Kuhn and modern mathematics

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15 July 2022

Is a Kuhnian treatment of modern mathematics possible?

Separate this into two questions:

- Is a historically-informed philosophy of modern mathematics possible?
- Would Kuhn's own concepts help with the production of this philosophy?

Consequences

If Kuhn led the philosophy of science away from later logical empiricism (theory T, evidence e), a parallel in mathematics would lead us away from:

- Mathematics as timeless.
- '$2 + 2 = 4$' is by and large enough for philosophy.
- Set theory is available if you want more.
- Quinean ontological commitments from first-order presentations.
- ...

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- Should Kuhn's own concepts help with this philosophy?

Less clear.

Sources

I started out from

- Albert Lautman (1908-44): the same Idea manifests in many mathematical situations - e.g., covering spaces and Galois extensions, reciprocity/duality, global/local (my article).
- Category theory: structure similarity across mathematics; logic as part of mathematics.
- Imre Lakatos: *Proofs and Refutations*.

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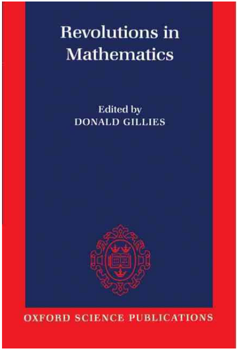
Returning to the UK, I encountered the "2+2=4 problem," then in the shape of Field's fictionalism,

- '$2 + 2 = 4$' is false because numbers don't exist.

Fortunately other things were going on:



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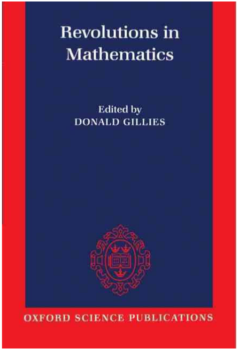

However, it contained very little past 1900 and that concerned foundational topics rather than mainstream interconnections.

(E.g., Herbert Breger, *A restoration that failed: Paul Finsler's theory of sets*, cf. Sec. 3 of my article.)

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Plenty more such change occurred over the following decades:

- Marshall Stone's 'quiet revolution' (1961): "**The unity of mathematics**. *The characterization of mathematics as the study of systems comprising certain abstract elements and certain abstract relations prescribed among them shows very clearly the essential unity of mathematics. Nevertheless it cannot adequately suggest* **the intimate structural connections which have actually been found among the different branches of mathematics, as a result of modern researches**. *During the last fifty or sixty years much has been done to identify and compare the mathematical systems dealt with in algebra, number theory, geometry, and analysis.*"

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Key is to look beyond mathematics as generating proofs of single propositions, to think about the larger picture.

Worries

What scale, extent and duration should you choose for your case studies?

The complexity of any specific passage of thought makes it very difficult to reconstruct as it occurred. Generally, it's much easier to learn from a later reworked perspective.

Unclear where to stop. Easier to talk to living mathematicians.

Isn't any (rationally reconstructed) case of discovery good enough? If so, then look back to simpler times.

Case studies need to generate great rewards to repay investment.

A Lakatosian step (DC 2003, Ch. 8)

Algebraic topology (1930s onwards, from Poincaré's *analysis situs*): the assignment of algebraic entities to topological spaces to resolve questions about the latter.

- Lakatos's own suggestion of a mathematical research programme
- Continues on from *Proofs and Refutations*
- Launches the use of category theory
- Spins off concepts used everywhere: **cohomology**, **homotopy**

Rivalry with point-set topology looks unlike Lakatos's scientific research programmes. Because these subjects have different goals? Is there more 'room' in mathematics than in science?

The space of mathematics

There appears to be no possibility of exhaustion in mathematics. More like technology with no end of possible functions and for each function no best solution.

Are we left with mathematics post-1930s as anything expressible in set theory?

But the participation in Stone's "intimate structural connections" accords with a clearly professed value in mathematics. Prizes are won for discovering these connections.

A bold choice: Langlands program, 1967 onwards

- Langlands program: the uniting of number theory, harmonic analysis, representation theory, algebraic geometry,...

Its story appeals to great names of the 19th century: Gauss, Riemann, Abel, Lie, Klein, Hilbert, etc.

Continuity with the past, and yet unthinkable to formulate it back then.

It was involved in the solution of Fermat's last theorem, and has a bearing on three millennium prize problems.

It's an excellent exemplification of Lautman's position.

Langlands program

The Langlands Program has emerged in recent years as a blueprint for a Grand Unified Theory of Mathematics. Conceived initially as a bridge between Number Theory and Automorphic Representations, it has now expanded into such areas as Geometry and Quantum Field Theory, weaving together seemingly unrelated disciplines into a web of tantalizing conjectures. The Langlands correspondence manifests itself in a variety of ways in these diverse areas of mathematics and physics, but the same salient features, such as the appearance of the Langlands dual group, are always present. This points to something deeply mysterious and elusive, and that is what makes this correspondence so fascinating.

One of the prevalent themes in the Langlands Program is the interplay between the local and global pictures... (Edward Frenkel, Langlands Correspondence for Loop Groups, CUP, 2007)

Flavours of Langlands correspondence

- global
- local
- geometric
- quantum
- ...

Geometric Langlands – S-duality of 4D supersymmetric gauge theories.

Ongoing mini-revolutions within the program

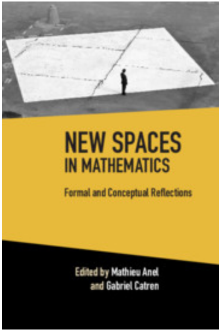
Frenkel's 'blueprint' has something of the sense of 'paradigm'. It's easy to find mention also of radical change:

> For mathematics as a whole, there's a sense of awe and possibility in the reception of the new work: awe at the way the theory of *p*-adic geometry Scholze has been building since graduate school manifests in the Fargues–Fontaine curve, and possibility because that curve opens entirely new and unexplored dimensions of the Langlands program.
>
> "It's really changed everything. These last five or eight years, they have really changed the whole field," said Viehmann.

Kevin Hartnett, *New Shape Opens 'Wormhole' Between Numbers and Geometry*, Quanta Magazine, 2021

New spaces

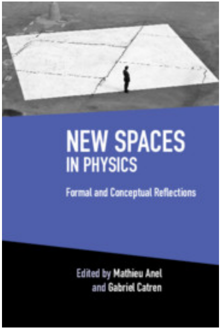
A historical depiction of the Langlands program would certainly convey a rich picture of recent mathematics. On the other hand, work on new geometry suggests that as a topic one chooses a meta-concept, such as *space*:



(Anel and Catren (eds.) 2021, CUP)

Physics

Associated book for physics with plenty of interweaving:



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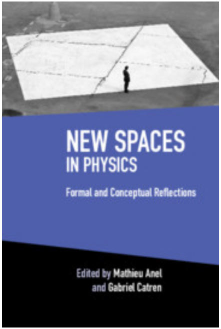
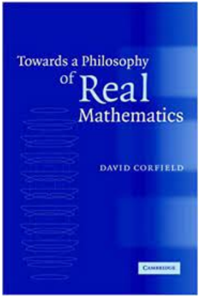


Choose a unifying quest (Langlands) or choose a concept (space) and you have the opportunity to convey something of what modern mathematics is like: its goals, its sense of progress, its values, its reality.

My latest choice

In my 2003 book, I end the final chapter on higher category theory by calling for a historically-informed philosophy of mathematics, but also suggest that all of philosophy might gain by looking to exploit the new logic then emerging from this field.



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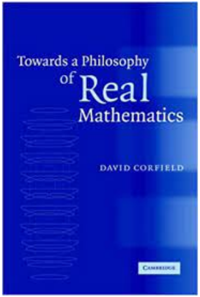
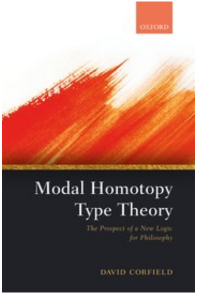


I mentioned there the *FOLDS* of Makkai, but since that time there has emerged *Homotopy type theory*.

New tools for philosophy

I've since taken this up:

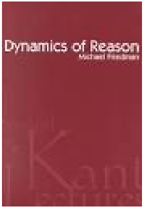


Philosophy failed to emulate Russell by not keeping abreast of attempts to extract a logic adapted to contemporary mathematics. (Important ingredients of the logic go back to the 60s.)

For some introductory slides, see I and II.

Back to Kuhn via Friedman

Now we can plough this case back into integrated history and philosophy of science, via Michael Friedman.



Friedman on Kuhn: Repair his irrationalism by a prospective rationality driven by meta-scientific philosophy.

New mathematics allows the formulation of new physical principles: Newton-Kant; Einstein-Vienna Circle; ???

The Third Exemplar

Witten on new geometry:

> *String theory at its finest is, or should be, a new branch of geometry... I would consider trying to elucidate this proper generalization of geometry as the central problem of physics, certainly the central problem of string theory. (Witten 1984)*

At the very least it seems this requires *homotopical mathematics*:

> *homotopical mathematics (whose foundations could be said to be the homotopy theory of type developed by Voevodsky and al.)... reflects a shift of paradigm in which the relation of equality relation is weakened to that of homotopy.*[1]
>
> [1] *It is very similar to the shift of paradigm that has appeared with the introduction of category theory, for which 'being equal' has been replaced by 'being naturally isomorphic'. (Toën 2014, p. 3)*

This isn't now about allusive examples. The scale is too large.

A new logic for a new conception of space for a new physics.

Friedman recovers

Having drawn a sharp distinction between natural science and mathematics (2001, p. 99), he later writes

> *This picture is deeply problematic, I now believe, in at least two important respects: it assumes an overly simplified "formalistic" account of modern abstract mathematics, and, even worse, it portrays such abstract mathematics as being directly attached to intuitive perceptible experience at one fell swoop [...]*
>
> *Our problem, therefore, is not to characterize a purely abstract mapping between an uninterpreted formalism and sensory perceptions, but to understand the concrete historical process by which mathematical structures, physical theories of space, time, and motion, and mechanical constitutive principles organically evolve together so as to issue, successively, in increasingly sophisticated mathematical representations of experience. (2010, p. 697-698)*

The Third Exemplar

1. An instance of a **cohomology** theory has been used to formulate (Hypothesis H) a long-sought fundamental physical theory.
2. In physics, **cohomology** is fundamental to modern formulations.
3. Mathematically, the ubiquitous **cohomology** is best treated as functions in **higher toposes**.
4. Logically, **higher toposes** may be reasoned about in the language of (modal) **HoTT**.
5. Philosophically, the language (modal) **HoTT** makes sense of many currents of thought in metaphysics, philosophy of language, etc.
6. Metaphilosophically, we need to situate Friedman within the story of philosophy.

(For more details see these slides, I and II.)

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Less clear. Almost all findings of historical philosophy of science have some counterpart in mathematics. Radical change occurs on different scales. Communication is not always easy, yet there is endless reworking and unification. If not anomalies, then heuristic falsifiers.

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Thank you!

Notes

Langlands program relates Galois groups in algebraic number theory to automorphic forms and representation theory of algebraic groups over local fields and adeles.