quantum, higher-groupoidal and dependent/fibred settings—all ask the same question: which parts of the Sweedler architecture survive when the notion of ambient state is changed again?

4 The ambient category is part of the theory

Seen retrospectively, the later Parts were not a sequence of unrelated generalisations. They were testing how much of the construction survives when the ambient world changes. The abstract starting point is a symmetric closed duoidal category

$$\mathcal{D} = (\mathcal{D}, \mathrm{Id}, \circ, J, \star)$$

in which the Sweedler homs required by $\mathbf{Run}(\mathcal{D})$ exist [7]. Nothing in the definition of $\mathbf{Run}(\mathcal{D})$ requires $\mathcal{D}$ to be an endofunctor category, and polynomial, species-like and classical degenerate-duoidal examples already show that the ambient Sweedler universe can arise in different ways.

The construction

$$\mathcal{C} \longmapsto [\mathcal{C}, \mathcal{C}]_{\mathrm{acc}}$$

is nevertheless the most important source of examples in the tutorial, and its status should be stated precisely. For $\mathcal{C}$ locally presentable symmetric monoidal closed, composition and Day convolution give the accessible endofunctors the relevant duoidal structure. For accessible monads T, R , McDermott–Rivas–Uustalu prove that the Sweedler hom $\mathcal{M}(T, R)$ exists and that its coalgebras are exactly the R -residual stateful runners of T ; dually, the Sweedler copower exists [8, Thms. 1–2]. These existence results come from the locally presentable categories of runners and residual structures, not from local presentability of $[\mathcal{C}, \mathcal{C}]_{\mathrm{acc}}$ itself: even $[\mathbf{Set}, \mathbf{Set}]_{\mathrm{acc}}$ is not accessible in general. The full internal hom of comonads, and hence a general tensored–cotensored enrichment of monads in comonads, is not established for this endofunctor category.

Two cautions become important away from $\mathbf{Set}$. First, the Day unit is

$$JZ = \mathcal{C}(I, Z) \bullet I,$$

which agrees with Id in the familiar $\mathbf{Set}$ -case but not in general. Thus a Grothendieck topos still gives a natural ordinary endofunctor example, but typically a *non-normal* duoidal one; likewise

for module categories. Any argument using coincidence of the two units must therefore be checked rather than transported automatically.

Second, several Parts were really asking for strong monads. This leads to the enriched category

$$\mathcal{C}\text{-}[\mathcal{C}, \mathcal{C}]_{\text{acc}},$$

or the corresponding strong accessible endofunctors. Enrichment restores the expected unit and packages compatibility with the tensoring of the ambient world; this is why it arose naturally for graded resources, smooth structure and related settings. But here one must be cautious: the enriched accessibility theory needed for unrestricted Sweedler homs is largely open [8, §8]. Part VI’s fixed-rank construction is therefore evidence for this direction, not yet a general existence theorem.

The additive world also requires a sharper distinction. Ordinary accessible endofunctors on $R\text{-}\mathbf{Mod}$ are not “linear theories” in any simple sense, while the enriched R -linear endofunctors are so restrictive that genuinely non-linear arities disappear. The tensor-algebra monad, for example, is not an R -linear strong endofunctor of the required kind. In the linear world, operads and species with substitution are often a better home for non-linear algebraic operations than

$$[R\text{-}\mathbf{Mod}, R\text{-}\mathbf{Mod}]_{\text{acc}}.$$

The differential graded case is considerably richer. Anel and Joyal develop a full Sweedler theory for dg-algebras and dg-coalgebras ([1]): dg-coalgebras form a symmetric monoidal closed category, while dg-algebras are enriched, tensored and cotensored over them, and this structure reconstructs in particular the bar–cobar adjunction. Thus a substantial homological Sweedler theory already exists. What remains distinct is the genuinely derived question: how much of this structure descends after passing to weak equivalences or to an appropriate model- or ∞ -categorical localisation, and what notion of weak equivalence on the coalgebraic side is required.

The condensed discussion should be read with the same care. Light or suitably κ -condensed sets fit the locally presentable regime more directly than unrestricted condensed sets. Solid and liquid module categories are promising analytic bases only after the relevant closedness and accessibility properties of their tensor structures have been checked. Thus Part XB is deliberately programmatic at this point: it identifies an analytic direction rather than claiming a completed ambient theorem.

A second generalisation runs in parallel rather than above the duoidal one. Aravantinos-Sotiropoulos and Vasilakopoulou develop Sweedler theory for locally presentable monoidal closed fibrant double categories, obtaining enrichments of monads in comonads and of modules in comodules [2]. Their monoidal double categories have invertible interchange, whereas the duoidal endofunctor examples relevant here have a generally non-invertible interchange. Consequently the two theories should be viewed as parallel many-object and one-object/lax-interchange extensions of the classical measuring-comonoid story [5], not as instances of one another. A common extension to lax or oplax monoidal double categories is precisely the sort of structure in which $\mathbf{Run}(\mathcal{D})$ ought eventually to live.

The module/comodule enrichment of [2] is structurally suggestive, but it is not yet a theorem identifying our categories of Eilenberg–Moore handlers with their modules. Induction and delegation should therefore be compared with that fibred module/comodule theory, not simply subsumed by it.

The landscape that emerged is not one universal recipe but a family of related regimes:

$\begin{aligned} &\text{symmetric closed duoidal } \mathcal{D} \rightsquigarrow \mathbf{Run}(\mathcal{D}), \\ &\mathcal{C} \text{ locally presentable monoidal closed} \rightsquigarrow [\mathcal{C}, \mathcal{C}]_{\text{acc}} \text{ with runner-based Sweedler homs,} \\ &\mathcal{V}\text{-}[\mathcal{C}, \mathcal{C}]_{\text{acc}} \rightsquigarrow \text{strong/enriched theories, presently with extra existence questions,} \\ &\text{monoidal closed fibrant double categories} \rightsquigarrow \text{the many-object Sweedler theory of [2].} \end{aligned}$

However, a recurring limitation of the present development is now visible. There is a substantial general theory of Sweedler enrichment once an appropriate monoidal or duoidal ambient category has been supplied, ranging from ordinary and dg settings to higher categories. What is much less systematic is the preceding step needed throughout this tutorial: given a base category $\mathcal{C}$, or an enriched base $(\mathcal{V}, \mathcal{C})$, when does

$$[\mathcal{C}, \mathcal{C}]_{\text{acc}} \quad \text{or} \quad \mathcal{V}\text{-}[\mathcal{C}, \mathcal{C}]_{\text{acc}}$$

carry the closed duoidal and measuring structure required by **Run**?

The Set-based result of McDermott–Rivas–Uustalu shows that one should not expect the answer simply from local presentability of the endofunctor category: the needed Sweedler homs can exist even when that category itself falls outside the usual locally presentable hypotheses. The open problem is therefore to develop an existence theory specifically for endofunctor worlds, giving checkable hypotheses on $\mathcal{C}$ and $\mathcal{V}$ under which the measuring comonoids, copowers and enriched variants required by runner theory exist. Such a theorem would turn many of the changes of ambient world explored in Parts VI–XB and the appendices from case-by-case constructions into instances of a single principle.

Such a theory is central to our program: Changing ambient world changes what can count as state, parameter, resource, convergence, path, symmetry or infinitesimal variation. It may prove more feasible through a second strategy.

4.1 From endofunctor worlds to arity-generated worlds

One qualification to the preceding discussion has become increasingly important. The natural ambient category for a new base $\mathcal{C}$ need not be the whole endofunctor category

$$[\mathcal{C}, \mathcal{C}]_{\text{acc}},$$

nor, in the enriched case, the whole category

$$\mathcal{V}\text{-}[\mathcal{C}, \mathcal{C}]_{\text{acc}}$$

of accessible strong endofunctors. Those are useful maximal targets, but they may be much larger than the systems under study require, and their accessibility and closedness properties can be difficult to control.

Part VI provides a more economical alternative. For

$$\mathcal{E} = C\text{-}\mathbf{Set}$$

the elementary cost objects $y_c = C(c, -)$ form a small tensor-closed enriched system of arities

$$j : \mathcal{J} \hookrightarrow \mathcal{E},$$

with

$$y_c \otimes_D y_d \cong y_{c+d}, \quad [y_c, X]_{\mathcal{E}}(p) \cong X(p+c).$$

The second identity says that homming out of a cost arity is just a colimit-preserving shift. This allows the resource-sensitive theory to be developed inside the much smaller category

$$\mathcal{D}_C = \text{End}_{\mathcal{J}}^{\mathcal{E}}(\mathcal{E}) \simeq [\mathcal{J}, \mathcal{E}]_{\mathcal{E}}.$$

Here composition becomes substitution, while the second tensor is enriched Day convolution. The latter is biclosed, substitution is accessible, and the two products form a normal duoidal structure. Under the standard enriched local-presentability hypotheses, the general measuring-comonoid theorem therefore supplies the required Sweedler homs and copowers.

This example suggests that the search for new ambient Sweedler universes should be organised around triples

$$(\mathcal{V}, \mathcal{C}, \mathcal{J})$$

rather than around $\mathcal{C}$ alone. The category $\mathcal{C}$ specifies the geometry in which states and behaviours live; the enriching category $\mathcal{V}$ specifies the structure respected by maps and theories; and the small arity category $\mathcal{J}$ specifies the admissible shapes through which operations may probe that geometry. The resulting candidate

$$[\mathcal{J}, \mathcal{C}]_{\mathcal{V}}$$

may support a duoidal Sweedler theory even when the full enriched endofunctor category is inaccessible or poorly understood.

This changes the emphasis of one of the main open problems. There is already a substantial theory of Sweedler enrichment once a suitable monoidal or duoidal ambient category has been supplied. What is missing for the systems programme is a practical existence theory for the ambient categories themselves. The resource calculation suggests the following more focused question:

$$(\mathcal{V}, \mathcal{C}, \mathcal{J}) \rightsquigarrow [\mathcal{J}, \mathcal{C}]_{\mathcal{V}} \text{ as a Sweedler universe?}$$

Eleutheric systems of enriched arities provide one natural source of the required substitution structure, while tensor closure of $\mathcal{J}$ gives access to Day convolution. The remaining task is to formulate useful hypotheses ensuring local presentability, biclosedness, accessibility and interchange in sufficient generality.

This perspective also changes how the later parts should eventually be read. The resource representables of Part VI are the first worked example, but analogous choices suggest themselves elsewhere: representable cubes for concurrency, suitable smooth or infinitesimal probes in smooth settings, compact probes in condensed or analytic settings, and species or operadic arities in linear settings where purely linear endofunctors lose non-linear operations. These cases have not all been established, and some remain programmatic. The point is rather that the tutorial now suggests a common method for investigating them:

$$\text{geometry} + \text{arities} \implies \text{small duoidal world}.$$

Thus “changing the ambient category” has acquired a more precise meaning than it had at the beginning of the series. We are not merely replacing **Set** by another category and hoping that all endofunctors behave equally well. We are choosing both a world of states and a calculus of admissible operations inside it. The success of the cost example suggests that this smaller, arity-generated route may be the most practical way of extending **Run** beyond its original Set-based setting. There may be analogues of **Poly** worth pursuing too.

5 Behaviour, wiring and closure

Two themes that initially looked external to Sweedler theory became increasingly internal to the programme. The first was wiring. Parts II–III already showed that tight maps, coproducts, tensor products and parameterised theories provide substantial compositional structure; the later comparison with DOTS should therefore not be read as an admission that Sweedler theory lacks wiring. Rather, DOTS and IntRun suggest additional interface organisation, especially for paired and dependent interfaces, on top of a runner calculus that already composes systems algebraically.

The second was feedback. Part XI’s direct trace calculation should not be read as withdrawn: under the guardedness and dualizability conditions isolated there, one obtains the corresponding traced structure without compact closure. The behaviour construction supplies a complementary semantic route. If

$$\text{Beh} : \mathbf{Run}(\mathcal{D}) \longrightarrow \mathbf{CofProf}$$

can be made into the appropriate double functor, then the shadow on the profunctorial target computes loop behaviour after passing from implementation to observation. The two constructions answer different questions: one asks when runners themselves admit a trace; the other asks how closed observable behaviour is obtained in a target where shadows are already native.

This is conceptually attractive because it separates implementation from observation. A runner describes how one theory is realised through another; its behaviour records what is externally visible; only then does one close the observable interface into a loop. The same separation underlies the reachability–observability material of Part IX and the wheeled constructions of Part XI.

6 What remains open

The tutorial intentionally crossed the boundary between established theory and research programme. The main unfinished tasks now look rather different from those at the midpoint of the series.

1. **State the ambient existence results cleanly.** Separate at least three levels. For ordinary $[\mathcal{C}, \mathcal{C}]_{\text{acc}}$, Sweedler homs and copowers are known by the runner argument of [8], but local presentability of the endofunctor category and the full comonoid-hom story fail or remain open. For enriched $\mathcal{C}\text{-}[\mathcal{C}, \mathcal{C}]_{\text{acc}}$, normality and strength improve the algebraic fit but the required accessibility theory is not yet available in general. At the many-object level, [2] covers monoidal double categories with invertible interchange; extending this to lax or oplax monoidal double categories is the natural bridge to $\mathbf{Run}(\mathcal{D})$.
2. **Understand change of ambient world.** Endofunctor categories are not simply covariantly functorial in their base. Given an adjunction $L \dashv R : \mathcal{C} \rightleftarrows \mathcal{E}$, the assignment $F \mapsto RFL$ is lax monoidal for composition and transports monads from $\mathcal{E}$ back to $\mathcal{C}$, whereas $G \mapsto LGR$ is oplax and transports comonads in the opposite direction. Theories and servers therefore already move asymmetrically under base change. This suggests that change of ambient world for $\mathbf{Run}$ may require spans, equipments or profunctorial comparison rather than an ordinary covariant functor.
3. **Complete the general behaviour double functor.** The tutorial has developed a candidate double functor

$$\text{Beh} : \mathbf{Run}(\mathcal{D}) \longrightarrow \mathbf{CofProf},$$

extending Garner’s behaviour construction for comodels: a residual runner $T \rightsquigarrow R$ determines a profunctor between the corresponding behaviour categories, while composition of runners is represented by profunctor composition. Concrete cases, including the free-unary/NFA example, support the construction, and the measuring comonoid supplies its general form. What remains is to state the precise hypotheses on $\mathcal{D}$, prove all double-functorial coherence results, and identify systematically how Garner’s B_T , the shadow of profunctors and the trace constructions of Part XI are recovered from it.

4. **Develop calculi inside the new bases.** The point of changing ambient world is not merely existence. Predicates should yield verification calculi; C -sets an amortisation calculus; precubical sets a calculus of concurrency and deadlock; probabilistic bases a stochastic calculus of runners; condensed and solid bases an analytic calculus of well-posed systems.
5. **Clarify the relation among Run, IntRun, DOTS and optics.** The current evidence suggests complementarity rather than replacement: Sweedler theory supplies stateful implementation and theory-level wiring, while paired, fibred or prism-like interfaces organise richer open-system boundaries. The exact comparison remains to be made.
6. **Build the internal language.** Part XII identified a possible syntax of theories, handlers, runners, resource modalities and implementation composition. A genuine Sweedler type theory should recover the external double-categorical semantics and explain which logical principles correspond to the different ambient bases.
7. **Pass to higher and derived worlds.** Groupoidal state, stable and derived categories, and quantum or correspondence-enriched settings are not merely exotic examples. They test whether the architecture depends essentially on ordinary sets and 1-categories or belongs to a more general homotopical form of systems theory.

These problems are not independent. A clean account of the several ambient existence regimes should reveal the correct notion of change of base; comparison with double-categorical Sweedler theory may clarify which parts of the handler calculus admit a module-theoretic reformulation; behaviour semantics may determine the right treatment of feedback; and the internal language should expose which pieces of duoidal, enriched and double-categorical structure are actually indispensable.

7 A final perspective

The original attraction of Sweedler theory was that it answered a concrete question with a universal object:

what lies between two theories?

The answer was richer than a set of maps. There is a measuring comonoid, and its coalgebras can be read as stateful mechanisms implementing one theory through another.

Across the series that observation connected comodels with residual effects, handlers with implementations, theory maps with wiring, predicates with certification, grading with resource, cubes with concurrency, behaviours with observability, smooth structure with differential dynamics, probabilistic bases with stochastic semantics, condensed mathematics with infinite-dimensional analysis, and profunctorial shadows with closed execution. The appendices suggest that paired

interfaces, actegories, higher state and quantum semantics may belong to the same expanding picture.

The strongest lesson is therefore not that one particular category of systems has been found. In the Set-based theory and in the presheaf, predicate and other cases worked out explicitly, the architecture survives substantial changes of ambient world. In the condensed, solid, quantum and genuinely derived directions the evidence is still programmatic. Theories and runners are the candidate stable algebraic skeleton; changing the base changes what the systems are allowed to be, while each new base still has to earn the required Sweedler structure.

The proposal in one sentence. Categorical systems theory may profit from treating a system as a *stateful realisation of one theory of interaction inside another*, organised by Sweedler theory and transported across geometric, logical, probabilistic, smooth, analytic, linear and higher ambient worlds.

This is not a claim that every useful systems formalism should be replaced by $\mathbf{Run}(\mathcal{D})$. The more modest and more interesting hope is that several successful formalisms are manifestations of a common algebraic architecture. If that is right, Sweedler theory supplies not a universal language of systems, but a reusable grammar for moving between theories, implementations, behaviours and the structured worlds in which they run.

References

- [1] M. Anel and A. Joyal *Sweedler Theory for (co)algebras and the bar-cobar constructions*, arXiv:1309.6952, 2013.
- [2] V. Aravatinos-Sotiropoulos and C. Vasilakopoulou, *Sweedler theory for double categories*, arXiv:2408.03180, 2024.
- [3] R. Garner, *The costructure–cosemantics adjunction for comodels for computational effects*, Math. Struct. Comput. Sci. 32 (2022), 374–419.
- [4] R. Garner and I. López Franco, *Commutativity*, J. Pure Appl. Algebra 220 (2016), 1707–1751.
- [5] M. Hyland, I. López Franco and C. Vasilakopoulou, *Measuring comodules and enrichment*, arXiv:1703.10137, 2017.
- [6] S.-Y. Katsumata, E. Rivas and T. Uustalu, *Interaction laws of monads and comonads*, LICS 2020, 604–618.
- [7] I. López Franco and C. Vasilakopoulou, *Duoidal categories, measuring comonoids and enrichment*, arXiv:2005.01340, 2020.
- [8] D. McDermott, E. Rivas and T. Uustalu, *Sweedler theory of monads*, FoSSaCS 2022, LNCS 13242, 428–448.
- [9] J. Power and O. Shkaravska, *From comodels to coalgebras: state and arrays*, Electron. Notes Theor. Comput. Sci. 106 (2004), 297–314.
- [10] T. Uustalu, *Stateful runners of effectful computations*, Electron. Notes Theor. Comput. Sci. 319 (2015), 403–421.
- [11] T. Uustalu and N. Voorneveld, *Algebraic and coalgebraic perspectives on interaction laws*, APLAS 2020, LNCS 12470, 186–205.