The Structure Behind Dependent Runners:
Part B: Refined Worlds, Polynomials, and Change of Context

Research note for the categorical systems programme

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Abstract

This part explains the structure behind the concrete machines of the preceding gallery. We study stateful interpretations of operational theories whose commands and continuation arities depend on a world or typestate. The central observation is that a residual-state family $p : S \to W$ is itself a refinement of worlds. State passing admits the factorisation

$$\mathsf{St}_S(R) = \Pi_p\big(p^* R \Sigma_p\big) p^*,$$

so a runner can be presented as a monad opfunctor across the refinement p. For free dependent polynomial theories over a set of worlds, we give an explicit endofunctor whose coalgebras are runners, and calculate the two-world alternating-path example. We then study the comparison between forming a free monad and changing worlds, proving a tree-lifting criterion for its invertibility in **Set**. We distinguish these results from the still-open construction of a global dependent Sweedler double category, and explain the relevance to typestate verification and proof-relevant assume–guarantee contracts.

Scope and status. Sections 2–5 contain direct Set-based calculations and proofs; Section 6 gives checked examples and a proposed general existence argument requiring a separate categorical verification; Sections 7–8 are programme statements. This part presupposes the elementary dependent runner gallery, not the earlier research discussion. The global dependent-double-category programme is deliberately confined to a short outlook.

1 From the gallery to the underlying structure

The preceding part began with things we recognise: an alternating machine with states S_0, S_1, a Moore sensor with different observation types in its modes, and a file protocol in which `read` can return a datum or EOF. None required a new abstract double category. But three questions arose naturally: how can one describe all such typed command signatures, why is residual state itself indexed by mode, and why can a complete `read;reset` program survive hiding EOF even when the primitive `read` command does not?

The present part answers those questions in that order of motivation. The elementary examples remain the reference points: the alternating-path system for typed substitution, the file protocol for changes of context, and the nondeterministic and lottery targets for the limits of polynomiality. Readers interested only in concrete machines can stop with Part A. Here we derive the constructions that unify them.

2 The systems question

Ordinary Sweedler runners translate commands of a source theory T into programs of a target theory R, while passing residual state. But many protocols admit different commands

at different stages. A file can be read only once it is open; a network connection may be active, half-closed or terminated. A continuation may inhabit a different protocol state from the current command. The intended dependent runner should translate such well-typed operations, carry implementation state and ultimately transport evidence of safety or correct protocol use.

Let W be a set of worlds. An object of $\mathcal{E}_W := \mathbf{Set}/W \simeq \mathbf{Set}^W$ is a family $X = (X_w)_{w \in W}$. For a map $f : W' \to W$, there is the familiar adjoint triple

$$\Sigma_f \dashv f^* \dashv \Pi_f.$$

This is the entry point for both dependent operations and residual state. In a locally cartesian closed category the same expressions are meaningful in slices; in an ∞-topos they are understood homotopy-coherently. Our theorems below, unless otherwise specified, concern **Set**.

3 Residual states as refined worlds

Fix a monad R on $\mathcal{E}_W$ and a family of residual states $p : S \to W$. We write $S \times_W -$ for fibrewise product and $(-)^S$ for exponential in $\mathcal{E}_W$. The ordinary state-passing construction is

$$\mathsf{St}_S(R)(X) := \big(R(S \times_W X)\big)^S.$$

It is a monad without requiring R to carry a cartesian strength: this is the conjugate of R along the adjunction $S \times_W - \dashv (-)^S$. Unit and multiplication are obtained by adjunction from unit and multiplication of R, with the intervening counit implementing evaluation and state threading.

Lemma 3.1 (Refined-world factorisation)**.** *For $p : S \to W$ there is a canonical identification of state-passing monads*

$$\boxed{\mathsf{St}_S(R) \cong \Pi_p \, (B_p R) \, p^*, \qquad B_p R := p^* R \Sigma_p.}$$

Proof. The standard slice identities are $S \times_W - \cong \Sigma_p p^*$ and $(-)^S \cong \Pi_p p^*$. Hence

$$(-)^S R(S \times_W -) \cong \Pi_p p^* R \Sigma_p p^* = \Pi_p (B_p R) p^*.$$

The identifications come from adjunctions and therefore respect the induced conjugate-monad structures. □

The base S of residual states can therefore be regarded as a new world type, refined over W. A runner is not just a stateful computation attached to the old worlds: it is an operational interpretation across this refinement.

Proposition 3.2 (Runner–opfunctor correspondence)**.** *For a monad T on $\mathcal{E}_W$, monad morphisms*

$$\alpha : T \longrightarrow \mathsf{St}_S(R)$$

correspond bijectively to natural transformations

$$\bar{\alpha} : p^* T \longrightarrow (B_p R) p^*$$

satisfying the monad-opfunctor unit and multiplication laws.

Proof. Put $F = p^*$, $U = \Pi_p$, and $Q = B_p R$. Then $F \dashv U$, and Lemma 3.1 identifies $\mathsf{St}_S(R)$ with UQF. The mate is

$$FT \xrightarrow{F\alpha} FUQF \xrightarrow{\varepsilon_{QF}} QF.$$

The unit and multiplication laws, after taking mates, are respectively

$$\bar{\alpha}\, F\eta^T = \eta^Q F, \qquad \bar{\alpha}\, F\mu^T = \mu^Q F\, Q\bar{\alpha}\, \bar{\alpha}_T.$$

These are exactly the opfunctor axioms; the reverse construction uses the unit of $F \dashv U$. This is an application of the adjunction mate calculus, including its compatibility with the composite-monad multiplication. □

This suggests a later generalisation to distinct source and implementation world types: for a span $W \xleftarrow{p} S \xrightarrow{q} V$, seek an opfunctor $p^*T \to (q^* R \Sigma_q) p^*$. We treat this as a proposed generalisation, not a finished runner double category.

4 Free dependent polynomials and their runners

A dependent polynomial signature on W is a diagram

$$W \xleftarrow{s} E \xrightarrow{r} A \xrightarrow{t} W, \qquad P(X)_w = \coprod_{a \in A_w} \prod_{e \in E_a} X_{s(e)}.$$

An operation a lives at world $w = t(a)$, but its continuation positions may have worlds other than w. Its polynomial action is $\Sigma_t \Pi_r s^*$; composition and free polynomial monads belong to the Gambino–Kock theory [1].

Let y_v be the family concentrated at world v, with singleton fibre there and empty fibres elsewhere. For $a \in A_w$ define its continuation object

$$B_a := \coprod_{e \in E_a} y_{s(e)} \in \mathcal{E}_W, \qquad \mathcal{E}_W(B_a, X) \cong \prod_{e \in E_a} X_{s(e)}.$$

Suppose $T = P^*$ is the free monad and R is a target monad on $\mathcal{E}_W$.

Theorem 4.1 (Free-polynomial runner classification)**.** *For each residual family $S \in \mathcal{E}_W$, monad morphisms $T \to \mathsf{St}_S(R)$ correspond naturally to coalgebra structures $S \to H_{P,R}(S)$, where*

$$\boxed{H_{P,R}(S)_w = \prod_{a \in A_w} R(S \times_W B_a)_w.}$$

With residual-state morphisms preserving the interpretations, the runner category is $\mathrm{Coalg}(H_{P,R})$.

Proof. By freeness, a monad morphism $P^* \to \mathsf{St}_S(R)$ is determined by a natural transformation $P \to \mathsf{St}_S(R)$. Yoneda on the continuation object B_a identifies the interpretation of operation a with an element of $\mathsf{St}_S(R)(B_a)_w$, that is, a map $S_w \to R(S \times_W B_a)_w$. Taking the product over the operations at each world yields the claimed coalgebra map. Naturality in S identifies structure-preserving residual maps with coalgebra morphisms. □

An interpreter of a branching source operation chooses a target computation; when it returns, its residual state belongs to the world of the chosen continuation position. The world typing is enforced by $S \times_W B_a$, not by an extra side condition.

Remark 4.2 (Accessibility)**.** *If P is a small dependent polynomial and R is accessible, $H_{P,R}$ is accessible (possibly for a larger regular cardinal). Standard accessible-coalgebra results then give a cofree comonad $G_{P,R}$ whose coalgebras are $H_{P,R}$-coalgebras. This is a* fixed-world *measuring classification; it is not an application of a global monoidal-double-category Sweedler theorem.*

5 The two-world experiment

Let $W = \{0, 1\}$ and take the unary commands

$$0 \xrightarrow{a} 1 \xrightarrow{b} 0.$$

The signature is $P(X)_0 = X_1$, $P(X)_1 = X_0$. Its free monad is the monad of alternating paths:

$$T(X)_w = \coprod_{n \geq 0} X_{w+n \bmod 2}.$$

Take $R = T$. Paths $0 \to 1$ and $1 \to 0$ have odd length, so Theorem 4.1 says that a runner on $S = (S_0, S_1)$ is exactly

$$k_a : S_0 \to 2\mathbb{N} + 1, \qquad d_a : S_0 \to S_1,$$
$$k_b : S_1 \to 2\mathbb{N} + 1, \qquad d_b : S_1 \to S_0.$$

To interpret a at $s \in S_0$, execute an alternating target path of odd length $k_a(s)$ and return state $d_a(s)$ at world 1; similarly for b. The runner on the composite source path ab has

$$\boxed{K_{ab}(s) = k_a(s) + k_b(d_a(s)), \qquad D_{ab}(s) = d_b(d_a(s)).}$$

Operational consequence. The implementation of a cannot be a no-op: the empty target path ends at world 0, whereas the continuation of a has world 1. This is a typing constraint, not an imposed inequation on path lengths.

The associated one-step runner endofunctor is

$$H(S)_0 = (2\mathbb{N} + 1) \times S_1, \qquad H(S)_1 = (2\mathbb{N} + 1) \times S_0.$$

Its cofree comonad describes alternating streams of residual-state labels and odd implementation lengths. This provides a concrete benchmark for any global dependent measuring construction.

6 Change of context and the free-monad comparison

Let $f : W' \to W$. The canonical restriction of a polynomial endofunctor in the polynomial equipment is

$$B_f P = f^* P \Sigma_f.$$

Unfolding its fibres gives

$$(B_f P)(X)_{w'} \cong \coprod_{\substack{a \in A_{f(w')} \\ \lambda : E_a \to W',\ f\lambda = s|_{E_a}}} \prod_{e \in E_a} X_{\lambda(e)}.$$

Thus a lifted operation consists of an old operation and a chosen lift of every continuation world. This is a precise sense in which a changed context can create, remove or distinguish operational choices.

The induced construction on monads is $B_f T = f^* T \Sigma_f$; the adjunction $\Sigma_f \dashv f^*$ supplies its monad structure. Base-changing the generator inclusion $P \to P^*$ and using freeness gives a canonical monad morphism

$$\boxed{\kappa_{f,P} : (B_f P)^* \longrightarrow B_f(P^*).}$$

It generally fails to be invertible: programs built after lifting the primitive commands need not account for programs that leave a restricted context and later return, and collapsing worlds can erase intermediate lift choices.

Example 6.1 (An excursion survives restriction)**.** *Let $i : \{0\} \hookrightarrow \{0,1\}$ in the two-world example. The restricted signature B_iP is zero, since the command a has no destination-world lift; hence $(B_iP)^* = \mathrm{id}$. But forming the path monad first retains the paths $\mathrm{id}_0, ab, abab, \ldots$, giving $B_i(P^*)(X) \cong \mathbb{N} \times X$. Here κ_i is the monad unit $\mathrm{id} \to \mathbb{N} \times -$.*

Example 6.2 (File read with a handled EOF response)**.** *Take worlds $W = \{C, O, E\}$ and include $U = \{C, O\}$ by $i : U \hookrightarrow W$. In the free file signature, the one-step operation* `read` *at O requires continuations at both O (a datum) and E (EOF), so it does not survive in B_iP. However, the completed program*

$$\mathsf{read}(d \mapsto \mathrm{return}_O,\ \mathsf{reset}; \mathrm{return}_C)$$

has its only variable leaves in U: the EOF branch passes through E internally before resetting to C. Thus it belongs to $B_i(P^*)$ but not to the image of $(B_iP)^*$. This is the concrete failure of surjectivity already seen in Part A.

Example 6.3 (Nullary branches matter)**.** *Take $P(X)_0 = X_0 \times X_1$ and $P(X)_1 = 1$, and restrict to $\{0\}$. The term $c(x, z)$ has root and only variable leaf in world 0, but has a nullary subtree z at world 1. It survives in $B_i(P^*)$, not in $(B_iP)^*$. Thus for branching theories, controlling paths between boundary worlds does not suffice.*

Theorem 6.4 (Tree-lifting criterion in Set)**.** *For a dependent polynomial P and a map $f : W' \to W$, the comparison $\kappa_{f,P}$ is an isomorphism if and only if f is injective and its image $U \subseteq W$ is* tree-convex*: every well-founded P-tree with root and all variable leaves in U has every internal-node world in U.*

Proof. If f is not injective, a zero-operation term based at $w' \in W'$ in $B_f(P^*)$ may return a variable in $X_{v'}$ for $v' \neq w'$ but $f(v') = f(w')$. Such a term cannot lie in the image of the zero-operation sector of $(B_fP)^*$, and κ_f preserves underlying tree size. Thus f must be injective. For injective f, a lifted tree is uniquely determined whenever every node world belongs to U. The codomain allows all trees with root and variable-leaf worlds in U. Surjectivity is therefore exactly tree-convexity; injectivity is automatic. □

A usable sufficient condition is *forward closure*: every continuation world of every operation rooted in U already belongs to U. It is stronger than tree-convexity: a command may leave U harmlessly if no completed tree returns its variable leaves to U.

For successive maps $U \xrightarrow{g} W' \xrightarrow{f} W$, base-change functors satisfy $B_gB_f \cong B_{fg}$. Uniqueness of extension from free generators gives the coherence identity

$$\boxed{\kappa_{fg,P} = B_g(\kappa_{f,P}) \circ \kappa_{g,B_fP}}$$

under this canonical identification, and $\kappa_{\mathrm{id},P} = \mathrm{id}$. Thus the comparisons are coherent even when not invertible. Sweedler measurement, where it exists, reverses their direction in the source theory.

7 Equations and fixed-world measuring: checked examples and an open proof

Equations in the source theory become conditions on translated programs, including residual updates. They may be conditions on iterated rather than one-step behaviour.

Example 7.1 (Idempotent source operation)**.** *Let T have a unary operation u with $u^2 = u$, and take target $R(X) = \mathbb{N} \times X$. A runner is $k : S \to \mathbb{N}$ and $d : S \to S$. The equation gives $d^2 = d$ and $k \circ d = 0$. The cofree runner on X has carrier $X \times \mathbb{N} \times X$ and generator structure*

$$d(x, n, y) = (y, 0, y), \qquad k(x, n, y) = n.$$

The universal map from a runner labelled $h : S \to X$ is $s \mapsto (h(s), k(s), h(d(s)))$.

Example 7.2 (Commuting unary operations)**.** *Two unary generators satisfying $uv = vu$, into the identity target, give commuting endomorphisms of residual state. The cofree runner is the usual cofree $\mathbb{N}^2$-action, with underlying functor $X \mapsto X^{\mathbb{N}^2}$.*

A prospective general argument starts from a small equational presentation $T = \mathrm{coeq}(E^* \rightrightarrows P^*)$. A runner for T is an $H_{P,R}$-coalgebra for which the two interpretations of the equation signature agree. One expects an accessible equifier of coalgebra categories, closed under colimits and under the required split equalisers; then a right adjoint to the forgetful functor and the dual Beck theorem would give a measuring comonad.

Status. This is a *proof programme*, not a theorem established here. The accessibility of the relevant equifier, creation of colimits, split-equaliser descent and infinitary presentation step should be proved carefully in a single categorical argument. The concrete examples above do not rely on that argument.

8 Categorically indexed Moore machines

The familiar Moore-machine theory admits a many-sorted generalisation that clarifies the relationship between world-indexed polynomial dynamics and category actions.

Let G be a small directed graph with vertex set W, let $\mathcal{K} = G^*$ be its free category, and fix observation sets O_w for $w \in W$. Define a many-sorted algebraic theory $T_{\mathcal{K},O}$ with sorts indexed by W, unary operations

$$[a] : X_v \longrightarrow X_w \qquad (a : w \to v)$$

and observation operations

$$\mathsf{read}_w : X_w^{O_w} \longrightarrow X_w.$$

The action operations satisfy the identity and composition equations of $\mathcal{K}$, while the observation operations satisfy the ordinary reader equations. No additional equations relate actions to observations.

Proposition. The category of Set-valued many-sorted comodels of $T_{\mathcal{K},O}$ is naturally equivalent to the category of coalgebras on $\mathbf{Set}^W$ for

$$F_O(S)_w = O_w \times \prod_{\substack{a \in G_1 \\ s(a) = w}} S_{t(a)}.$$

Equivalently, these comodels consist of functors $S : \mathcal{K} \to \mathbf{Set}$ equipped with arbitrary observation maps $o_w : S_w \to O_w$.

Proof. The unary action operations induce maps $\delta_a : S_w \to S_v$ satisfying the functoriality equations. The reader equations force the state component of each observation operation to be the identity, leaving precisely $o_w : S_w \to O_w$. Since $\mathcal{K}$ is freely generated by G, specifying its action is equivalent to specifying the transitions associated with generating edges. These

are exactly the components of an F_O-coalgebra. The constructions preserve morphisms and are mutually inverse.

For a category presented by generators and relations, the corresponding coalgebras must additionally satisfy the transition equations induced by those relations. When W has one element and G consists of loops indexed by an alphabet A, the theory recovers the ordinary Moore-machine theory $T_{\mathrm{Moore}}(A, O)$ and its comodels. This comparison identifies category-indexed Moore machines as a special case of the dependent polynomial systems considered here. It does not introduce a new kind of dependence: the graph of admissible actions is fixed, and the resulting category organises their composable histories.

The next question is whether context formation, extension, and substitution can themselves become intrinsic to the source theory. In that setting, a command may refer to an individual introduced only after execution has extended the available context. This motivates the passage from many-sorted theories to genuinely dependently sorted algebraic theories.

9 Outlook: beyond shared worlds

The calculations in this part concern a common world set W. A more realistic implementation may have its own low-level worlds V. If a residual state refines both, it determines a span $W \xleftarrow{p} S \xrightarrow{q} V$. One possible runner structure then has the shape of a monad opfunctor

$$p^*T \longrightarrow (q^*R\Sigma_q)p^*.$$

This is a promising *proposal*, not a completed general theorem. An implementation of the file protocol using distinct buffer, connection and device modes would be a suitable test. Transporting dependent safety certificates and handlers would be the next operational questions.

The larger programme asks for measuring comonads compatible with those world-changing arrows, possibly through a matrix-indexed or accessible-polynomial equipment. Neither the closedness hypotheses of the general Sweedler-double-category theorem nor the required universal constructions have been verified for that equipment. We collect these structural questions in a separate research appendix, leaving the main tutorial with the established calculations.

Conclusion

The central insight is modest but reorganising: *a residual state is a refined world.* The identity

$$\mathsf{St}_S(R) \cong \Pi_p(B_pR)p^*$$

shows that state passing and context change arise from the same dependent adjunctions. Dependent polynomials then supply the operations and their world-indexed continuations; for free theories, runners admit an explicit coalgebra presentation. The alternating-path example from Part A demonstrates the operational force of typing, and the comparison κ_f explains why completed programs may survive a context restriction even when their primitive commands do not. These results justify a compact dependent Sweedler research programme. The full global double-categorical construction and its verification semantics remain important aims, but they need not be assumed in order to use the calculations already available.

References

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[3] T. Fiore, N. Gambino and J. Kock, *Monads in double categories*, Journal of Pure and Applied Algebra **215** (2011), 1174–1197. https://arxiv.org/abs/1006.0797.