Chapter II

Dependent types

II.1 The need for types

Once upon a time, there was a kingdom where everyone was sad. The townspeople were sad. The farmers were sad. Even the King and Queen were sad.

From the standard untyped perspective, were I to seek to render the opening sentence in formal terms, I might first rewrite it as:

- There is a time and there is a place such that everyone in the place at that time is sad.

I can then represent this sentence as something like:

$$\exists x \exists y \forall z (Time(x) \ \& \ Place(y) \ \& \ Person(z) \ \& \ InAt(z, y, x) \to Unhappy(z, x))$$

for a predicate of timed location $InAt$. The question then arises as to what it is I am supposed to be quantifying over in this proposition. It would appear to be a domain whose elements include at least all times, places and individuals. What else should be thrown into the pot to make a coherent domain out of this disparate collection of items? While some philosophers, notably Timothy Williamson (2003), advocate unrestricted quantification over 'everything', merely possible entities included, other researchers in philosophical and mathematical logic have sought highly restrictive forms.

For instance, the type theorist of a certain stripe will take there to have been preparatory work already completed before the formation of the proposition. We would have declared the existence of some types: Time, Place, Person. We may well have distinguished between a type of Time *intervals* and a type of Time *instances*.[1] Type declarations provide criteria for the behaviour of their elements, including importantly identity criteria. Then

[1] We will do precisely this in Chap. IV.

there would perhaps be a function from Person to Time intervals, marking the life of the individual, and a function locating their whereabouts at times in their lives. Further, one could construct a type family of people alive at a certain time and living in a certain place, which varies with time and place, $Person(t, p)$. The original proposition could then be constructed itself as a type, where its truth would amount to producing an element of that proposition, so here an element of the type of triples of the form (t, p, f), where $t : Time$, $p : Place$, and $f : \prod_{z:Person(t,p)}(Unhappy(z, t))$. This last element, f, is the kind of thing which when given a person from time t and place p, spits out evidential warrant of their unhappiness then. Notice that each instance of quantification takes place on a specified type, whether a simple type, such as as Time or Place, or one formed from other types, such as $\prod_{z:Person(t,p)}(Unhappy(z, t))$.

So one very important difference of type theory from first-order logic is that we don't have untyped variables, that is, ones ranging over some unspecified domain. When we consider the classic '*All ravens are black*' as $\forall x(Raven(x) \rightarrow Black(x))$, its truth depends on whether it really applies to *everything*. With unrestricted quantification this really should mean everything you take to exist: a bird, a table, the Queen of England, a number, perhaps even your filial duty. The failure of all non-bird entities to satisfy $Raven(x)$, along with all non-raven birds, means that the implication is classically true in their case whatever their colour. Of course, we typically smuggle in 'sensible' restrictions with talk of a *domain*, but this is a situation where we ought to avoid such vagueness. Such imprecision troubles us in the so-called 'Ravens paradox' where the negations of $Raven(x)$ and $Black(x)$ are thought to make sense, and the supposedly logically equivalent '$\forall x(\textit{non-Black}(x) \rightarrow \textit{non-Raven}(x))$' is constructed. Once again we should reflect on what the domain is taken to be. Are the times, places and people needed for the previous proposition included? For the type theorist, on the other hand, $Black(x)$ must be properly typed. It is a proposition depending on an element of some other specified type. How to delimit this specified type is then an important question, whether here a broad type of space-time continuants or a narrow type of birds. In the solution to the Ravens paradox which proposes that an instance of a non-black non-raven *does* act marginally to confirm the universal statement, it is commonly argued that the relative numbers of non-ravens to ravens matters. But, as we shall see, we only have the individualisation permitting such counting when we work with a reasonably prescriptive master type over which quantification is taking place.

Why logical formalisms, one of whose aims is to capture the structure of natural language, should have foregone typing is worth pondering. Natural languages are quite evidently marked by typing information to greater or lesser degrees. When I ask a question beginning *Who* or *When*, I expect an answer which is a person or a time, even if I hear 'April'. Some languages are more thoroughly lexically typed than English. In Japanese, when counting objects, aspects of their nature are reflected in the choice of appropriate counter words, *hai* for cups and glasses of drink, spoonsful, cuttlefish, octopuses, crabs; *dai* for cars, bicycles, machines, etc. Swahili has *eleven* classes of lexically marked noun. But even without explicit lexical markers for kinds of noun, the depth of type structuring is revealed by subtleties in the grammatical legitimacy of sentences. This is shown clearly by similar pairs of sentences, the second in each of which is problematic:

- I ran a mile in six minutes.
- *I ran a mile for six minutes.
- I had been running a mile for six minutes, when I fell.
- *I had been running a mile in six minutes, when I fell.
- I spilt some coffee on the table, but it's been wiped clean.
- *I have spilt some coffee on the table, but it's been wiped clean.
- She brushed the crumbs off the table.
- *He broke the plate off the table.

As we shall see later, this may be explained by the compositional structure of event types. The ontology of events as distinct from objects will feature significantly through this chapter.

Type theory has found many supporters in computer science for a number of reasons, not least that a desirable feature of a program is that it processes data of specific kinds, and calculates answers and performs actions of specific kinds. At a basic level we would like our machines to tell us that something is amiss when we enter a number on being asked our name. But we would also like the internal operations of the program to respect the types of the data. Type-checking, either while being compiled before the program is run ('static') or while the program is running ('dynamic'), provides assurance of the correctness of the procedure, as expressed by the slogan "well-typed programs cannot go wrong".

Typing usefully wards off category mistakes in natural language and is often supported by syntax. However, plenty of jokes play on the reverse phenomenon, arising from ambiguities of different type readings. A collection of words which may be parsed in two ways when spoken, and sometimes also when written, can give rise to very different readings. So when the joke sets off with *A termite with a toothache walks into a pub and asks "Where's the bar tender?"*, we are typically hearing a request for an individual's location, as if a reasonable answer is *"The bar-tender is in the cellar."* When we discover the joke is supposedly complete from the expectant look on the comedian's face, we rapidly engage in some re-parsing to take into account the inclusion of the detail of the termite and its ailment, and duly realise that the sort of answer the question requires is *"The bar is tender over by the wall."*

Type theories come in many shapes and sizes, one important axis of variation being the richness of the type structure. On one end of this axis we find the kind of type theory used by Richard Montague (1974). As developed from Alonzo Church's work, Montague worked with a sparse type theory with basic types of individuals, e, and truth values t. From these one can build up compound types, such as properties, of type $(e \to t)$, i.e., assignments of 'True' to individuals with that property. This then allows quantifiers to be of type $(e \to t) \to t$, taking properties to truth values, and so on, such as when a property holds for at least one individual.

But if the 'Battle of Hastings' and 'Vlad the Impaler' are two terms of type e, it seems as though we will be able to ask of them whether or not they are identical. We are marginally better off here than when working in a completely untyped setting, for example, by not being able to ask whether the entity *Julius Caesar* is equal to the property *crossed the Rubicon*. However there is a venerable philosophical tradition, associated in recent decades with the name of Donald Davidson, which argues for a fundamental distinction in our ontology between *objects* and *events*.

Davidson was led to take *events* as a basic ontological category for many reasons, not least because we can individuate them and yet apply a range of predicates to them which differ from those suited to objects. Indeed, events are plausibly not objects. We don't ask of an event what its colour is, but where it took place, its duration and so on. We don't ask of an object whether it has occurred or has ended, whether it was expected, and so on. A type theory which allows for several basic types seems desirable then. The type theory we consider in this book will allow for many basic types along with those derived from them. This will raise the question of how many basic types there are. Vlad the Impaler is apparently an element of the kind of rulers of Wallachia, the kind of political leaders, the kind of humans, the kind of primates, the kind of mammals, the kind of living things, the kind of objects. The battle of Hastings is an element of the kind of mediaeval battles on English soil, the kind of battles, the kind of events, and so on. It might seem that subtypes are being formed by increasing specification. The type *Human* will be something like *Primate of a certain sort*, primates will be mammals of a certain sort, mammals will be animals of a certain sort, and so on, right up to objects of a certain sort. If we carve out more specific types by some kind of operation on the general types *Object* and *Event*, why not take the further step and unite these types, and maybe others, in a supertype of individuals, as Montague does? Then objects are individuals which are object-like and events are individuals which are event-like. Let us see why we might want to resist this move.

Psychologically, at least, we seem not to conflate across this boundary.

One useful way to test this is to see when we can apply 'or' to them. Gazing on a distant shoreline, I might wonder whether I'm seeing a seal or a smooth rock. Someone looking on in the Senate asks 'Is Brutus greeting Caesar, or is he stabbing him?' These latter are events, and indeed events taking place at a certain time and place and involving the same protagonists. On the other hand, we don't say 'that is either a chair or a battle', but looking at a rowdy group of youths in the distance we say 'it's either a street party or a fight'. Widening the range of types to include colours and places, appears to make things worse. Imagine saying 'That's either the colour red or it's a marriage or it's the corner of my bedroom.'

When someone simulates a fire with a piece of red cloth being billowed about from below and I ask 'Is it a piece of red cloth or a fire burning?', it might seem that I'm crossing the object/event divide, but then to the extent that a fire is considered an event by being a process, it could be better to construe my question as asking which event type is occurring: 'Is that a piece of red cloth billowing or a fire burning?'

As Casati and Varzi (2015) explain, some have suggested that an object is a degenerate process, "a thing is a monotonous event; an event is an unstable thing" (Goodman 1951, p. 286), or more generally for a spectrum between the "firm and internally coherent" object (Quine 1986, p. 30) and the rapidly unfolding event. It has also been suggested that events have a more definite time demarcation, but may have loose spatial boundaries, while objects tend to have better defined spatial boundaries. Matters are made more complicated by entities whose constituents change, preserving to some extent shape or place. Animals maintain a certain body shape while replacing their constitutive elements slowly, yet may move place rapidly. Rivers replace their constitutive elements rapidly, yet are more static than animals. Fires can vary from the steady flame of a candle to the raging forest fire.

One revolutionary approach is looking to replace talk of objects by *processes*, at least in the biological realm:

> ...the world – at least insofar as living beings are concerned – is made up not of substantial particles or things, as philosophers have overwhelmingly supposed, but of processes. It is dynamic through and through. (Nicholson and Dupré 2018, p. 3)

Now, rather than take stability of living entities for granted, the focus is on how such stationarity is produced in the flux. This requires a radical revision to the way we think and indeed speak. However, since I will be drawing heavily on natural language use through this chapter, I will follow those who argue for the equal priority of occurrents and continuants.

Another approach to type classification comes from individuation and counting. We will come to see that our type theory contains some types which are *sets*. This is meant in the technical sense that for any two elements of such a type, their identity forms a proposition, again in a technical sense. This in turn means that there can be no multiplicity of how two elements are the same, merely a question of whether or not they are the same. So a set is a type such that it is only a matter of *whether* its elements are the same, not *how*. This will be expressed formally later, but let us note here that in everyday life a kind of thing generally allows some form of individuation. Typically the more specific the kind, the clearer this becomes. Many of the kinds of types we employ in ordinary language behave as sets in the technical sense. As might be expected, due to this property of individuation, any set has a cardinality. Relative to a specified type which is a set, individuation should be such as to allow counting to take place. Even if this is difficult, as with the types 'Hair on my head' and 'Leaf on that tree', it makes sense to imagine that there is a total number. On the other hand, other types, and typically less specific ones, do not allow easy individuation. Until one specifies a variety of *object*, it is generally the case that counting is problematic. You don't ask, looking in on a kitchen, how many objects there are in there, but to ask for the number of plates is fine.

It has been proposed (see, e.g., Wellwood, Hespos & Rips 2017) that there is an important analogy between objects and events in terms of their relationship with their respective constituents:

$$\text{object} : \text{substance} :: \text{event} : \text{process}.$$

In the case of substances, we tend to identify portions as constituting objects and apply plural noun forms when we see them as reasonably regularly shaped. A football on a soccer pitch possesses this feature very clearly, a sandcastle on a beach somewhat less clearly, while a puddle on a pavement may be very hard to identify. Similarly, to consider a process as constituting an event requires some order, in particular a temporal boundary. Sharp boundaries such as provided by a football match make for easy identification and counting.

Discriminating and counting *events* thus requires our carving out discrete parcels of processes of a kind. Gazing over an Olympics gymnasium, we may ask how many events are occurring now: the 2012 Summer Olympics, a gymnastics competition, women's gymnastics, the vault, an athlete is running, a step is taken, and so on. Again, to be asked how many events occurred in the Second World War makes little sense, but one could imagine being asked to estimate the number of a better specified type, such as rifle shots on Juno beach during the first day of the Normandy landings. We now obsessively monitor the steps we take in a day, even though for any particular foot movement its status as a step may not be clear, such as when you make a small adjustment of stance as someone brushes past you. Similarly we make a reasonable attempt to individuate hurricanes and tropical storms, and even name them alphabetically, all the time waiting for them to 'lose their identity'. We will return to the details of the structuring of events after we have understood ways to express their composite nature.

As for entities on the cusp between events and objects, in a hot and dry summer there may be a number of forest fires in a locality. One threatening my house right here may be taken as different from another one raging ten miles away, but whether to count two patches of the same burning hillside as the same fire or different may not be obvious. One reason for the difficulty here is that there is little internal cohesion to a fire – it spreads, merges and separates all too easily. On the other hand, some kinds of dynamical processes do lend themselves to individuation and counting. For Collier, *cohesion*, "an equivalence relation that partitions a set of dynamical particulars into unified and distinct entities..." (2004, p. 155) is responsible for this.

> Cohesion is an invariant, by definition, but more important, it determines the conditions under which something will resist both externally and internally generated disruptive forces, giving the conditions for stability. (2004, p. 156)

Such cohesion comes in degrees, so that one can take a flock of birds to be *weakly* cohesive. Counting can thus be seen to be no easy matter, and all but impossible if taken to range over some domain which includes objects, events and other kinds. We need then to distinguish kinds or sorts, and to do so we need to specify the conditions for membership of these kinds. Taking his departure from Collier's ideas, DiFrisco claims that

> ...when the interactions of the relevant type meet the cohesion condition, we get access to a dynamical explanation grounding the *sortal* for the individual in question. (2018, p. 85)

This then allows for individuation:

> ...processes are identical iff they have all of the same cohesive properties, including cohesion profile and cohesion regime, and they occupy the same spatio-temporal region. (p. 85)

Let us now consider what other kinds we must contend with.

We must be careful not to rely solely on words to extract the relevant types. A point often noted by linguists is that terms apparently designating single entities of one type may be better understood as falling under different but related ones. This is the phenomenon of *copredication*, such as when we say *Lunch today was long but delicious.* On the one hand, we take lunch here as an eating event, on the other, as a collection of foodstuffs. The word 'book' displays a similar ambiguity. *I have two books on my shelf. Each contains the three books of 'The Lord of the Rings'. How many books are there: two, three or six?* Of course, it depends on whether we mean by 'book' – a physical tome, an information source, or an instance of an information source. Such ambiguity renders the following sentences strange:

- John has an extensive library of one thousand books. But he only has one book. They're all copies of *War and Peace*.
- I have *War and Peace* as an electronic book. My reader is light, and so is the book.

In other cases we may use different terms to disambiguate. Rather than ask at an airline's board meeting how many people have been carried this year, it is likely to be of greater interest to know how many trips have been taken, usually phrased as the number of passengers. Of course, it may be important to know as well how many different people travelled on your airline to understand frequency of flying.

The ease with which a restricted kind of middle-sized object allows for counting must surely be due to the underlying solidity and cohesion of its substance. If we have defined the type *Cup* in terms of objects which are hand-sized and used to contain a liquid for consumption, the identity condition for *Cup* can piggyback on that for object. But there is a part-of relation on objects, so a handle is part of a cup and also then an object in its own right. Given two elements of the same specific kind, identification is more straightfoward, how many cups in the cupboard, how many handles on this cup.

Sometimes an object may support more than one instance of another type. The artist Ivon Hitchens painted many semi-abstract landscapes from around his dwelling in Sussex. His choice of colour made it quite feasible to turn his elongated rectangular works upside down to present a different image. Indeed, he joked that his clients were receiving two paintings for the price of one. Here an individual painting is an object plus an intended orientation. If I hang the Mona Lisa the wrong way up, this is a subversive act.

Of course, other types most definitely are not supported as material objects: melodies, manners of walking, colours. George Harrison could be found guilty of using the melody from *He's so fine* when composing *My sweet Lord*. The type *Colour* has its own identity criteria in terms perhaps of side by side comparison of instances. Two colours are the same if when juxtaposing patches of these colours in good lighting conditions, we cannot tell them apart. We also use colour charts to provide standard reference points. When you say "I love chicken Bhuna," you are referring to an element of $Dish$, which generates the expectation that all instances of $Plateful\ of\ chicken\ Bhuna$ should be liked by you. What kind of thing is a dish? It doesn't have an associated place, yet it has a flavour.

Places rely on the fixity of environments. We can bury a pile of treasure at a spot and return to find it decades later. We can also have a place on a boat to keep track of our possessions while circumnavigating the world. Place is not generally a space-time point, but rather a path traced out by some stable configuration of objects. *"To think we were in the same place as each other on New Year's Eve without knowing."* Individuation occurs here (My trip to the US has me visiting three places: Boston, New York City and the Niagara Falls), but inevitably there will be a spatial structure to this type. Places can be closer or further from each other, and I'm unlikely to say that I visited ten places in France, namely, Paris and nine sites in the city of Bordeaux. The formal introduction of spatial features in our type theory will be deferred until a later chapter.

Alongside evidently spatial types, it seems plausible that many everyday types contain further structure relevant for counting. Presented with a collection of seven oil paints, I may be disappointed as a beginner looking to equip myself with the basics to find that these are: Royal blue, Cobalt blue, French blue, Prussian blue, Ultramarine blue, Manganese blue and Cerulean blue. The type of colours would seem to have a richer structure than being a mere set. Indeed, perhaps 'blue' and 'red' are better taken as designating a range of colours. It is common also to use spatial descriptions here, so that it is hard to imagine anyone taking Cobalt blue to be 'closer' to Burnt Umber than to Prussian blue. Similarly with animals, we are disappointed when visiting a zoo said to hold animals from 91 different species, if these are close relatives. We want to see dramatically different kinds, and not representatives of the 91 species of antelope.[2]

Looking beyond what seem to be basic types, we will also have to consider functions, such as a matching of my knives and forks, an assignment of soldiers to their regiments, a grading of the work of a cohort of students. Once these are admitted, the production of a master type of all individuals would seem to push us towards an intolerable inhomogeneity. We never would ask

> Is that Burnt Umber or a game of football or an elephant or a preference order of candidates or a declaration of war?

More radically, we are going to look to have *propositions* as types too. Let us now see why we should want to view a propositions as a kind of type on the same footing as a set-like type.

II.2 The analogy between logic and arithmetic

One good way to motivate this novelty of our type theory in taking propositions as a kind of types is by showing how similarly propositions and sets behave. When I teach an introductory course on logic to philosophy students, I like to point out to them certain analogies between the truth tables

[2] For a mathematical treatment of species diversity measures, cf. Leinster and Cobbald 2012.

for the logical connectives and some simple arithmetic operations, some of which have been known since the nineteenth century.

If we assign the values 1 to True and 0 to False, then forming the conjunction ("and") of two propositions, the resulting truth value is determined very much as the product of numbers chosen from $\{0, 1\}$:

- Unless both values are 1, the product will be 0.
- Unless both truth values are True, the truth value of the conjunction will be False.

It is natural for students then to wonder if the disjunction ("or") of two propositions corresponds to addition. Here things don't appear to work out precisely. It's fine in the three cases where at least one of the propositions is False, for example, 'False or True' is True, just as $0 + 1 = 1$. But in the case of 'True or True', we seem to be dealing with an addition capped at 1, something such as would result in a third glass when the contents of two other full or empty glasses of the same size are emptied into it.

Setting to one side for the moment this deviation, the next connective usually encountered is *implication*. A very good way of conveying to students the meaning of an implication between two propositions, such as $A \to B$, is as an *inference ticket*, a process taking a guarantee for the truth of A into such a guarantee for B. Implications seem to act very much like functions. Now, the standard truth table for implication assigns True to each row except in the case of 'True implies False', corresponding to the lack of a map from a singleton to the empty set. So, once again the truth table is reflected in the Boolean arithmetic of exponentials, $1^1 = 1^0 = 0^0 = 1$, while $0^1 = 0$.[3]

We begin to see that propositions as understood by propositional logic behave very much as sets of size 0 or 1. But now we notice further that the connectives can be put into relation with one another. Denoting the conjunction of A and B by $A \& B$, we may further observe the analogy:

- $(A \& B) \to C$ is True if and only if $A \to (B \to C)$ is True.
- $c^{(a \times b)} = (c^b)^a$

We may reformulate the former as expressing the equivalence between deducing C from the premises A and B, and deducing B implies C from the premise A. This equivalence is understood by every child:

- If you tidy your room and play quietly, I'll take you to the park.

[3]In continuous settings, 0^0 is generally considered to be of indeterminate value, but what provides a rationale for the answer 1 in discrete, combinatorial settings is relevant here.

- If you tidy your room, then if you play quietly, I'll take you to the park.

So what accounts for this pleasant logic-arithmetic analogy? The idea of a proof of $A \to B$ as a function and its analogy to the arithmetic construction b^a should bring to mind the fact that the latter may also be interpreted as related to functions. Indeed, b^a counts the number of distinct functions from a set of size a to a set of size b, so that, for example, there are 9 distinct functions from a set of size 2 to one of size 3.

Now we can see a closer analogy between proposition and sets, first through a parallel between conjunction and product:

- Guarantees of the proposition $A\&B$ can be formed from pairing together a guarantee for A and one for B.
- Elements of the product of two sets $A \times B$ can be formed from pairing together an element of A and one of B.

And second, there's a parallel between implication and function space, or exponential object:

- Guarantees of the proposition $A \to B$ are ways of sending guarantees for A to guarantees for B.
- Elements of the function space of two sets B^A (or $Fun(A, B)$) are functions mapping elements of A to elements of B.

This approach to logic is very much in line with the tradition of Per Martin-Löf (1984), and in philosophy with the ideas of Michael Dummett (1991). What homotopy type theory does is take observations such as these not merely as an analogy, but as the first steps of a series of levels. Propositions and sets are both considered to be simple kinds of *types*. In Martin-Löf style type theories, given a type and two elements of that type, $a, b : A$, we can form the type of identities between a and b, often denoted $Id_A(a, b)$. The innovation of homotopy type theory is that this identity type is not required to be a proposition – there may be a richer type of such identities. But we can see now how the levels bottom out. If you give me two expressions for elements of a given *set*, the answer to the question of whether or not they are the same can only be 'yes' or 'no'. We say that the identity type is a *proposition* or *mere proposition*. On the other hand, in the case of a type which is a proposition, were we to have constructed two elements, in this case proofs, and so know the proposition to be true, these elements could only be equal.

So taking identity types of the terms of a type decreases the type's level, and this may be iterated until we reach a point where the type resembles a singleton point, known as a *contractible* type. For historical reasons connected to the branch of mathematics known as homotopy theory, this level is designated -2. Mere propositions are thus of level -1, and sets of level 0. The hierarchy then extends upwards.

...	...
2	2-groupoid
1	groupoid
0	set
-1	mere proposition
-2	contractible type

In the next chapter we will give a formal treatment of the hierarchy, and explore the consequences of allowing higher level types, ones for which there may be a *set* of identities between two terms, or a type of identities of yet higher level. Here for the moment we may think of an easy example of a type for which this is so. Take the type of finite sets.[4]. For any two elements, that is, any two finite sets, an element of their identity type is any map which forms a one-to-one correspondence between the sets. In the case of equally sized sets of cardinality greater than 1, there will be several of these correspondences. But then there is nothing further to say about any two such identities, other than that they are the same or different. The identity type of two finite sets is a set. The type of finite sets therefore has level $+1$, since its identity types are sets, of level 0. And so it goes on. One may have types of infinitely high level.

In Martin-Löf-style type theories, propositions are also types, the types of their warrants. In homotopy type theory a type with at most one element is called a *mere proposition* or simply a *proposition*. This terminological decision reflects the fact that Martin-Löf designed his type theories for mathematical use, and in mathematics the word *proposition* is typically used as a claim which is to be established. The decision of the HoTT people to call these 'mere propositions' comes from the fact that once a claim has been proved in a piece of mathematical writing, details of the proof are then generally forgotten. It's just the fact that the claim has been shown to be true that matters, and different proofs of a proposition are not distinguished.

There are two noteworthy types which are propositions that may be formed simply in our type theory. These propositions account for the 0 and 1 appearing in the logic-arithmetic analogy above. To define the empty type $\mathbf{0}$, there are no elements to construct, but there is an elimination rule which says that from an element of $\mathbf{0}$ then for any type, C, we can construct an element of C. This is the type theoretic version of *ex falso quidlibet* with $\mathbf{0}$

[4] This is defined via the type family $Fin(n)$ (see III.1.2 below and Ex 1.9 in (UFP 2014))

as the proposition *False*. On the other hand, to define the unit type, **1**, we specify a single element $\star : \mathbf{1}$. **1** plays the role of the proposition *True*.

Whether or not the terminology *mere proposition* lasts in the long term, it leads to an important finding: **There is no need to propose a separate logic for HoTT**. The general type constructions suffice. For instance, almost all type theories have a rule that takes two types and generates their product:

$$A, B : \mathit{Type} \vdash A \times B : \mathit{Type}.$$

Elements of the product type are formed by pairing elements of the separate types. When you give two coordinates for a point in the plane or place on a map, 43 degrees North *and* 13 degrees East, the 'and' there is just like the 'and' which conjoins two propositions. A proof of A *and* B is a proof of A *and* a proof of B.

In a similar vein, another type formation rules states that for any two types, there is a type of functions between them:

$$A, B : \mathit{Type} \vdash [A, B] : \mathit{Type}$$

An element here amounts to a construction taking an element of A to an element of B. In the case of mere propositions, an element of the function type is a proof of the implication, a mapping of a warrant for A to a warrant for B.

So we see that from the perspective of HoTT that there is no coincidence in some of the analogies we have just observed, and we shall shortly see that this applies to the others. It is a blindspot of other foundational systems, such as ZFC set theory, that conjunction of propositions and the cartesian product of sets, and that implication and function space, are treated so differently. HoTT explains these commonalities as arising from the *same* constructions. And, as we shall see after I have introduced *dependent* types, these commonalities may be greatly expanded. For the moment, though, let's consider why from the category theoretic perspective, we should expect product operations and function space objects to appear so very commonly. Taking type theories as the syntactical counterpart of categories, the prevalence of products of types coincides with the prevalence of cartesian products in categories. Since, as I shall now show, products in a category emerge as *adjoints* to a very basic duplication map, their prevalence should not surprise us.

As we should expect from the *computational trinitarianism* thesis discussed in the previous chapter, an account of the appearance of constructions whose instances include logical connectives may be given from a category theoretic perspective. We owe this account to William Lawvere who in *Adjointness in foundations* remarked:

> One of the aims of this paper is to give evidence for the universality of the concept of adjointness, which was first isolated and named in the conceptual sphere of category theory, but which also seems to pervade logic. (Lawvere 1969, p. 283)

A few years later, he explained in his paper, *Metric spaces, generalized logic and closed categories*, that we should identify logic with a "scheme of interlocking adjoints" (Lawvere 1973, p. 142). We will be treating adjunctions in detail in Chap. IV, but for the moment let me remark that they constitute an essential part of category theory. As one of the founders of category theory, Saunders Mac Lane, remarked, "Adjoint functors arise everywhere" (Mac Lane 1998, p. vii).[5] From a certain perspective, it is quite surprising how little knowledge of adjunctions has penetrated into philosophical consciousness when it has been known for fifty years how integral they are to logic, and all the more so in the case of modal logic, as we shall see later.

We can consider a category, $\mathcal{C}$, to be a collection of objects related by arrows between pairs of them. That is, for any pair of objects, S and T, in the category there is a collection of arrows between them, $Hom(S, T)$. Think of these arrows as processes, transformations, or inferences running from S to T. Then at any object there is an identity process, 1_S, and any two matching processes, $f : S \to T$ and $g : T \to U$, where the target of the first matches the source of the second, may be composed $g \circ f : S \to U$. Composition of a process with the relevant identity process on either side leaves it unchanged, $f \circ 1_S = f$ and $1_T \circ f = f$. Finally, it doesn't matter in which order we calculate the composition of three or more composable processes, so that for appropriate maps $h \circ (g \circ f) = (h \circ g) \circ f$. A functor between two categories, $F : \mathcal{C} \to \mathcal{D}$, sends identity processes to identity processes, and composites of processes to composites of the images of processes, $F(g \circ f) = F(g) \circ F(f)$.

In the case of a *deductive system*, we have a category with propositions as objects, P and Q, and a single arrow in $Hom(P, Q)$ if and only if P entails Q. Here composition of arrows represents the transitivity of the entailment relation. A functor between deductive systems we might consider to be a translation from one to the other, where the entailment relation is preserved.

A pair of adjoint *functors*, F and G, between two categories $\mathcal{C}$ and $\mathcal{D}$ is such that for any A in $\mathcal{C}$ and B in $\mathcal{D}$,

$$Hom_{\mathcal{C}}(A, G(B)) \cong Hom_{\mathcal{D}}(F(A), B).$$

Here F is the *left* adjoint and G is the *right* adjoint, and one may think of them of providing something approximating an inverse to one another. So now, take a single category, $\mathcal{C}$, and then form the cartesian product of $\mathcal{C}$ with itself. This is a category which has pairs of objects of $\mathcal{C}$ as objects, and pairs of morphisms of $\mathcal{C}$ as morphisms. Clearly there is a *diagonal* map, a *functor*,

[5] See also the Notes therein on p. 107.

$\mathcal{C} \to \mathcal{C} \times \mathcal{C}$, which sends any object of $\mathcal{C}$, X, to (X, X). We now look to form the right adjoint of this functor. If it exists, this will have to send a pair of objects of $\mathcal{C}$ to a single object. Now, given any three objects, A, B, C, in $\mathcal{C}$, we know that

$$Hom_{\mathcal{C}\times\mathcal{C}}((A, A), (B, C)) \cong Hom_{\mathcal{C}}(A, B) \times Hom_{\mathcal{C}}(A, C).$$

So for there to be such an adjunction we need a construction, which we call $B \times C$, such that

$$Hom_{\mathcal{C}}(A, B) \times Hom_{\mathcal{C}}(A, C) \cong Hom_{\mathcal{C}}(A, B \times C).$$

This is precisely what the product construction achieves. Loosely speaking, we could say that as far as logic goes, as soon as one admits the possibility of making more than one assertation, it is natural to arrive at the concept of conjunction. Very similarly, the left adjoint to this diagonal functor corresponds to the sum or *coproduct*.

$$Hom_{\mathcal{C}\times\mathcal{C}}((B, C), (A, A)) \cong Hom_{\mathcal{C}}(B + C, A).$$

Pushing on one more step in the web of interlocking adjoints, if our category $\mathcal{C}$ has such binary products, then for any object B there is a functor from $\mathcal{C}$ to itself which takes A to $A \times B$. Then for there to be a right adjoint, G_B, of *this* functor, it would need to be such that

$$Hom_{\mathcal{C}}(A \times B, C) \cong Hom_{\mathcal{C}}(A, G_B(C)).$$

The image of such a functor we denote $G_B(C) = C^B$. Exponentials thus readily appear in many contexts, and the connective '*implies*' is seen to occur naturally. Again, with the notion of an assertion following from another, it is natural to arrive at the concept of implication.[6]

II.3 Dependent sum and 'and'

Ask philosophical logicians to show you their wares with respect to the meaning of a favoured connective, and they'll often plump for conjunction. 'Or' suffers the ambiguities as to whether it is to be understood as *inclusive* or *exclusive*. Implication suffers worse from the commonsense assumption that 'A implies B' indicates some relevance between the propositions. But good

[6]See Patrick Walsh's paper (Walsh 2017) for more along these lines concerning the relationship between logical connectives and adjunctions, including the appearance of identity types in an adjunction. Technical aside: the *counit* generated by the diagonal-product adjunction is a map in $Hom_{\mathcal{C}\times\mathcal{C}}((A \times B, A \times B), (A, B))$ which represents the paired elimination rules for the product type, and in particular for the conjunction of propositions. Similarly, the *unit* of the diagonal-coproduct adjunction represents the paired introduction rules for the sum type. See IV.3 for more on this.

reliable 'and' appears to work as it is supposed to. $A\&B$ is true if and only if A is true and B is true.

Indeed, when so-called *logical inferentialists* explain how they take the rules governing logical connectives to encapsulate their meaning, they frequently illustrate this with the case of conjunction, writing the *introduction* and *elimination* rules something like this:

$$\frac{\mathrm{P} \quad \mathrm{Q}}{\mathrm{P}\ \&\ \mathrm{Q}} \qquad \frac{\mathrm{P}\ \&\ \mathrm{Q}}{\mathrm{P}} \qquad \frac{\mathrm{P}\ \&\ \mathrm{Q}}{\mathrm{Q}}$$

For Paul Boghossian,

> ...it's hard to see what else could constitute meaning conjunction by 'and' except being prepared to use it according to some rules and not others (most plausibly, the standard introduction and elimination rules for 'and'). (2011, p. 493)

When Timothy Williamson presented an inference that a three-valued logician would take to be valid, but which runs contrary to conjunction elimination. Boghossian replied that he does not believe that this case "presents us with an intelligible counterexample to the analyticity of conjunction elimination..." (2011, p. 493), he was professing a commonly held view of the meaning-constitutive power of natural deduction rules.

It was a provocative choice then when Bede Rundle devoted his paper on conjunction (Rundle 1983) to the limitations of taking 'and' as the logicians' & or $\wedge$.

> On the one hand, I am inclined to think that the standard philosophical treatment of conjunctions like 'and', 'but' and 'although' has been grossly inadequate, no concern being shown for anything more than a narrow aspect of their use, and no investigation of that use being conducted on the right principles. On the other hand, the preoccupation with truth-conditions which has resulted in this defective approach is one towards which it is easy, and proper, to be sympathetic, at least initially. I shall begin by indicating the considerations which might invite our sympathy, and then call upon the example of 'and' to show how the approach is defective. (1983, p. 386)

Rundle had long campaigned in the ordinary language tradition against formalisation:

> ...it is not uncommon to find a philosopher showing an exaggerated respect for what he describes as the 'logical form' of a class of sentences, seeing in his own artifice a fundamental linguistic or logical pattern when in practice the process of extracting it from actual sentences more often than not results in their distortion. (Rundle 1979, p. 8)

But one might have expected '*and*' to be ceded to the opposition.

In his 1983 paper Rundle runs through a number of objections, not the least important of which is that we use '*and*' to connect commands, expressions of wishes and questions, even producing mixtures of these:

- Where do you come from and what do you do?
- Put out the dog and bring in the cat.
- He was decent enough to apologize, and make sure you do too.

He does so to cast doubt on the logicians' reading:

> ...given that the conjunctive role of 'and' is quite indifferent to mood, we should surely be suspicious of any account of its use that makes reference to truth essential. (1983, p. 388)

Well, a die-hard 'conjunctivist' may suggest that we should take a command to be an instruction to bring about the truth that a states of affairs obtains, at which point it seems reasonable that conjunctions of commands are commands to bring about the truth of the conjunction of propositions that the corresponding states of affairs obtain. But looking further into Rundle's arguments we see that the main charge he brings to bear is that a crucial semantic aspect of '*and*' has been overlooked, one which underpins its conjunctive use:

> ...if I can show that, as far as it exists, the truth-functional character of 'and' can be accounted for by having regard to more general features of its use, a use that is essentially the same whatever the sentence-types it unites, then the supposed priority [in declarative cases - DC] can surely be considered illusory. (p. 388)

In this section, we consider a central element of these "more general features of its use", which is that '*and*' only conjoins on certain occasions:

- We typically do not use '*and*' to connect disparate assertions, still less in the case of the supposedly logically equivalent '*but*'.

As we saw with '*is*' in the previous chapter, there is no reason to commit ourselves to treating a single word of natural language formally in the same way. It is no different here with '*and*.' First, we may use '*and*' to judge of two elements that they each belong to a single type:

- Fido and Bella are poodles
- $\vdash \mathit{Fido}, \mathit{Bella} : \mathit{Poodle}$

Or two elements may jointly be judged to satisfy a relation.

- Adam and Eve are married
- $\vdash p : married(Adam, Eve)$
- $\vdash (Adam, Eve, p) : Married\ Couple$

Frequently we rely on ellipsis to avoid repetition when conjoining judgements, as with the conjunction:

- Adam is a doctor and Eve is an accountant,

which may be again be represented as the claim that two elements belong to a type:

- $(Adam, doctor), (Eve, accountant)$: *People and their occupation.*

In such a case we may omit grammatical parts, here the repeated '*is*':

- Adam is a doctor and Eve an accountant.

In the case of repeated surface structure, but not underlying type structure, such usage of ellipsis has a humorous element. Gilbert Ryle gives the example "She came home in a flood of tears and a sedan-chair" (Ryle 1949, p. 22). We see the beginnings of a witticism here, such as in the description of American GIs serving in Britain in the second world war as "overpaid, oversexed and over here." The grammar is suggesting a common typing through the parsing, which the meaning of the words resists.[7]

Rundle points out that *and* is also used to indicate temporal linkage, so that the following two propositions are not equivalent.

- He used to lie in the sun and play cards.
- He used to lie in the sun and he used to play cards.

It seems here in the first sentence that we mean something like:

- He used to lie in the sun, and at that time play cards.

The cardplaying corresponds to a reasonable concentration of subintervals of sunbathing intervals.

Indeed, it is generally the case that when we conjoin two or more distinct propositions, there is some kind of thematic linkage. Rundle writes

[7]Ryle says of other similar typing mistakes, "A man would be thought to be making a poor joke who said that three things are now rising, namely the tide, hopes and the average age of death. It would be just as good or bad a joke to say that there exist prime numbers and Wednesdays and public opinions and navies; or that there exist both minds and bodies." (Ryle 1949, p. 23)

> With conjunctions generally some relation will be implied, and with ‘and’ it is surely that the word is used to signal, rather than to effect, the relevant linkage or association, for all it could provide in the latter capacity is some form of phonetic bridging. (1983, p. 391)

This makes sense of the observation that the former of the following two sentences is clearly the more natural:

- Jack fell down and broke his crown, and Jill came tumbling after.
- Jack fell down and is twenty years old, and Jill is holidaying in Thailand.

There are thematic connections right through the first of these sentences – Jack cracks his head because of his fall, and then Jill also falls down after Jack. Rundle claims of this sentence that “the first ‘and’ gives us a compound predicate based on the possibility of thinking in terms of a single, if complex, event” (1983, p. 389). We will take up this point in a later section. As for the second ‘and’, he speaks of ‘things that Jack and Jill then did’ or more generally of ‘episodes which occurred at that time’. This would be a case, as above, of two terms belonging to the same type.

Let me offer a slightly different diagnosis. Had it scanned satisfactorily, we might have seen ‘Jill came tumbling after *him*’ appear in the nursery rhyme to emphasise the linkage. The use of such a pronoun is what grammarians call *anaphora*. When Aarne Ranta (1994) treated such anaphoric reference type-theoretically, he deployed Martin-Löf’s *dependent sum* construction, which I will now introduce.

It is common to have a situation where for any element of a given type, it is possible to form a type which depends upon that element. In the case of days of the month, each time we consider a month, $m : Month$, we judge that the days of that month, $Days(m)$, form a type. Formulating things this way makes sense for the computer scientist who will want to specify that different types $Day(m)$ as m varies have different sizes. Indeed, give the behaviour of leap years, it may prove useful to produce $Month(y)$ depending on $y : Year$. In the former simpler case, elements of the dependent sum are *pairs*[8] of month and day in that month, such as (February, 24) and (August, 5). We may like to imagine the days of a month lined up on top of a marker for their respective month. We write $\sum_{m:Month} Day(m)$ for the type of dates. But this construction works just as readily for dependent types which are propositions, such as for any month m, whether it is the case that this is a summer month, $summer(m)$. Now the pairs are constituted by a month and a warrant that that month is in the summer, so (August, warrant that August is a summer month).

[8]This is why some authors refer to the construction as ‘dependent *pair*.’

In the case of a series of propositions where a later one may depend on a component of an element of an earlier one, the same dependent type structure is in play. For example, *It's raining now, and doing so heavily* can be represented as involving the following judgements:

- $\vdash a : It\ is\ raining$
- $x : It\ is\ raining \vdash Heavily(x) : Prop$
- $\vdash Heavily(a) : Prop$
- $\vdash b : Heavily(a)$
- $\vdash (a, b) : \sum_{x:It\ is\ raining}(Heavily(x))$

where a and b are suitable warrants for their respective propositions. The final judgement expresses the truth of *It's raining now, and doing so heavily.* The statment might arise as a response to being posed linked questions of the form: P? And if so, Q? So here, *Is it raining? If so, is it raining heavily?* The second question only makes sense on condition of a positive response to the first. Without the element a of $It\ is\ raining$, there is no proposition $Heavily(a)$ of which to discover its truth. So, similarly 'Jill came tumbling after (him)' only makes sense if we already have that Jack has fallen.

This phenomenon also arises when we form a proposition by 'truncating' a type. Think of a questionnaire:

- Qu. 1: Do you have children? Yes □ No □.

  If you answered 'No' to Qu.1, go to Qu. 3.
- Qu. 2: Are any of your children aged 5 or under? Yes □ No □.
- Qu. 3...

There are only three legitimate ways to answer these two questions, not four as is the case with two independent questions. The type of my children has been truncated to the proposition as to whether I have children. A property which may be assessed for each child, 'being aged 5 or under', now needs to be modified to a proposition relating to the single element of 'I have children'. Here it is altered to some child having this age, but another question might have asked whether *all* are similarly youthful.

Now if we take the special case where types have no genuine dependence, then forming dependent sum results in a product, $\sum_{x:A} B \simeq A \times B$. A product type such as 'day and hour' is such a dependent sum, since each day has 24 hours invariantly through the week and so $\sum_{d:day} Hour(d)$ collects pairs such as (Monday, 15:00) and (Saturday, 09:00). In the case of propositions, this non-dependent dependent sum amounts to a conjunction, its element a proof of the first conjunct paired with a proof of the second. In a sense then, conjunction is a degenerate form of 'and', and indeed it is typically inappropriate to use the word 'and' in such cases. On the other hand with proper dependence, there is often a progressive flavour to the assertions.

- Pam took the key out of her bag and opened the door.

Does this imply that Pam used the key to unlock the door? You would certainly be surprised if, on being shown footage of the event, you saw that she took out the key, made no attempt to use it, but just pushed open the door, or used another key that happened to be in her hand. On the other hand, she might give off signs after withdrawing the key that she just recalled that she had in fact left the door unlocked. But then this would be mentioned in the report. The script strongly suggests the use of the key she took out of the bag. Indeed it's reasonable to think the sentence above is elliptical for

- Pam took the key out of her bag and unlocked the door *with it*.

Now we see the dependence. We introduce a key, which is taken out of a bag and then used to enter, $k : Key, p : TakeOut(Pam, k, her\ bag), q : OpenWith(Pam, door, k)$,

$$((k, (p, q)) : \sum_{x:Key} TakeOut(Pam, x, her\ bag) \& OpenWith(Pam, door, x).$$

We see then again the reliance on anaphoric reference, even if implicit, means that it makes no sense to ask of a dependent statement whether it is true, as it does in the case of separate conjoined propositions. If Jack did not in fact fall down the hill, then Jill cannot come tumbling after him. If there is no key for Pam to take out of her bag, then she cannot open her door with it.

Now, if '*and*' demands linkages between parts, then '*but*' does so even more strongly. Here there is often a contrast between clauses or a denial of typical consequence.

- I take sugar in my tea, but sweeteners in my coffee.
- Phil tripped over, but he still won the race.

Suitable versions with '*and*' are:

- I take sugar in my tea and in my coffee.
- Phil tripped over and lost the race.

Similarly, *while*, *although*, *whereas*, *despite the fact that* are all forms of *and* with more stringent conditions:

- Jay likes beer, whereas Kay prefers wine.
- Despite the fact that Jane fell over, she still won the race.

Thus we have found that our '*and*', which appeared first to arise from the product of types, both the product of sets and the conjunction of propositions, has now taken on some extra refinement. Indeed, it seems to be used especially in cases where we have to bear in mind some aspect of what precedes the connective to tie it to some feature of what follows. Dependent sum allows us to represent precisely this.

Of course, not any thematic connection will work with '*and*'. As Rundle points out, we don't use it to join these sentences "You can't expect him to be here yet. The traffic is so heavy." (1983, p. 389). This would be to run against the direction of any dependency. It is working backwards to give a reason for what ought to be expected. Viewing an explanation as the provision of premises from which the explanandum is derived, then we see there is a way to employ '*and*' here, namely, by saying "The traffic is so heavy, and so you can't expect him to be here yet."

Similarly, a challenging '*and*' is a request to hear what follows next.

- You haven't tidied your room yet. *And?* Well, I won't give you your pocket money then.

With these constructions we appear to have shifted away from the succession of pieces of information to the succession of some process of inference. One way to construe this is as an explicit proposition A and an implicit hypothetical $A \to B$ driving inference to B. When someone delivers you a loaded proposition, you can be sure there are consequences in the air.

But we might take there to be a dependency present again,

- You haven't tidied your room and so I won't give you your pocket money as a punishment for not tidying your room.

Similarly, '*and*' appears explicitly in an implication in Winston Churchill's plea to the United States for arms in World War II:

- Give us the tools, and we will finish the job.

We might rephrase this:

- There is a job. If you give us the tools for this job, we will finish the job with these tools.

Recall that the reason I gave for starting this section with '*and*' was that it was thought to provide the logician the means to showcase their semantics for connectives. The reason for hesitancy about '*implies*' was the issue of the relevance of antecedent to consequent. Now that we have revealed that '*and*' is heavily mired in relevance considerations, yet emerges unscathed after the adoption of the dependent sum construction, perhaps we will find a satisfactory way to treat '*implies*' type-theoretically.

Earlier we saw that a proof of 'A implies B' behaves like a function mapping proofs of A to proofs of B. Let's see what this says about the standard classical interpretation of the truth of $A \to B$ as A being false, B being true, or both. So if A is false, or in other words, if there are no proofs of A, then there is a function defined on this empty set whatever the target. If this comes as news to the reader, it is the easiest function to define since nothing needs to be specified.[9] On the other hand, if B is true and we have a proof, p, of it, then we can define a function on proofs of A as sending any of them to p. This is a constant function which ignores any information about A.

We see now where the Logic 101 student is surprised. 'If London is the capital of France, then P' turns out true whatever P, because of this strange map from the empty set, and 'If Q, then Paris is the capital of France' turns out true whatever Q because we already know that Paris is the capital of France. The student senses that tricks are being played on them. What of genuine connections between the propositions?

Well no stronger connection can be found than identity, so it is somewhat less surprising to be told of the truth of $A \to A$ whatever A. It may not be a particularly interesting result, but at least we have to rely on any proof we might have of A to send it to itself via the identity map. A may be turn out to be false, but every eventuality is covered by that presence of that identity map. But 'if P, then P' is still a degenerate form of the use of this connective, even if it can play the kind of rhetoric function of 'If this is (indeed) what we must do, then this is what we must do'. Let us then consider some more substantial examples of the consequent relying on the antecedent:

- If you see something you like, then you should buy it.
- If we miss the last train, we will have to stay here for the night.
- If he says something to Jane and upsets her, I will be very cross with him.

Take the first of these. The type of things you like is a dependent sum, $\sum_{x:Thing} like(you, x)$, pairs of things and evidence that you like them. Then

[9] For the category theorist, this is similar to the universal property of the empty set in the category of sets being the *initial object*, as such there being a unique map to any object in the category.

we are being told that there's some function, q, which takes any instance of this type, say, (a, r) with a a thing, and r a warrant that you like a, to a warrant, $q(a, r)$, for 'You should buy a'. As we will see, such a passage is provided by an element of a dependent product, denoted

$$q : \prod_{z:\sum_{x:Thing} like(you,x)} You\ should\ buy(p(z)),$$

where p picks out a, the first component of (a, r). It is possible that such a q for '*you*' has been derived from a function defined on a collection of people, justifying their purchase of anything they like.

Just as dependent sum generalises away from the cases of conjunction and product, so dependent product generalises away from implication between non-dependent propositions and functions between sets, to maps between types and types dependent upon them. If the standard logical '*and*' was a degenerate form of dependent sum, the standard logical '*implies*' is a degenerate form of dependent product. The connective '*or*' does not generally arise in a similar way. We don't say 'Jack fell down or Jill came tumbling after' since the necessary dependency is not present, although we do use '*or*' in the form of an implication from the negation of a proposition. This is generally dependent as in 'This fine should be paid by midnight or it will be doubled.'

We need now to take a closer look at these dependent constructions, but before doing so let me draw a lesson from this section. In his *Stanford Encyclopedia of Philosophy* article on *Moral Particularism*, Jonathan Dancy (2017) uses the example of '*and*' to provide an instructive analogy of how moral reasoners cope with variation in the "practical purport" of a concept such as *cruelty*:

> In knowing the semantic purport (= the meaning) of 'and', one is in command of a range of contributions that 'and' can make to sentences in which it occurs. There need be no 'core meaning' to 'and'; it would be wrong to suggest that 'and' basically signifies conjunction. If you only know about conjunction, you are not a competent user of 'and' in English, for there are lots of uses that have little or nothing to do with conjunction. For example: two and two make four; 'And what do you think you are doing? (said on discovering a child playing downstairs in the middle of the night); John and Mary lifted the boulder; the smoke rose higher and higher. Those competent with 'and' are not unsettled by instances such as these, but nor are they trying to understand them in terms of similarity to a supposed conjunctive paradigm or core case.

Dancy points to a "manageable complexity" in this case, and claims this to be so in moral cases too. From what we have seen we can take the manageability of the variation in 'and' to be explained by the limited range of ways it can be taken up by our type theory. For instance, 'The smoke rose higher and higher' would seem to employ 'and' with its sense of dependency so that it might be written less briefly as: *The smoke rose higher than it had been, and then higher again than that.*

II.4 Dependent types

Dependent types are a very important part of Martin-Löf type theory. As we saw, these are denoted $x : A \vdash B(x) : Type$, where the type $B(x)$ *depends* on an element of A. In mathematics we may have a collection of types indexed by the natural numbers, such as the type of $n \times n$ matrices over the real numbers, $n : \mathbb{N} \vdash Mat(n, \mathbb{R}) : Type$. But let us work with an example from ordinary language as in $Players(t)$ for $t : Team$. So we express this as

$$t : Team \vdash Player(t) : Type.$$

These dependent types are sets, in a sense to be defined later, but we can have examples where they are propositions, such as

$$t : Team \vdash Plays\ in\ UK(t).$$

Quantification then takes place in these dependent type situations, where we find that domains of variation are the indexing types. This relies on the type formation rules for *dependent sum* and *dependent product*. For the dependent type $B(x)$ depending on $x : A$,

- The dependent sum (sometimes known as dependent *pair*), $\sum_{x:A} B(x)$, is the collection of pairs (a, b) with $a : A$ and $b : B(a)$. When A is a set and $B(x)$ is a constant set B, this construction amounts to the product of the sets. Likewise if A is a proposition and $B(x)$ is an independent proposition, B, dependent sum is the conjunction of A and B. In general we can think of this dependent sum as sitting 'fibred' above the base type A, as one might imagine the collection of league players lined up in fibres above their team name.

- The dependent product (sometimes known as dependent *function*, $\prod_{x:A} B(x)$, is the collection of functions, f, such that $f(a) : B(a)$. Such a function is picking out one element from each fibre. When A is a set and $B(x)$ is a constant set B, this construction amounts to B^A, the set of functions from A to B. Likewise if A is a proposition and $B(x)$ is an independent proposition, B, dependent product is the implication $A \to B$. In terms of the picture of players in fibres over their teams, an element of the dependent product is a choice of a player from each team, such as $Captain(t)$.

Dependent sum	Dependent product
$\sum_{x:A} B(x)$ is the collection of pairs (a, b) with $a : A$ and $b : B(a)$	$\prod_{x:A} B(x)$, is the collection of functions, f, such that $f(a) : B(a)$
When A is a set and $B(x)$ is a constant set B: The product of the sets.	When A is a set and $B(x)$ is a constant set B: The set of functions from A to B.
When A is a proposition and $B(x)$ is a independent proposition, B: The conjunction of A and B.	When A is a proposition and $B(x)$ is an independent proposition, B: The implication $A \to B$.

Here then we can see how close this foundational language is to mainstream mathematics and physics. Dependent sum and dependent product correspond respectively to the total space and to the space of sections of fibre bundles, which appear in gauge field theory often in the guise of principal bundles. A fibre bundle is a form of product, but a potentially *twisted* one where as we pass around the base space, a fibre may be identified under a nontrivial equivalence. An easy example is the Moebius strip, where an interval is given a 180 degree twist as it is transported around a circle. Of course, for physics one needs some geometric structure on the base space and total space. We shall see more about this in Chap. V.

Quantification is associated to the second example in each case where the dependent type is a proposition. The dependent sum being inhabited amounts to the existence of a team that plays in the UK, and the dependent product being inhabited amounts to all teams playing in the UK. We can now see that these constructions allow us to formulate the quantifiers from first-order logic, at least as defined over types. So consider the case where A is a set, and $B(a)$ is a proposition for each a in A. Perhaps A is the set of animals, and $B(a)$ states that a particular animal, a, is bilateral. Then an element of the dependent sum is an element a of A and a proof of $B(a)$, so something witnessing that there is a bilateral animal. Meanwhile an element of the dependent product is a mapping from each $a : A$ to a proof of $B(a)$. There will only be such a mapping if $B(a)$ is true for each a. If this were the case, we would have a proof of the universal statement 'for all x in A, $B(x)$', in our example, 'All animals are bilateral.'

Returning to the dependent sum, this is almost expressing the usual existential quantifier 'there exists x in A such that $B(x)$', except that it's gathering all such a for which $B(a)$ holds, or, in our case, gathering all bilateral animals. As we have seen before in the capped addition of the Boolean truth values in a disjunction, to treat this dependent sum as a proposition, there needs to be a 'truncation' from set to proposition, so that we ask merely whether this set is inhabited, in our case 'Does there exist a bilateral animal?' This extra step should be expected as existential quantification resembles forming a long disjunction. That we don't need to adapt for universal quantification tallies with the straightforward form of the product of Boolean values.

What emerges from this line of thought is that the lower levels of homotopy type theory have contained within them: propositional logic, (typed) predicate logic and a structural set theory. Coming from a tradition of *constructive* type theory, one needs to add classical axioms if these are required, such as various forms of excluded middle or axiom of choice.

The difference with an untyped setting is very apparent when we look to express something with multiple quantifiers, such as 'Everyone sometimes finds themselves somewhere they don't want to be'. In type theory there will be dependency here separately on types of people, times and places, and not variation over some universal domain, requiring conditions that specify that some entities in the domain be people, times or places.

We saw a simplified version of Göran Sundholm's resolution of the puzzle of the Donkey sentence in the previous chapter. A treatment of the full version should be illuminating now since it combines the constructions we have just seen (Sundholm 1986):

- If a farmer owns a donkey, then he beats it.

Recall that the problem here is that we expect there to be a compositional account of the meaning of this sentence, in particular, one where the final 'it' appears in the representation. It appears that an existential quantifier would be involved in a representation in first-order logic because of the indefinite article, and yet a simple-minded attempt to use one is ill-formed, the final y being unbound:

$$\forall x(Farmer(x) \& \exists y(Donkey(y) \& Owns(x, y)) \to Beats(x, y)).$$

It is standard then in first-order logic to rephrase the sentence as something like: 'All farmers beat any donkey that they own', and then to render it formally as

$$\forall x(Farmer(x) \to \forall y(Donkey(y) \& Owns(x, y) \to Beats(x, y))).$$

But now we have radically transformed the original sentence, and the 'it' does not seem apparent.

Sundholm's solution uses the resources of dependent type theory:

- $\vdash Farmer : Type$
- $\vdash Donkey : Type$
- $x : Farmer, y : Donkey \vdash own(x, y), beat(x, y) : Prop$

- $x : Farmer \vdash Donkey\ owned\ by\ x : Type$
- $\vdash (\sum_{x:Farmer} Donkey\ owned\ by\ x) : Type$

Then the statement is true if we have a function b such that

- $z : (\sum_{x:Farmer} Donkey\ owned\ by\ x) \vdash b(z) : Beats(p(z), p(q(z)))$

Elements of the dependent sum of donkeys owned by farmers are pairs formed of a farmer and then a pair formed of a donkey and a warrant that the donkey is owned by that farmer. From such an element, z, we project to the first component of the pair, $p(z)$, to extract the farmer and then project to the first component of the second component, $p(q(z))$, for the donkey, so at to be able to express the beating of one by the other. It is this last term that is being referred to in natural language as 'it'.

The b above provides a proof of the relevant beating for any such z and as such is then an element of the following dependent product:

$$b : \prod(z : (\sum(x : Farmer) \sum(y : Donkey) Owns(x, y))) Beats(p(z), p(q(z))).$$

Now sometimes the referents of these anaphoric pronouns are not so obvious. Consider the song '*If you're happy and you know it, clap your hands*'. What does the 'it' refer to in the antecedent?

- You're happy and you know it.

Following Vendler, who claimed that we *believe* propositions but *know* facts, we might say this is

- You're happy and you know the fact that you're happy.

So how is 'X knows P' as a type formed? Well, for P to be known it had better be true, so we can make a precondition that the type which is P is inhabited:

$$X : Person, P : Prop, x : P \vdash know(X, P, x) : Type.$$

We might take this in turn to be a proposition if we consider the singular fact of X knowing a true P, without distinguishing variety of warrants for ascribing this knowledge to X. Typically in natural language we merely write 'X knows P' without indicating P's element. But with the implicit dependence on x, we can say that the formation of this proposition *presupposes* the truth of P, in a sense that will be explained in the next section. Our epistemology will dictate here a complex set of requirements for judging a knowledge assertion, likely dependent on the kind of proposition P at stake.

According to this way of framing matters, knowing that one knows becomes quite involved, and the KK principle (that knowing implies knowing that one knows) won't hold. To form 'X knows that X knows P' we need not only that P be true, but also that it's true that X knows P:

$$P : Prop, x : P, y : Know(X, P, x) \vdash Know(X, Know(X, P, x), y) : Prop.$$

To establish this latter proposition as true we will need further to demonstrate that this latter type is inhabited, and this will now depend on the kind of proposition at stake and the kind of warrant for its being known.

Once we have the judgements:

- $\vdash you : Person$,
- $\vdash you're\ happy : Prop$,
- $\vdash h : you're\ happy$
- then, $\vdash know(you, you're\ happy, h) : Prop$
- and we may have $\vdash k : know(you, you're\ happy, h)$,
- and so $\vdash (h, k) : \sum_{x:You're\ happy} know(you, you're\ happy, x)$

The 'and' of '*You're happy and you know it*' has again been represented by a dependent sum. The 'it' is the pair $(you're\ happy, h)$. In accordance with Vendler we might contrast

$$X : Person, P : Proposition, x : P \vdash know(X, P, x) : Type$$

with

$$X : Person, P : Proposition \vdash believe(X, P) : Type$$

So we don't sing 'If you're happy and you believe it, clap your hands.' Of course, 'I believe it (her claim)' is fine, and the claim may in fact be false.

A very similar and much more extensive treatment of this believe/know distinction in terms of dependent type theory is given in Tanaka et al. (2017). The authors note in an earlier version of the paper (Tanaka et al. 2015) that the distinction is lexically marked in Japanese. For example, in the case of knowledge

- John-wa Mary-ga kita *koto*-o sitteiru.
- John-TOP Mary-NOM came COMP-ACC know.
- 'John knows (the fact) that Mary came.'

whereas for belief

- John-wa Mary-ga kita *to* sinziteiru.

- John-TOP Mary-NOM came COMP believe.
- 'John believes that Mary came.'

They continue

> In general, *koto*-clauses trigger factive presupposition, while *to*-clauses do not. This contrast can be captured by assuming that a factive verb like *sitteiru* takes as its object a proof (evidence) of the proposition expressed by a *koto*-clause, while a non-factive verb selects a proposition denoted by a *to*-clause. (Tanaka et al. 2015, p. 6)

Before ending this section, in the spirit of *computational trinitarianism*, I shall first give the type theoretic rules for dependent sum and product, and then give a category theoretic account of their universal properties in terms of adjoints. First then, dependent sum:

$$\frac{\vdash X : Type \quad x : X \vdash A(x) : Type}{\vdash (\sum_{x:X} A(x)) : Type}$$

$$\frac{x : X \vdash a(x) : A(x)}{\vdash (x, a) : \sum_{x':X} A(x')}$$

$$\frac{t : \sum_{x:X} A(x)}{p_1(t) : X \quad p_2(t) : A(p_1(t))}$$

$$p_1(x, a) = x \quad p_2(x, a) = a$$

Then the rules for dependent product

$$\frac{\vdash X : Type \quad x : X \vdash A(x) : Type}{\vdash (\prod_{x:X} A(x)) : Type}$$

$$\frac{x : X \vdash a(x) : A(x)}{\vdash (x \mapsto a(x)) : \prod_{x':X} A(x')}$$

$$\frac{f : \prod_{x:X} A(x) \quad x : X}{x : X \vdash f(x) : A(x)}$$

$$(y \mapsto a(y))(x) = a(x)$$

Recall from above that a logical inferentialist such as Boghossian understands the meaning of the conjunction '*and*' to be given by its introduction and elimination rules. With our more adequate treatment of '*and*' as dependent sum for propositions, we can maintain a similar stance, but now with these new versions of the rules.

Let's now look at the category theoretic understanding, presented as usual in terms of *adjoints*. It should not be surprising to find that dependent sum and product can be formulated in terms of adjunctions in view of a similar treatment for conjunction and implication shown in section 2. As I mentioned there, adjunctions are a vital part of category theory, and provide a way of dealing with something as close to an inverse as possible. When such an inverse does not exist, left and right adjoints are approximations from two sides.

In our type theory, given a plain type, A, we can turn any type C into one trivially dependent on A by formulating $x : A \vdash (A \times C)(x) :\equiv C : Type$. If A and C are sets, think of lining up a copy of C over every element of A, the product of the two sets projecting down to the first of them, $A \times C \to A$. Thus we have made C depend on A, but in a degenerate sense where there is no real dependency. Now, we can think of approximating an inverse to this process, which would need to send A-dependent types to plain types. Such approximations, or *adjoints*, do indeed exist. Left adjoint to this mapping is dependent sum, and right adjoint is dependent product. The fact that these are adjoints may be rendered as follows, for B a type depending on A:

- $Hom_{\mathcal{C}}(\sum_{x:A} B, C) \cong Hom_{\mathcal{C}/A}(B, A \times C)$
- $Hom_{\mathcal{C}}(C, \prod_{x:A} B) \cong Hom_{\mathcal{C}/A}(A \times C, B)$

$\mathcal{C}/A$ indicates the slice of A. Objects in this slice are objects of $\mathcal{C}$ equipped with a mapping to A. A morphism between two such objects is a commutive triangle. These correspond to the A-dependent tyes. Think of a couple of types dependent on $t : Team$, the type of teams, say $Player(t)$ and $Supporter(t)$. Then an example of a map from $Supporter$ to $Player$ in the slice over $Team$ is *favourite player*, assuming that each supporter's favourite player belongs to the team that they support.

This category-theoretic formulation of dependent sum and product will reappear in Chap. IV when a type of possible worlds comes to play the role of A. Let me end here by noting that, in light of the adjunctions above, we can see further developments of the logic-arithmetic analogy from earlier in this chapter. Take A, B and C as finite sets, with B fibred over A. This just amounts to a map from B to A, the elements of B fibred above their images in A. Let a be the cardinality of A, c be the cardinality of C, and b_i be the cardinality of the subset of B which is the fibre over i in A. Then taking the cardinalities of the two sides of each adjunction above yields further recognisable arithmetic truths:

- $c^{\sum_i b_i} = \prod_i c^{b_i}$
- $(\prod_i b_i)^c = \prod_i (b_i)^c$

So a pupil being taught, say, that $3^4 \times 3^5 = 3^9$ or that $3^5 \times 7^5 = 21^5$ is being exposed to the shadows of instances of important adjunctions, which in turn, as with all of the discussion above of dependent sums and products, works for types up and down the hierarchy of n-types. These n-types in the guise of higher groupoids we turn to in the next chapter. Let us now consider the nature of the dependence relation between types.

II.5 Context and dependency structure

Consider what might be the beginning of a story, or a play:

> A man walks into a bar. He's whistling a tune. A woman sits at a table in the bar. She's nursing a drink. On hearing the tune, she jumps up, knocking over the drink. She hurls the glass at him. "Is that any way to greet your husband?", he says.

For Ranta (1994), this kind of narrative should be treated as the extension of one long *context*, with its dependency structure, which begins as follows:

> $x_1 : Man, x_2 : Bar, x_3 : WalksInto(x_1, x_2), x_4 : Tune, x_5 : Whistle(x_1, x_4), x_6 : Woman, x_7 : Table, x_8 : Locate(x_7, x_2), x_9 : SitsAt(x_6, x_7), x_{10} : Drink, x_{11} : Nurse(x_6, x_{10}), x_{12} : Hear(x_6, x_5), ...$

Note how one could easily populate this sketch with plenty of '*and*'s, especially where there is dependency, suggesting that there are dependent sums about.

In general, a context in type theory takes the form

$$\Gamma \;=\; x_0 : A_0,\; x_1 : A_1(x_0),\; x_2 : A_2(x_0, x_1),\; \ldots x_n : A_n(x_0, \ldots, x_{n-1}),$$

where the A_i are types which may be legitimately formed. As we add an item to a context, there may be dependence on any of the previous variables. A context need not take full advantage of this array of dependencies. For instance, in the case above, $Whistle$ only depends upon x_1 and x_4 and not upon x_2 or x_3. But it certainly cannot depend on a variable ahead of it.

The idea of a context can help us make sense of R. G. Collingwood's theory of *presuppositions*, described in his *An Essay on Metaphysics* (1940). There he argues against the idea that there can be freestanding propositions. Rather any statement is made as an answer to a question. That question in turn relies on some further presuppositions. These can be traced back to further questions, until one reaches the bedrock of *absolute* presuppositions. These presuppositions are not truth-apt, that is, are not to be assessed for their truth value.

> Absolute presuppositions are not verifiable. This does not mean that we should like to verify them but are not able to; it means that the idea of verification is an idea which does not apply to

> them, because, as I have already said, to speak of verifying a presupposition involves supposing that it is a relative presupposition. If anybody says 'Then they can't be of much use in science', the answer is that their use in science is their logical efficacy, and that the logical efficacy of a supposition does not depend on its being verifiable, because it does not depend on its being true: it depends only its being supposed. (1940, p. 32)

It is possible, however, for the *constellation* of absolute presuppositions of a science to change. Their *logical efficacy* may be found wanting relative to a different constellation.

Collingwood (1940, p. 38) illustrates the role of presuppositions in an everyday situation by considering the 'complex question' *Have you left off beating your wife?* Notoriously, to answer '*yes*' or '*no*' to this question exposes you to censure, and yet what are you to do if either you have no wife, or you have one but have never beaten her? Collingwood's response is to say that questions only arise in certain circumstances, and to find out these circumstances it is necessary to work out a series of questions to which a positive answer allows the next question. He arrives at the following series:

1. Have you a wife?
2. Were you ever in the habit of beating her?
3. Do you intend to manage in the future without doing so?
4. Have you begun carrying out that intention?

Similar to the opening narrative of this section, we can tell a simple story here.

> A man and a woman met and got married. At some stage in their marriage, he began to beat her. In time he came to see this had to stop, and so he formed the intention to desist. He acted on this decision and to this day has abstained.

As I mentioned, Collingwood claimed that his method, which he called the *logic of question and answer*,[10] would allow us to reveal the *absolute presuppositions* in operation in some walk of life. The work of the metaphysician is to perform just this kind of revelation for bodies of organised thought, '*sciences*' as he calls them, at different epochs. This is to allow us to see how our deepest foundational concepts, such as *cause*, have been transformed through time. Even in the everyday case we just considered, the analysis might clearly probe deeper. Imagine the child pestering its parents with a series of questions.

[10] See also Collingwood (1939, Chap. V).

- *What is a wife?* A woman to whom one is married. *What is marriage?* When two people agree to a legally binding union. *What is it for two people to agree*?...

Type theoretically, all of the uncovered presuppositions need to appear in the context right up to the formation of the type $Person$.

Type theory with its resources of context and type formation may allow us better to keep track of our background theoretical assumptions. For instance, as I encounter different jurisdictions, I must keep a tally of the regulations concerning marriage. The rules of marriage are ever evolving, often in contested ways, both in terms of the conditions that need to be in place for it to be recognised that two people be married, and in terms of what follows, morally, legally, financially, etc., from the marriage. We have seen substantial changes over the past few decades, and should only expect more to follow, such as perhaps to the presumption that only two people may be married, as put in question by the polyamory community.

Another philosopher arguing along similar lines to Collingwood in the context of the natural sciences is Michael Friedman. Friedman (2001) contrasts his own views with those of Quine who famously proposed that we operate with a connected web of beliefs (Quine and Ullian 1970). For the latter, when observations are made which run against expectations derived from these beliefs, we typically modify peripheral ones, allowing us to make minimal modification to the web. However, we may become inclined to make more radical changes to *entrenched* beliefs located at the heart of the web, perhaps even to the laws of arithmetic or the logic we employ. Friedman insists, by contrast, that there is a hierarchical structure to our (scientific) beliefs which entails that, without the availability of fundamental modes of expression, including the resources of mathematical languages, and constitutive principles which deploy these resources, statements concerning empiricial observations and predictions cannot even be meaningfully formed. In his (2002), while discussing the formulation of Newtonian physics, he says

> It follows that without the Newtonian laws of motion Newton's theory of gravitation would not even make empirical sense, let alone give a correct account of the empirical phenomena: in the absence of these laws we would simply have no idea what the relevant frame of reference might be in relation to which the universal accelerations due to gravity are defined. Once again, Newton's mechanics and gravitational physics are not happily viewed as symmetrically functioning elements of a larger conjunction: the former is rather a necessary part of the language or conceptual framework within which alone the latter makes empirical sense. (Friedman 2002, pp. 178-179)

To see how similar this case is to that of the beaten spouse, we might imitate Friedman thus,

> It follows that without the concepts of personhood, of marriage, of action, of intention, etc., the idea of someone leaving off beating their wife would not make any sense, let alone give a correct account of the empirical phenomena: in the absence of these concepts we would simply have no idea what the relevant moral-legal-ontological frame of reference might be in relation to which the claimed cessation of violence has taken place. Once again, facts about personhood, intention and action are not happily viewed as symmetrically functioning elements of a larger conjunction: the former is rather a necessary part of the language or conceptual framework within which alone the latter makes empirical sense.

What neither Collingwood nor Friedman went on to do, however, is to give an appropriate formal treatment of this dependency. Let us see how we fare with dependent type theory.

With the idea of a context available, type theory should provide us with a way to distinguish between *consequences* and *presuppositions*. Whereas 'James has stopped smoking' *presupposes* that at some point he was in the habit of smoking, from the proposition 'n is the sum of two odd numbers' we may *conclude* that n is even. Presuppositions are what need to be in place for a type to be formed or a term introduced. Consequences result from application of rules to existing judgements. For example,

- Presupposition: from $A \times B : Type$, we must have that $A, B : Type$.
- Consequence: from $c : A \times B$, we can conclude that $p(c) : A$ and $q(c) : B$.

Despite this neat distinction we are inclined to see an inference occurring in the case of presupposition just as much as in the case of consequence. It is easy to think that we infer from 'James has stopped smoking' that he was in the habit of smoking. What seems to be occurring is that we are presented with a proposition P, such as 'James has stopped smoking'. Then to interpret it, it is our job to provide the minimal context, Γ, which can support the formation of P:

$$\Gamma \vdash P : Prop.$$

Then say $\Gamma = x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n$, from which we extract one clause of the context

$$x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n \vdash x_j : A_j, for\ 1 \leq j \leq n$$

We have trivially

$$x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n, y : P \vdash x_j : A_j,$$

at which point we forget the context and think we have an entailment

$$y : P \vdash x_j : A_j.$$

This is to be contrasted with the case when we have the means in a context to derive a new proposition, such as with

$$x : A, f : [A, B] \vdash f(x) : B.$$

Ranta in *Constructive type theory* (2015, p. 358) proposes along similar lines that we distinguish these operations as follows:

- B presupposes A means $x : A \vdash B(x) : Type$
- A semantically entails B means $x : A \vdash b(x) : B$

The centuries-old disagreement as to the status of *Cogito ergo sum* would appear to be relevant here. Where Carnap in *The Elimination of Metaphysics* (1932) takes Descartes to task for a faulty piece of deductive reasoning, one might rather say that a presupposition is being revealed – *I think* presupposes *I am*. Heidegger observes

> Descartes himself emphasizes that no inference is present. The *sum* is not a consequence of thinking, but vice versa; it is the ground of thinking, the fundamentum. (Heidegger 1967, pp. 278-279)

Perhaps the 'I', rather then refer to an element of a type, plays more of a performative or expressive role in our assertions,[11] as represented in the judgement symbol, ($\vdash$), but a third-person proposition such as *Jill thinks* certainly presupposes that there be a *Jill* who is one of a kind that can think.

Certainly worth exploring here is whether forms of inferentialism could be well-represented type-theoretically as above. In the case of Brandom's version of inferentialism with his idea of *commitments* and *entitlements*, see, e.g., *Making it explicit* (Brandom 1994), we see a connection not only with the pairing of introduction and elimination rules, but also with the distinction between constructing implicit context and calculating consequents. There, we are told, to understand the meaning of a proposition (something approximating what we are calling a judgement in the sense of a thing to be

[11] Cf. 'The "emptiest of all representations," the "'I think' that can accompany all representations" expresses the formal dimension of responsibility *for* judgments.' (Brandom 2000, p. 160)

judged, rather than the act of judging) we should understand what conditions entitle us to assert it, as well as what follows from asserting it. Perhaps in most forms of not fully-spelled out inference, what we are 'making explicit' is the context.

With such a rich notion of context we may also be able to provide support for a philosopher of language such as Jason Stanley in *Language in Context* (2007), who believes that contextualists have overstated the case, and that reliance on the context of utterance is limited to indexical aspects. Clearly when it comes to indexicals, we do rely on *situational* context. Where I point is evidently vital to establishing the truth of the claim 'This is a kangaroo'. Stanley looks to limit the role of this situational context to the indexical aspects of an assertion. The remainder is borne by what we might call the *conceptual* context of the utterance. Where he sees his task as "attributing hidden complexity to sentences of natural language, complexity which is ultimately revealed by empirical enquiry" (2007, p. 31), we may describe this as revealing all aspects of the type-theoretic context.

A further consideration will be to analyse ill-formed constructions. We will look at the case of ill-formed definite description in the following chapter, but let us consider briefly here the attempt to generate the proposition which supposedly leads to the liar paradox

- *This proposition is false.*

Now a reasonable rule for forming terms of the kind 'this A' for some established type A is that we have just presented a term of that type, say, $a : A$. Then we may define

$$\vdash \textit{this } A :\equiv a : A.$$

But in the case of the liar sentence, we haven't yet produced a proposition, an element of the type *Prop*, in order to be able to form 'this proposition'. In other words, the liar statement presupposes that a proposition has just been presented, but we don't yet have one. Without it we cannot introduce the term 'this proposition'. Once again we find an answer which follows the Wittgensteinian line, but we can present it formally.

Let us note one final feature in Collingwood's wife-beating example – the choice of the perfect tense. This tense requires that at the time of asking the question an affirmative answer indicates that the intention to stop is still being adhered to. The English perfect always has this connotation of present relevance. Mention of complex temporally structured events brings us back to an outstanding topic which we can address now we have the tools provided by dependent types.

II.6 Events as basic types

Martin-Löf type theory was designed to be a language for expressing mathematical reasoning. Very few basic types need to be postulated, since most can be built up from the type formation rules. In a type theory which allows for *inductive types* and also *higher* inductive types, described in the next chapter, we have a type of natural numbers, and can then build up rationals and real numbers in something like the usual way. We can also form quotients of existing types and types with more complicated identity structures, such as n-spheres for all n and algebraic entities.

However, if we hope to develop type theory to capture ordinary language and everyday inference, we will have to assert that there are some types composed of elements from our world.[12] We considered in the first section of this chapter the prospect of employing a type of entities or individuals, and then extracting out of it subtypes of entities of different kinds. This follows the current standard practice of representing quantifiers over restricted domains by including conditions, for instance, $\forall x(A(x) \to B(x))$ for 'All A are B'. I provided some reasons for refusing to adopt this approach and instead taking there to be a number of basic types. Now that we have available the tools to represent dependency, we can say more.

Among the active research groups working on the representation of natural language by dependent type theory we find both the position that each common noun denotes a type (Luo 2012) and the rival position that these nouns should be formed by predication on some master type $Entity$ (Tanaka et al. 2015). One reason we might look to the latter is that otherwise a simple proposition such as '*John is a man*' presents problems. Since this is a proposition, and can be negated and appear in conditionals, it needs to be represented as a type. This would seem to rule out the syntactical form $John : Man$, since in type theory this is not a proposition. We might decide instead then to use $Man(x)$ as a predicate defined for some master type of entities, so that 'John is a man' is rendered as $(John, r) : \sum_{x:Entity} Man(x)$, where $r : Man(John)$. Then 'John is not a man' is the proposition $\neg Man(John)$.

Now Luo and colleagues have formulated solutions to this challenge within the 'each common noun denotes a type' paradigm.[13] But even were we to find these solutions unconvincing and consider Tanaka to be closer to the mark, there is good reason to resist ascent to too high-level a master type. Let us see why by considering how our sentence – *John is a man* – could come to be asserted, denied or included within a conditional.

First off, the interlocuter of someone making such a claim about 'John' must be expected to know something about the referent of 'John' prior to

[12] These may be termed 'kinds'. "Types are on the side of mind, kinds are on the side of the world." (Harré, Aronson & Way 1994, p. 27).

[13] See Chatzikyriakidis and Luo (2017), Xue et al. (2018).

its assertion. For it to be meaningful to assert or deny the humanness of John, it has to be thinkable already that John might not be a man, so he cannot have been initially presented as such. By way of comparison, it is difficult to make sense of '*If red is not a colour, then I will wear a blue shirt*' since it seems we could not have been introduced to *red* other then as a colour. On the other hand, for a subtype of $Colour$ this is fine, as in '*If red is not a colour that you like, then I will wear a blue shirt*'. It seems then that if '*John is a man*' is to make sense as an assertion, we would need some context as to how the term *John* has been introduced. Perhaps the speaker has explained how John was working in the fields all of yesterday, that John pulled heavy loads and that John was given just a bowl of oats in the evening. We might reply '*If John is a man, then you're not treating him well*'. We have accepted *John* as a something, and are wondering about predication as a man. However, even here, instead of some master type of individuals or entities, there is good reason to opt for separate basic types.

> Note that although we do not take common nouns to be types, it is possible to refine type entity by introducing more fined-grained types such as ones representing animate/inanimate objects, physical/abstract objects, events/states, and so on. (Tanaka et al. 2015, p. 3)

The characterisations given of John strongly suggest at the very least that he is an animate being. John cannot be referring to, say, a meteorological event.

But don't we sometimes hear a term and then labour under some misapprehension as to its type? I recall once embarrassing myself by speaking of the *hermeneutic circle* as though it were a group of intellectuals such as the *Vienna Circle*. Shouldn't we say that I knew that the term had singled out an individual, typed to a very minimal extent such that greater specification could class it under 'processes to understand texts' or 'groups of intellectuals'. And as far as I knew, perhaps it referred to a geometric figure. All I knew confidently was that it purported to pick out some individual entity.

Indeed, I might mutter to myself 'If the hermeneutic circle is not a group of intellectuals, then I have made a complete fool of myself', and I could replace *group of intellectuals* with any number of types, and this sentence still makes sense. One way to think about this situation, however, is that this comment is taking place at the *meta*-level. I had assumed as a part of the common context of conversation the judgement that *hermeneutic circle* was an element of the type $Intellectual\ group$, but I was wrong to do so. My muttering in full is 'If I was wrong to assume the judgement that the hermeneutic circle is a group of intellectuals, then I have made a fool of myself.'

With this preamble over, let's see if we can say more about the type *Event*, a type with an intricate internal structure and with a reasonable claim to be basic. Those, such as Peter Hacker, dubious of the possibility of formalisation as a tool to reveal meaning, have taken events as an important case in support of their views:

> The ideal of displaying the meaning, in particular the entailments, of sentences about events as wholly or even largely a function of structure as displayed in the canonical notation of the predicate calculus is chimerical. (Hacker 1982, p. 485)

Hacker is correct to draw our attention to the kinds of inference we need to be able to represent with our formalism, and in his claim that we need to fare better than is possible with the predicate calculus. If we can show that it is possible to draw inferences from complex event statements, and if the inference is formalisable in type theory, it must be because there is an implicit type structure operating behind the scenes. One is either inferring presuppositions, deducing consequences or both. So from

- Meg has wiped all the crumbs off the table.

we can reasonably conclude amongst other things that

- Meg moved her hand in a sweeping motion.
- Meg was in contact with the table.
- There were crumbs on the table.
- There are no longer crumbs on the table.

Now, the reasons Donald Davidson gives for postulating that events make up their own ontological category include that, on the one hand, as with objects, descriptive details can be added endlessly, and on the other, they have their own distinct form of identity criterion which differs from the identity criterion for objects. Let us take these in order.

When I consider a particular book in my office, I see that it has a blue cover, was written by Jane Austen, is in contact with a work by Charles Dickens. I know that it was given to me by my wife, we read it together last year, and that it contains a story about the difficulties of finding a suitable husband in the years around 1800. Further investigation reveals it has 346 pages, including an introduction of 24 pages, and 61 chapters. So now you have several details about the book, but I have only just begun. I haven't told you yet about its production method or its publication details. I haven't mentioned the font used in the text or the design of the frontispiece, and so on.

Now *prima facie* this description does not cause a problem for a predicate calculus representation. Taking the constant a to designate the item, we have

$$Book(a)\&BlueCover(a)\&WrittenAusten(a)\&...,$$

which allows us to infer any part of the conjunction, such as

$$Book(a)\&WrittenAusten(a).$$

Davidson noted that a similar device could be used with events. Let us modify an example devised by Peter Hacker:

- P: The stone hit the car at the crossroads at midnight with a bang.

This might be rendered by a relation as

$$hit(\textit{the stone}, \textit{the car}, \textit{at the crossroads}, \textit{at midnight}, \textit{with a bang}).$$

However, the problem Davidson raised with such an elaborate description of an event is that it seems as though we need related predicates of different *arities*. Simple consequences of the proposition are: the stone hit the car; the stone hit the car at midnight; the stone hit the car at the crossroads at midnight; the car was hit at midnight. If we imagine that there is a 5-place predicate, $hit(u, v, x, y, z)$, expressing

- u hit v at location x at time y in manner z,

then either we also have to have predicates of other arities such as $hit(\textit{that stone}, \textit{the car})$, $hit(\textit{that stone}, \textit{the car}, \textit{midnight}, \textit{with a bang})$, etc., or else we allow dummy place holders so that 'the stone hit the car' is $hit(\textit{the stone}, \textit{the car}, *, *, *)$.

The former solution is enormously profligate, each predicate occurring in a great number of forms. As for the latter solution, it seems implausible that we conceive of the *hit* predicate in this way, with many of its instances having unfilled slots. In any case, as Hacker suggests, there is nothing to prevent us extending the description by adding extra details to the event, for instance, that the stone hit obliquely.

Davidson's solution was to imitate our treatment of the book above and so allow quantification over events

$$\begin{aligned} P :\equiv\ & \exists e(hitting(e)\ \&\ agent(e) = \textit{the stone}\ \&\ object(e) = \textit{the car} \\ & \&\ place(e) = \textit{the crossroads}\ \&\ time(e) = \textit{at midnight} \\ & \&\ manner(e) = \textit{with a bang}). \end{aligned}$$

This solution allows for the inference to the reduced descriptions by removing the relevant conjuncts. So we may infer from the proposition, for instance, that 'the stone hit the car at the crossroads'.

This solution might suggest that we take events along with objects as falling under the larger class of *individuals*, so for example,

- $\exists x(object(x)$ & $book(x)$ &...
- $\exists x(event(x)$ & $hitting(x)$ &...

But objects and events have such different individuation criterion that Davidson preferred the solution which distinguishes them as of fundamentally different kinds. His identity criterion for events is:

- Events are the same if they have the same causes and are caused by the same things.

Hacker raises several objections, however, to Davidson's account. First, he observes,

> If the stone hit the car then surely, as a matter of logic, the stone and the car must exist. (1982, p. 484)

Of course, he's right that these are not derivable from P, however they are *presupposed* by it. Rather like the liar sentence case treated above with its use of 'this', and as we shall see in greater detail in the next chapter, we cannot form sentence P with its use of the definite article, 'the', without having introduced an element of the type *Stone* and an element of the type *Car*. We will already have judged the existence of these entities. Even in the case of indefinite articles, "A stone hit a car...", while here no objects are presupposed, the relevant *types* are presupposed.

As a broader objection, Hacker writes

> There are no straightforward rules for translating ordinary event-recording sentences into the canonical notation in advance of displaying and analysing their logical structure, not in the forms of the predicate calculus, but in terms of the verbs (and their specific meanings), the qualifying adverbs (and their specific significance, and hence effect upon the overall meaning of the expression or expressions they qualify), the application of the nominalizing operation to different types of adverbially qualified verbs, etc. (1982, pp. 485-486)

So when B is said 'to have wisely apologised', we cannot merely characterise this as an event which is act of apology, whose agent is B and which was a wise event. Instead we must look into the specificity of the meaning of the verb 'to apologise' to bring to light what we mean by someone being wise to do this. Hacker concludes

> Failure of Davidson's programme does not reveal something mysterious, let alone awry, with our ordinary forms of discourse, but the poverty and narrowness of the predicate calculus. (1982, p. 487)

But what if instead we look to represent event statements as types, preferably ones with a rich enough structure to allow appropriate inferences to be made? Well, linguists have taken to heart the proposal made by the philosopher Zeno Vendler that the basic category of events is composed of four subcategories: *activities*, *achievements*, *accomplishments* and *states*. Without entering into a close discussion of this proposal, the idea here is that we have

1. **Goalless activities**, such as playing in a park, or repeatedly jumping;
2. **Achievements** marking moments of reaching some goal, such as arriving in Beijing, or closing the deal;
3. **Accomplishments** include the activity leading to the culmination of a goal, such as running three miles, or emptying one's plate;
4. **States** involve no activity, such as to be square, or to feel love for something.

Since Vendler, extra details have been added, such as subdivision of achievements to account for differences in implicatures depending on whether or not the achievement is realised by an agent or not. For instance, in the case of the second sentence of the following pair but not the first, we may ask whether the event was deliberate or not:

- The lava flow reached the city wall.
- Petra lifted up her arm.

Furthermore, to Vendler's list, Bach (1986) added 'momentaneous' non-goal-directed happenings, such as 'Jane sneezed' or 'the horse stumbled'.

Linguists have realised that there is no fixed association between verb and event type. For instance, we often use an accomplishment and the imperfect tense to express an activity: *I am climbing Mount Snowdon. I am baking a cake.* This does not have as an implication that the achievement will come to pass. I may have to stop my ascent, or a power cut my interrupt my baking. So it is easy to think of cases where a verb appears in an accomplishment and an activity, such as 'climb' in the following sentences:

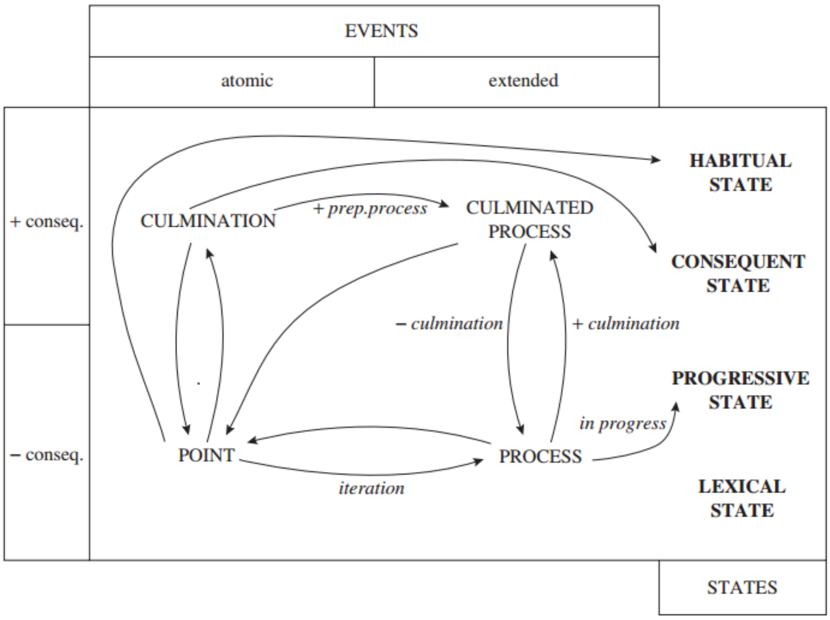
- Kit climbed the mountain peak.
- Kit was climbing the mountain, but didn't make the peak.

The saliance of this accomplishment-activity distinction is marked linguistically by the use of 'in' and 'for' when time intervals are assigned:

- Kit climbed the mountain peak *in* six hours.

- Kit was climbing the mountain *for* six hours, but didn't make the peak.

Moreover, sometimes work has to be done by the reader or listener to force or 'coerce' an interpretation of the use of a verb to a suitable event type. The *sneezed* in 'He sneezed as she entered the room' may be a momentaneous happening, but if I read 'He was sneezing while the choir finished the oratorio', to make sense of the use of the imperfect I need to blow up this point-like event by iteration to an activity in progress. Here 'sneezing' means having a bout of sneezes. This is represented in the lowermost arrow of the following diagram from Moens and Steedman (1988):



To interpret this diagram in Vendler's terms, we must make the following identifications: Process = Activity; Culminated process = Achievement; Culmination = Accomplishment; Point = Momentaneous happening. Arrows in the diagram are regular assignments, so that, for instance, the map from Culmination to Consequent state sends an achievement to the state of having achieved it, e.g., *reach Cairo* to *having reached Cairo*. We can say that the *iteration* and the *consequent state* arrows, as well as the others, are mappings between types. My travelling to Cairo last year is an element of the type of such culminated processes, *Travel to Cairo*. This latter type is an element of the Type *Culminated process*. *Travel to Cairo* is mapped by the arrow −*culmination* to *Proceeding towards Cairo*.

Moens and Steedman proposed that we see these event categories in relation to what they term an *event nucleus*. Typical basic events are composed of three components: a preparatory phase involving some activity, culminating in some accomplishment, which results in a change of state for some period. For instance, I reach out to a switch and flick it on, thereby lighting the room. These three components comprise an *event nucleus.*

- Event nucleus = "an association of a goal event, or *culmination*, with a *preparatory process* by which it is accomplished, and a *consequent state*, which ensues." (Moens and Steedman 1988, p. 15)

This tallies with our discussion in the first section of this chapter of the need to perceive a firm temporal boundary to a process or activity to count it as constituting an event. Its culmination in a change of state provides a first-rate way to identify an event.

In the context of this chapter, we might say that this event nucleus takes on the form of an iterated dependent sum, like the context associated to a very short story.

$$Event\ nucleus :\equiv \sum_{\substack{x:Activity\\ y:Achievement\\ z:State}} (Culminate(x,y) \& Consequent(y,z))$$

Since

$$Accomplishment :\equiv \sum_{\substack{x:Activity\\ y:Achievement}} Culminate(x,y)$$

we can see that

$$Event\ nucleus :\equiv \sum_{\substack{w:Accomplishment\\ z:State}} Consequent(q(w),z),$$

where q projects onto the second component of an accomplishment, namely, the achievement.

'*The stone hit the car*' is a happening. If it were part of an intended action '*John hit the car with a stone*', this would be an achievement representing the successful culmination of an activity. We therefore expect there to be a description of this activity giving rise to it, such as John planning his attack, selecting a stone and launching it. We also expect effects, consequent changes of state, such as a dent made on the car. If we learn that it is true that this event happened, then we infer that the stone was moving relative to the car right before the moment of impact, that they came together, and this produced a noise.

Evidence that we operate with the event nucleus schema in our minds as we look to understand the description of events comes from our rejection of some sentences as ungrammatical, and also from the effort we experience in making sense of a sentence which is difficult to parse. Take the first of these cases. We find it awkward or even wrong to say

- Paul broke the plates off the table.

where we happily say

- Meg brushed the crumbs off the table.

Linguists explain this difference by observing that the verb 'broke' already depicts a change of state, and so there is no place to add 'off the table'. On the other hand, 'brushed' merely depicts an activity. Completion of the event nucleus requires a resulting change of state.

As for difficulty in parsing, consider

- It took me two days to learn to play the Minute Waltz in 60 seconds for more than an hour.

'For' and 'in' are used to mark periods of time in, respectively, activities and accomplishments. 'Play the Minute Waltz in 60 seconds' is therefore an accomplishment, and yet it takes part in an activity, something lasting *for* more than an hour. This can only happen by iteration transforming an accomplishment, reduced to a point, into an activity. Finally there is the accomplishment of learning to be able to perform such a feat. There is a preparatory process lasting two days which culminates in the acquisition of this ability.

Our desire to complete an event nucleus makes sense of requests for explanation. Along with deductive-nomological, inductive-statistical and causal explanations, when it comes to animals we often seek a teleological explanation, a sought after change of state brought about by some activity. *'Why is that animal digging?'* 'It's burrowing. It wants to prepare a place to lay its eggs safely.' We look to place an activity with the setting of an event nucleus.

Similarly with humans and designed instruments:

- *Why are you flicking that switch?* I want to turn on the light. I believe this switch is for the light.
- *Why is the thermostat clicking on now?* It's to start up the boiler. It's designed to do so when the temperature drops.

Upstream there will some disposition, evolved or designed, or belief-desire bringing about the behaviour.

When someone poses a question of the form '*Why* b?' where $b : \mathit{Activity}$, it is likely that they are wanting to know if b is preparatory for some achievement so as to bring about some change of state? So in response I may offer an element of *Achievement*, *Accomplishment* or *State*. *Why are you going up into the attic? To fetch my tennis racket. I want to have my tennis racket.*

I may also be asked to fill in the first components of an event nucleus. *How did you get hold of a tennis racket? I went into the attic to fetch one.*

Such explanations rely on plausibly reliable connections between their parts, sometimes in the form of rules:

> Suppose you find yourself in a situation of a given type S; and suppose you want to obtain a result of a given type R, and there is a rule that in a situation of type S the way to get a result of type R is to do action of type A. (Collingwood 1939, p. 103)

Collingwood goes on to point out the limitations of this framework for action, that situations may be too specific to have rules applicable, etc. This points to a distinction between particular activities culminating in particular achievements, and activity subtypes regularly culminating in achievement subtypes, e.g., dropping an egg from high above a hard floor generally culminates in a broken egg. For the mathematical physicist Hermann Weyl it is at this level that we can speak of causal relations:

> The phenomena must be brought under the heading of concepts; they must be united into classes determined by typical characteristics. Thus the causal judgment, "When I put my hand in the fire I burn myself," concerns a typical performance described by the words "to put one's hand in the fire," not an individual act in which the motion of the hand and that of the flames is determined in the minutest detail. The causal relation therefore does not exist between events but between types of events. First of all–and this point does not seem to have been sufficiently emphasized by Hume–*generally valid relations must be isolated by decomposing the one existing world into simple, always recurring elements.* The formula "*dissecare naturam* [to dissect nature]" was already set up by Bacon. (Weyl 1932, p. 56)

Of course not all activities are directed to goals, and some activities are directed to multiple goals. The type

$$\sum_{y:Achievement} Culminate(b, y)$$

for some activity, b, is a set which need not be a singleton, and may also be empty.

Events are clearly closely related to time. We expect a particular activity to take place in an interval of time, and any culmination to happen at an instant, or at least within a relatively short time interval. '*The orchestra completed the full performance of Beethoven's ninth at 10 pm*'. We would reasonably expect the orchestra to be playing at 9.30 pm since there is mention here of an achievement. Some act has been accomplished, and we may know the preparatory process takes a little more than an hour. But perhaps there was a bomb scare, and after the first three movements starting at 8 pm, the audience were evacuated around 8.45 pm and only returned at 9.35 pm. The orchestra nobly decided to complete the symphony despite the continuing threat. Should these be considerations with which logic is to concern itself? Well, we may robustly infer that the orchestra was playing Beethoven's ninth some time before 10 pm.

Time can certainly be used to distinguish events. The plague of 1348 cannot be the plague of 1665, even if the infectious agent is the same. But of course there's more to the identity of events. Is my flipping the switch the same event as my turning on the light? For Davidson, *Yes*, for Kim, *No*. With my type theoretic reading in place we can say that both are right in a sense. *Flipping the switch* is an accomplishment. As for *turning on the light*, one natural reading is as an achievement resulting in a change of state. On the other hand, it might be taken as a full event nucleus with accompanying accomplishment. Certainly if a begrudging person dawdles on their way to the switch, you might say exasperatedly '*How long does it take to turn on the light?*'. Either way, strictly speaking Kim is right, putting us also on the side of Goldman (2007, pp. 466-467) who counts himself like Kim as a *multiplier*, unlike the *unifiers*, such as Davidson.

A more protracted description separates out the parts of the event '*Oliver moves his finger, flips a switch, turns on a light, and alerts a prowler*'. For Goldman, the unifier sees one action, where the multiplier sees four distinct actions. From our perspective, we have a clear event nucleus, and then an unintended consequence. As mentioned in the first section, we should only count within a type, having individuation rules established there, whereas Davidson is taking events as a basic type and then individuating according to their causal relations.

Goldman appears to be a multiplier for something approach type-theoretic reasons:

> ...if action a is a token of action-type A and action b is a token of action-type B, then a is not identical to b, even if a and b are performed by the same agent at the same time. (Goldman 2007, p. 472)

But the unifier's opposite point of view can also be given a type-theoretic gloss. It certainly seems most reasonable to be a unifier in cases where the difference concerns fine-grained levels of descriptions of some activity or accomplishment, as made apparent by the presence or absence of an adverb. So unifiers and multipliers disagree as to whether, say, Caesar dying violently is a different event from his dying. Davidson's thought is that this adverbial modification resembles the adjectival modification of a noun. In the latter case we typically take an apple before us as the same thing as the red apple before us. Strictly, in *intrinsic* varieties of type theory, such as HoTT, since a judgement as to something being a red apple requires more than a judgement as to something being an apple, then these are simply different types. Indeed, the former type is defined as a dependent sum $\sum_{x:Apple} Red(x)$. So similarly, violent deaths are formed as a dependent sum on the type of deaths, a subtype of momentaneous events. But we should note there exist *extrinsic* varieties of type theory which do allow an element to belong to more than one type. We shall return to this issue in the following chapter.

We have seen then that a number of philosophers instinctively take a type-theoretic turn, without explicitly embracing the calculus fully. Given the intricate structure operating behind our ways of seeing and speaking about the world, if a formalism is to have any chance of capturing this structure, it will need to be at least as sophisticated as a dependent type theory. But through this chapter we have also seen the need for a spatial and a temporal dimension to be included. These are very much to the fore in further of Hacker's objections to the treatment of events by predicate logic. He notes that we rely greatly on contextually intricate adverbial constructions, as in '*A was often drunk on New Year's Eve*'. This clearly doesn't mean that '*A was often drunk*', since A may refrain from consuming alcohol for the rest of the year. To unpack the former claim we would say that there exist multiple events such that they involve A drinking alcohol in a limited time period and achieving inebriation. Relative to all final days of the year, inebriation was achieved frequently, but we have no information for other days. We clearly need to be able to represent rates of states within time slots, requiring perhaps a form of temporal logic. For a type-theoretic approach to temporal logic, the reader will have to wait until Chap. IV. Continuing here briefly, in a similar vein Hacker points out that the following analysis of a further case will not do:

> 'A scarcely moved' as 'there was a moving by A and it was scarce', or 'it was done in a small way'. (Hacker 1982, p. 485)

Along with time, we need here some underlying space, and positions within that space depending on the passage of time through some interval. Evidently these are some of the ingredients which type theory will need to represent when it comes to physics.

Finally, returning to 'B wisely apologised', we recognise an achievement involving an agent, B, where it is presupposed that there is some (here unnamed) recipient or recipients of the apology. Such an achievement type comes also with a presupposition that some prior offence by B has been recognised by the party to whom the apology is addressed, and that the apology will change the existing state of offence with likely beneficial results, such as the removal of the threat of revenge. Now perhaps one might suspect that any inference involved here is very much *material*, highly dependent on the specific meaning of the words involved, but, as with the example from the previous chapter of the red ball that is therfore coloured, it would be worth exploring whether we can use type thery to make sense of the structural features of even such sophisticated intentional actions as apologising.

II.7 Revisiting the philosophical literature

Before we move on to more mathematical waters in the next chapter and introduce the 'homotopy' element of homotopy type theory, let me end this chapter with a few comments about the future of dependent type theory in philosophy. I can with confidence predict that it would be immensely profitable to run through large swathes of the philosophy of language and metaphysics literature with dependent type theory in mind. Places to begin are not hard to find. Experience suggests that in a great many places thinkers are striving to manage without the proper resources of a type theory. Take for example the following claim:

> It is now a near-consensus view in both philosophical and psychological work on emotion that emotions are intentional in the phenomenological sense—in other words, emotions possess a directedness towards a situation *as* a situation of a certain class, which is also defined by a corresponding meaning. (Keeping 2014, p. 243)

Now one needs a calculus which can take a kind of thing, here a *situation*, and form a class of that kind, so 'a situation of a certain class'.

Type theory offers the advantage here in that in the world of types there are no naked particulars. Or, to put this thesis in Aristotelian terms as taken up by John McDowell, every 'This' is a 'This such'. Something being declared to be of a type means that we may already know much about it. To give the type, conditions need to be set as to the 'introduction' of terms of a type, what we can do with them, that is, their 'elimination' rules, and their identity criteria. As observed earlier, this is very close to the entitlements and commitments of Brandom's inferentialism, and I predict that his work would offer rich pickings for the type theorist.[14]

So in type theory we find two slogans apply:

1. No entity without type.
2. No type without identity.

In our type theory we will only ask of two terms whether they are equal if they belong to the same type.[15] And remember this isn't something to be

[14] Cf. Paul Redding (2007) on why neo-Hegelians might want to return to Hegel in his use of Aristotle's *term logic*.

[15] But see the discussion of the difference between intrinsic and extrinsic forms of typing towards the end of section 2.1 in the following chapter.

determined – terms always come typed. We pose the question of the identity of terms by construction of a type of identities:

$$A : Type, a, b : A \vdash Id_{A(a,b)} : Type$$

I can't even pose the question of the identity of two terms of different types.

In *Sameness and substance*, Wiggins (1980) argues for the 'Thesis of the Sortal Dependency of Individuation' and against Geach's assertions concerning the 'Relativity of Identity'. The latter claims that it is possible for two individuals that they be the same according to one sortal but different according to another. Against this relative identity, Wiggins looks to demonstrate principle W:

$$W : (\exists f)(a \underset{f}{=} b) \supset ((g)(g(a) \supset a \underset{g}{=} b))$$

If two individuals of the same sortal are the same, then they're the same under any other sortal to which they may belong.

In our type theory, an element strictly belongs to one type alone, so we cannot speak precisely in Wiggins' terms. We might have a base type A, and then dependent propositions, $x : A$, $F(x)$ and $G(x)$. By $(a \underset{f}{=} b)$ we might mean the identity type of two elements in the dependent sum, $a =_{\sum_{x:A} F(x)} b$, is inhabited. Then via projection to the A-component, we would have that $p(a) =_A p(b)$ is inhabited. Obviously then Wiggins' condition would hold. If $G(p(a))$ is true, then of course so is $G(p(b))$.[16]

Alternatively, the dependent sums are taken as subtypes of A, where we typically consider them as 'the A which are F' and 'the A which are G', and speak of their elements as though they are simply elements of A. So then we might understand by $(a \underset{f}{=} b)$ that $a, b : A$ and that $a =_A b$ and $F(a)$ and $F(b)$ are true. But then of course, when $G(a)$ is true, we also have $G(b)$ is true, and so $(a \underset{g}{=} b)$. In sum, then, if Wiggins' considerations are to be taken up by a dependent type theory, they appear to be rather trivially the case. So why the controversy?

Let us then consider a counterexample which Wiggins discusses, one apparently proposed by Geach and Anscombe (Wiggins 1980, p. 41n35), of the road between Athens and Thebes. Let x be the road from Athens to Thebes, and y be the road from Thebes to Athens. Then x and y are the same as roads, and yet x may be uphill and y downhill. x certainly doesn't equal y as uphill roads, since y isn't even an example. So it seems we might have a counterexample to Wiggins' principle W.

Wiggins points out that to speak of the road from Athens to Thebes being uphill, we are really speaking about a journey along a road rather than a

[16] In fact, the type theoretic account is more general. It may matter *how* a and b are the same. When we have $m : a =_A b$ then for any A-dependent type, P, for $c : P(a)$ we have $m^*(c) : P(b)$.

road *simpliciter*, and as such these two journeys are not the same. From our perspective he is right – there are two types involved, albeit related ones, namely, the type of roads between two towns, and the type of directed roads between towns. Of course there are mappings between these types: one may forget the direction, and on the other hand one may take a road to the two associated directed roads. We might then write $U : Directed\ Road \to Road$. Then the identity of $U(a)$ and $U(b)$ in *Road* does not entail identity of a and b in *Directed Road*. *Uphill* and *downhill* are not predicates for the type *Road*.

We are prone to introduce an entity, then refer to it by 'it', allowing us to ask 'Is *it* the same as something else?' Is the road *from* A to B the same as the road *between* A and B. Without type disciplining it's easy to get lost here and ask improper questions of identity across types. In cases where there is at most one element of type A associated to each entity of type B, in other words when we have a subtype, we can ask 'Is that red apple the same as the apple I was given yesterday?'. But there are two directed roads associated to a single road, so we do not have a subtype.

These considerations do not depend on the entities involved having some underlying spatial or spatial-temporal basis for their identity. I can tell a similar story for non-object based sortals. For any two musical works, if they are the same as pieces by Bach, then they're the same as Baroque concertos. But I should not propose the identity of a musical work and a particular performance.

Robert Brandom comes to similar conclusions when considering sortals (2015, Chap. 7), since for him membership of properly distinct sortals will involve different sets of modal behaviours. Thus

> ...when we appreciate the modal commitments implicit in the use of *all* empirical descriptive vocabulary, we see that strongly cross-sortal identity claims–those that link items falling under different sortal predicates with different criteria of identity and individuation–are *never* true. (2015, p. 27)

With type theory as our means of expression, it's not just that these claims are never true. Rather, such identity types are not well-formed, and as such are uninhabitable.

My expectation is that any such linguistic, or, if you prefer, metaphysical, puzzle finds its solution when codified in a dependent type theory, but we must leave such investigations now to take up the quest for our full type theory. So far we have seen the benefits of a dependent type theory, where dependent types are propositions or sets. Now we must motivate the passage to higher types.