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Understanding the Infinite I: Niceness, Robustness, and Realism[†]

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This paper treats the situation where a single mathematical construction satisfies a multitude of interesting mathematical properties. The examples treated are all infinitely large entities. The clustering of properties is termed 'niceness' by the mathematician Michiel Hazewinkel, a concept we compare to the 'robustness' described by the philosopher of science William Wimsatt. In the final part of the paper, we bring our findings to bear on the question of realism which concerns not whether mathematical entities exist as abstract objects, but rather whether the choice of our concepts is forced upon us.

1. Introduction

This is the first of a pair of papers on new ways of thinking about some of the infinitely large entities met with in mathematics. In the second paper [Corfield, forthcoming], we shall see how mathematicians and computer scientists were led to a new formulation in their dealings with a certain kind of possibly infinitary situation. One type of infinite entity frequently emerges as the collection of behaviours of a dynamical system as it unfolds. For example, the *extended* natural numbers, essentially the ordinary natural numbers with a single infinite element adjoined and a predecessor function, captures the behaviour of the simplest black box which, each time we press a button, emits a beep or falls forever silent. This so-called *coalgebraic* conception of potentially endless unfolding, decomposition, or destruction is very pervasive. We see it operating even in analysis in the treatment of a Taylor series as an infinite list of coefficients, where evaluation at 0 gives the head of the list and differentiation the tail.

The pervasiveness of high-level concepts, such as the coalgebraic, is itself rather common—examples include broad themes such as duality, symmetry, and deformation. For example, duality may be seen in: projective geometry between lines and points; platonic solids, *e.g.*, between the dodecahedron and icosahedron; Stone duality between certain spaces and algebras; Fourier analysis; Poincaré duality between the homology and

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cohomology of complementary dimensions; duality between syntactic theories and semantic models; Pontryagin duality for locally compact abelian groups, and so on (see [Corfield, 2010b, §2]).

In this paper I would like to consider what might be thought to be the converse, or better *dual*, kind of phenomenon, where rather than a single concept or theme manifesting itself time and again in a wide range of entities, instead we have a single entity in which many concepts manifest themselves at the same time. Following [Hazewinkel, 2009a], I shall call this phenomenon *niceness*. The examples of nice entities I will treat in this paper are always infinitely large. The superposition of many interesting properties in the same object explains why they crop us so frequently, and suggests an answer to the puzzle as to why, when there are many possible infinitely large structures that mathematicians could study, some of them act almost as 'attractors'. There is an inevitability to them in that they are encountered when one steps out in a certain direction. Now this 'attractor' phenomenon might be attributed to many factors. Possibly humans have a limited number of ways of thinking and so work with entities constructed out of choices from a restricted menu. Or perhaps research mathematicians have been socialised to work in a limited set of ways with the same result. What I wish to propose in this paper is that we explore a third type of answer, one which uses the resources of mathematics itself to argue that, insofar as mathematics deals with 'possible structures', there are certain privileged members.

As with many problems in the philosophy of mathematics, it is worth our while scouring the philosophy of science literature for their treatment of analogous problems. I shall begin this paper, therefore, with some thoughts on parallels with the notion of 'robustness' described by philosophers of science. I think it quite reasonable to attribute to a form of mathematical robustness the sense a mathematician may have that he or she is dealing with something of real importance. With the freedom afforded them by a lack of empirical constraint, it is very reassuring to mathematicians when their constructions are found to be derivable along multiple paths, especially if those paths display a mutual independence.

The major part of this paper then studies examples where mathematical entities enjoy many compatible properties. As far as I am aware there has only been one attempt to address the problem of why such properties tend to cluster in the same entity. So after giving some examples of some privileged structures, including well-known ones such as the integers and rationals, along with lesser known ones, I shall discuss this solitary account—Michiel Hazewinkel's 'Niceness Theorems' [2009a]. Here, as a mathematician addressing mathematicians, Hazewinkel describes and partially explains the occurrence of situations in which we find coexisting properties. *Niceness*, the coexistence of many properties, could then be said to be a form of mathematical robustness. Success in drawing the attention of philosophers to the existence of such phenomena, and to forms of explanation of this niceness would be reward enough, but perhaps we can push a little harder on the door Hazewinkel has opened for us.

2. The Robustness of Mathematical Entities

In his paper 'The ontology of complex systems' William Wimsatt explains how he chooses to approach the issue of scientific realism through the concept of robustness.

> Things are robust if they are accessible (detectable, measurable, derivable, definable, producible, or the like) in a variety of independent ways. [Wimsatt, 2007, p. 196]

It is worth placing this thesis into the context of philosophy of science of recent decades. Until the 'experimental turn', the philosophy of science largely treated science in terms of the representational capacity of its language through the relationship between theory and observation. Hilary Putnam's no-miracles argument [1975, p. 73] argued that our scientific theories were so successful that it would be a miracle if their terms, including those designating unobservable entities, did not successfully refer. Larry Laudan [1981] responded by pointing out that the historical record should lead us to be pessimistic that all the terms of our theories refer. Successful theories had spoken of 'caloric' and of the 'ether', and these were later dispensed with. A final moment in this dialectic was reached when realists argued that something real *had* been captured by successful theories, even if theoretical terms were later revised, and that it was on account of this something that the theory's empirical success was made possible. For some it was the 'structure' of the theory (*e.g.*, [Worrall, 1989]). The theory of caloric embodied a structure which later theories of heat have only refined, and likewise for the theory of electromagnetism and the ether.

With Ian Hacking's *Representing and Intervening* [1983], we were provided with a different point of emphasis. According to his account, we should believe in the existence of electrons not because the term 'electron' features in a very successful physical theory with many empirical confirmations, but because we have such a manipulative grasp on electrons that we can use them for a variety of ends. We can shoot them at television screens to make pictures, use them to cure skin cancer (beta radiation), and so on. This led to his famous motto—'If you can spray them, then they are real.' Although Wimsatt worked out his theory of robustness prior to Hacking's work, we can see him as integrating the manipulative and theoretical approaches to scientific realism. He notes, about his own criterion, that 'A related but narrower criterion (experimental manipulability via different means) has since been suggested by Hacking [1983]', but that 'the independent means of access are not limited to experimental manipulations but can range all the way from non-interventive observation or measurement to mathematical or logical derivation, with many stops in between.' [Wimsatt, 2007, p. 196].

We can illustrate this notion of robustness in the context of Jupiter's moons, whose reality was up for debate when Galileo let leading astronomers of his day look towards the planet through his telescope. Even if the telescope had proved its worth on Earth, allowing merchants to tell which ship was heading towards port beyond the range of the naked eye, this did not completely guarantee its accuracy as an astronomical instrument. How do we know that light travels and interacts with matter in the same way in the superlunary realm as down here on Earth? How could we trust this optical device when what appeared to the eye to be a single source of light, a star, was split in two in the telescope's image? Of course, by now the moons are very robust for us. We can send probes close to their surfaces to report back on phenomena such as the Masubi Plume on Io. Our knowledge of optics tells us that Galileo's telescope was sufficiently reliable. We have an array of means to tell us that its discrimination of the single star into two points of light reflected the fact that many stars are binary. We also have theories of the formation of planets and their satellites. We can predict so accurately where they will be in years to come that we can send probes millions of miles to meet them. We have expectations of their composition, and so on. In sum, the weight of all the theoretical and practical considerations bearing on the moons of Jupiter forces us to accept their existence.

Perhaps for our *mathematical* purposes we should look to scientific entities where robustness has been established in large part through mathematical or logical derivation. Aside from our ability to spray them, the behaviour of electrons is described and derived by quantum electrodynamics. From this theory the value of the anomalous magnetic dipole moment of the electron can be calculated to an accuracy of twelve places. Wimsatt is right to want to wed this kind of consideration with the manipulative control we have over electrons. The collection of theoretical, experimental, and technological accounts which in some respect bear on the electron would fill a very large library. We may still want to say, however, that for an entity to be rightfully called real it is essential that there be an ability to point to its physical effects. For example, a warrant for the reality of electrons will mention an early reference to them, perhaps even a baptismal moment along the lines of 'This phenomenon is caused by negatively charged particles—let us call them *electrons*'. Certainly enormous theoretical refinement has taken place, but at some stage there was a pointing out of something in the world as being brought about by the action of the proposed entity, and the list of such phenomena has only lengthened since. Even in situations where a theory predicts the existence of an entity before it has been observed or produced, such as Dirac's prediction of a positively charged counterpart to the electron from solutions to the Dirac equation, we might want to place priority in the warrant for its reality on the subsequent experiments, in this case those that revealed the existence of positrons.

So then, does an analogous mathematical robustness constituted solely by a number of derivations have a bearing on anything we may want to call 'mathematical reality'? Deprived of the opportunity to spray mathematical entities, it would appear that we are left with mere derivation, and, however many independent paths there are to derive an entity, this may well not be enough to persuade those unconvinced of the need to postulate abstract entities to change their minds. However, looking a little closer at the debate in the philosophy of science, we see there a distinction between the question of the ontological commitments a successful theory should require of us, and a question Ian Hacking raises in *The Social Construction of What?* [1999]. Hacking describes a 'sticking point' between those who believe that we could have just as successful a collection of natural sciences as we do today but which employ very different concepts, and those who think our success is dependent on our having found something approximating to our current concepts. Rather than wonder what our theoretical and practical grasp of the electron warrants us to believe about the referent of 'electron'—a particle, a structure, nothing—instead we ask whether we could have a successful physics without something closely approximating the notion of an electron and the technological practices known to rely upon the notion.

Now we can come to analogues of these two questions operating in the philosophy of mathematics. The first [Corfield, 2010a] I have called the question of *external* realism, the second the question of *internal* realism.[1] The former is the more common debate within analytic philosophy. It wonders whether our ontology can be stretched to include abstract objects, such as numbers and sets, and worries how we might come to know them. The latter debate concerns differences within mathematics as to why some entities and properties are central, and others are of insignificant interest. It asks whether we could have a mathematics as successful as the contemporary one but with very different concepts. Criteria for success here, as in science, are up for debate, but would include the resolution of outstanding problems, the melding together of apparently disparate mathematical theories, and the provision of powerful applications. Considerations of robustness in mathematics, it seems to me, pertain much more to the latter internal realist debate, and it is that one to which I wish to contribute. A library of all works bearing on, say, the complex numbers, including monographs

[1] The reader should note some affinity to Carnap's distinction between external and internal questions [Carnap, 1950].

on algebraic number theory, Hilbert space theory, complex analysis, Riemann surfaces, treatments of matrix groups over $\mathbb{C}$, Mandelbrot and Julia sets, applications in control theory via the Laplace transform, and signal analysis via the Fourier transform, 2-dimensional potential flow in fluid dynamics, spinors in general relativity, quantum field theory, and so on, would rival our library dedicated to the electron.

Now, when we read in the quotation above Wimsatt saying '*Things* are robust...', he might be thought to be speaking merely about physical *entities*, perhaps particles or genes or galaxies. But he is quick to note that

> ...robustness plays a similar role also in the judgement of properties, relations, and even propositions, as well as for the larger structures—levels and perspectives... [Wimsatt, 2007, p. 196]

By *levels* he means 'hierarchical divisions of stuff (paradigmatically but not necessarily material stuff) organized by part-whole relations, in which wholes at one level function as parts at the next (and at all higher) levels' [Wimsatt, 2007, p. 201]. *Perspectives* occur when causal relations between parts of complex systems become too rich for simple analysis by levels. What is clear then is that Wimsatt is not a reductionist aiming to take the ontology of the natural science to be composed of a single kind of thing.

When looking for robust mathematical things we might likewise consider more than mere entities. Considering the following division of the subject matter of mathematics (due to Albert Lautman [2006, p. 223]) into entities, facts, theories, and ideas. We might readily argue that there are robust *theories* which apply in a variety of settings, such as cohomology which may be applied to topological spaces, algebraic varieties, groups, and so on. We may also consider robust *ideas*, such as duality, whose extent we sketched above as an example of a pervasive idea. The robustness of theories and ideas may be thought to reveal itself through their pervasiveness. However, in this paper I shall mostly consider mathematical *entities*. Robust *facts* such as the fundamental theorem of algebra which states that any complex polynomial has a complex root, with its many diverse proofs, can be phrased in terms of one of the properties of the complex numbers, that of being algebraically closed. The very many different proofs of the theorem reflect relations between the many other properties of the complex numbers. This coincidence of properties in an entity is what we focus on in this paper.

Restricting ourselves to entities, we might have included numbers such as π. This number famously crops up extremely frequently, from the circumference of a Euclidean circle, to the normalising constant in the normal distribution, to values of the Riemann zeta function. Like the examples we shall cover, we can account partially for its multiple appearances, here in terms of the theory of periods (see [Kontsevich and Zagier, 2001]).

Other prevalent entities include special functions, elliptic curves, and simple finite groups. In what follows, however, I have decided to focus mainly on entities picked out by category theory. It is not completely common knowledge amongst philosophers, even those who agree that category theory has already proved itself to be a powerful organising language, that it can be used to pick out special entities. That story should be told.

3. Universal Objects

A good first example of a special entity is the collection of natural numbers.[2] There is little more to this entity than the notion of 'start and proceed'. Any time we have a set with a designated element and a self-mapping, there is a unique way to label elements of this set with natural numbers, so that the designated element receives the label zero and the sequence of images of this element are labelled by the natural numbers. The sequence may repeat itself after a certain point, and enter a cycle, but this is something the essential 'start and proceed' entity could not do. It has to be able to label sequences with cycles of all possible lengths, as well as a non-cycling sequence. Also, the set whose members we are labelling may very well include elements which are not contained in the sequence. This is something the natural numbers cannot have. They have just what they need and nothing more.

We can give more technical characterisations of the natural numbers, as may be seen in the companion paper [Corfield, forthcoming]. They form the free monoid on one generator. A monoid is a set with a binary associative operation for which there is an identity element. Being the *free* such monoid, given any monoid with designated element, m, there is a unique mapping of monoids so that 1 is sent to m. We should note here that the natural numbers as monoid has another property—its binary operation is commutative. For an alternative characterisation, let us consider the category, **2**, with two objects and a single non-identity arrow going between them. This is a representation of the notion of process. If we could glue together start and end points of this process then we could repeat it endlessly. We can do precisely this by identifying the inclusions of the objects in **2**. We say that the natural numbers form the *coequaliser* of the two distinct maps, or functors, from the category **1**, a single object with identity arrow, to the category **2**.

A first characterisation of the integers is as the group completion of the monoid constituted by the natural numbers, in other words we simply add inverses. To our set, its designated element, and its self-mapping, we are

[2] I shall be viewing anything which occurs as an object in a category as a single entity, even when from the perspective of set theory such objects are formed by bringing together a set of elements.

adding the ability to undo the mapping. It inherits the commutativity of the natural numbers; so we say that the integers form the free abelian group on one generator. This entity is neatly captured by the idea of winding a string around a peg. The different ways of wrapping the string are assigned different integers, where two windings are given the same integer if one can be wiggled to the other without taking loops off the peg. A more technical formulation takes the integers as the coequaliser of the two maps from **1** to **I**, where **I** is the category with two objects and an arrow in both directions, so that their composites are identities for the objects. The category **I** is the embodiment of the notion of an isomorphism.

Now the main observation of Hazewinkel is that simple things often come along with extra properties or structure. Let us show this in the case of the integers. After they introduce us to addition for two integers, our teachers show us that there is a multiplication, which is compatible in certain ways with addition, for example, $a \times (b + c) = (a \times b) + (a \times c)$. In sum, the integers form a *ring*. What we probably will not learn is that the multiplication and ring structure for the integers follow merely from the fact of their being the free abelian group on one generator. Let us now sketch why this is so. To do so I shall have to introduce another category-theoretic construction, that of an *adjunction*. Adjunctions are ways of comparing two categories via two functors, the maps which go between categories. So in this case we have a pair of functors between the category of sets and the category of abelian groups. Given a set, X, we can form the free abelian group with elements of X as generators, $F(X)$. Essentially, elements of this abelian group will be linear combinations of elements of X, that is, items of the form $3x_1 - 7x_2 + 5x_3$ of any finite length with integer coefficients. To do things properly we should also explain what happens to functions $f : X \to Y$. Fortunately the map from linear combinations of elements of X to linear combinations of elements of Y should be obvious, *e.g.*,

$$3x_1 - 7x_2 + 5x_3 \mapsto 3f(x_1) - 7f(x_2) + 5f(x_3).$$

The second functor, U, takes an abelian group, A, and strips off its group structure, leaving its underlying set. One important characteristic of adjoint functors corresponds in this case to the fact that for a set X and an abelian group A, the collection of functions between X and $U(A)$ is isomorphic to the collection of abelian-group homomorphisms between $F(X)$ and A, or $Hom_{Ab}(F(X), A) \cong Hom_{Set}(X, U(A))$. The resemblance between this isomorphism and an equation defining the adjoint of a linear operator is what gave the construction its name. Suffice it to say that 'adjoints occur almost everywhere in many branches of Mathematics' [Mac Lane 1971, p. 107].

Now in the special case where X is a singleton $\{*\}$ and A is the free abelian group on one generator $F(\{*\})$, we have

$$\begin{aligned} Hom_{Ab}(\mathbb{Z}, \mathbb{Z}) &= Hom_{Ab}(F(\{*\}), F(\{*\})) \\ &= Hom_{Set}(\{*\}, U(F(\{*\})))\text{(by the adjunction)} \\ &= U(F(\{*\})) \\ &= U(\mathbb{Z}). \end{aligned}$$

This says that there is one abelian-group map from $\mathbb{Z}$ to itself for every integer. For an integer n, this works by sending 1 to n, and in general m to $n \times m$. Then there is an evaluation map

$$Hom_{Ab}(F\{*\}, F\{*\}) \times F\{*\} \to F\{*\},$$

which corresponds to multiplication

$$\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}.$$

A fancy way of saying all of this is to say that $\mathbb{Z}$ is a monoid object in *Ab*, which is a category-theoretic definition of a ring. In fact the integers form the free ring on no elements, or, in other words, the initial ring.

This exposition of how the ring structure of the set of integers derives from its property of being the free abelian group on one generator has perhaps been too hasty for the reader. To give a detailed exposition of this result would, however, take up much more space than I have available. The essential lesson to be learned from this exercise is fortunately a simple one—it is sometimes possible to give a mathematical proof of the fact that entities defined by universal properties possess further properties. Special objects may well have further special properties.

Looking now beyond the integers we might want to provide an inverse to multiplication. To give it its technical name we form the field of fractions. This is part of a much more general construction of taking the *injective hull* of the integers amongst abelian groups. In the process the rationals pick up more of the integers' structure. They are not just a ring, but an *ordered* ring, inheriting an order from the integers. The order structure of the rationals has another characterisation. It is the Fraïssé limit of all finite linear orders. What does this mean? Consider all finite ordered sets with order-preserving injections. The order of the rationals is constructed as the direct limit of this filtered set of finite orders. This means that all finite ordered sets may be embedded in the rationals. Furthermore, any order-preserving mapping between two finite subsets of the rationals may be extended to an automorphism of all of the rationals. This property is termed *homogeneity*. The integers do not have this property as can be shown by the fact that there is an isomorphism between $\{0, 1\}$ and $\{0, 2\}$ as ordered subsets of the integers which cannot be extended to an automorphism of the whole set. We say that the integers as an order lack density, while the rationals are dense and homogeneous.

We shall see a universal characterisation of the closed real interval in the companion paper [Corfield, forthcoming]. Turning now to properties of the whole set of reals, we know that it is complete in two ways due to (i) Dedekind—every set with an upper bound has a least upper bound; (ii) Cauchy—Cauchy sequences converge. These two corresponding ways of completing the rationals are examples of more general constructions: the Dedekind-MacNeille completion of a lattice; and the Cauchy completion of a metric space. Through the former construction they inherit an order from the rationals. They are also Archimedean; every element has a multiple larger than 1, and indeed the reals form the largest Archimedean field. The reals have many other properties including forming a connected locally compact Hausdorff abelian topological group.

An important lesson, however, is that just because there is a strong force leading you in one direction from a given place, does not mean that there will not be others pointing elsewhere. Indeed, we can impose different metrics on the rationals, and complete them so as to form other fields. By doing so we must give up on the notion of an order. These are the p-adic fields, defined for each prime p. From this perspective the reals are the quirky completion; taking up this direction of thought mathematicians speak of the reals as the completion of the rationals at 'the prime at infinity'. Very many formulas are best derivable by bundling together the reals along with its sister completions.

For each prime p the norm of a rational number q is p^{-n} where $q = p^n \cdot a/b$, with a and b both integers coprime to p. The real norm is just $|q|$. One can easily show then that the product of the real and p-adic norms of a rational is 1, although this may not impress, as the p-adic norms may be thought to have been selected for this result. But there are surprising results which suggest that the set of completions of the rationals belong with one another. For example, that gem of a discovery by Euler that

$$\prod_p \frac{1}{1-\frac{1}{p^2}} = \frac{\pi^2}{6}$$

can be construed as expressing something about all the completions of the rationals simultaneously [Manin, 1996, p. 522]. The so-called *adèle ring* for the rationals is a (restricted) product of all of these completions, and its many good properties, including local compactness and Pontrjagin self-duality, allow for the techniques of harmonic analysis, such as Fourier analysis, to be applied right at the heart of number theory.

Why the reals have been privileged as a completion of the rationals is an important question. The coincidence of so many good properties must play a part, and especially the preservation of the rationals' ordering. Today, however, we hear of speculative proposals to model physical space with other completions than the reals, or rather with the product of all the completions, the adèle ring.

> On the fundamental level our world is neither real, nor p-adic; it is adèlic. For some reasons reflecting the physical nature of our kind of living matter (*e.g.*, the fact that we are built of massive particles), we tend to project the adèlic picture onto its real side. We can equally well spiritually project it upon its non-Archimedean side and calculate most important things arithmetically. [Manin, 1996, p. 523]

Kato poetically phrases his thoughts on the situation as follows:

> As the night sky, mathematics has two hemispheres: the archimedean hemisphere and the non-archimedean hemisphere. For some reasons, the latter hemisphere is usually under the horizon of our world, and the study of it is historically always behind the study of the former. [Kato, 1993, p. 50]

> Mysterious properties of zeta values seem to tell us (in a not so loud voice) that our universe has the same properties: The universe is not explained just by real numbers. It has p-adic properties... We ourselves may have the same properties. Are there physical meanings of zeta elements? [Kato, 1993, p. 159]

Manin alludes to our physical composition prompting us to see the reals first. This kind of consideration is clearly of huge importance. Our discovery first of Euclidean geometry must owe much to our perception of space approximating it. Similarly the appearance to us of living in a three-dimensional world must account for our mathematical treatment of knotted circles in 3-space before knotted spheres in 4-space. A thorough treatment of the constraints acting on mathematicians, including—beyond the physical—also historical, psychological, and sociological factors, is something called for in [Corfield, 2003, chap. 1]. The central message of this paper is that we should not overlook the constraints which mathematics itself can describe.

So far I have covered characterisations of well-known entities. Yet we can form others of equal importance to mathematics, but lesser known. For example, **Symm**, the collection of symmetric polynomials in a countably infinite set of variables, is the free 'λ-*ring*'. We shall see in the next section that it possesses a bewildering array of properties, making it an extremely robust entity. But there are many other special structures. To name three: the Rado random graph, the Fraïssé limit of finite graphs, which contains every countable graph as an induced subgraph and possesses a compatible group structure; the von Neumann algebra Type II hyperfinite factor, which is closed under the formation of 2×2 matrices with entries from the factor; and, the infinite-dimensional complex projective space, $\mathbb{C}P^{\infty}$, the direct limit of finite-dimensional complex projective space as the dimension approaches infinity, which can be taken as the space of all pure states of the quantum system whose Hilbert space has countable dimension and as the classifying space for complex line bundles. It is important to note that all entities mentioned so far have been infinitely large; as we shall see, it is easier for infinitely large structures to possess many properties.

4. Hazewinkel and Niceness

Let us now turn to the one treatment that I have managed to find on the topic of this paper, mathematical reasons for the appearance and utility of certain pervasive mathematical structures. Michiel Hazewinkel has this to say in his paper 'Niceness theorems':

> It appears that many important mathematical objects (including counterexamples) are unreasonably nice, beautiful and elegant. They tend to have (many) more (nice) properties and extra bits of structure than one would a priori expect . . . [2009a, p. 107]

Just as with the integers and the reals, we appear to get out of them more than we put in. Hazewinkel continues

> These ruminations started with the observation that it is difficult for, say, an arbitrary algebra to carry additional compatible structure. To do so it must be nice, i.e., as an algebra be regular (not in the technical sense of this word), homogeneous, everywhere the same,. . . It is for instance very difficult to construct an object that has addition, multiplication and exponentiation, all compatible in the expected ways. [2009a, pp. 107–108]

This points us to something we have seen concerning the Fraïssé construction of the limit of finite linear orders. This limit, the rationals as ordered set, is sufficiently homogeneous that it can support a compatible addition, multiplication, and indeed a whole field structure. The reals are also homogeneous and support a field structure, inheriting these properties from the rationals.

Next Hazewinkel lists five phenomena:

(1) Objects with a great deal of compatible structure tend to have a nice regular underlying structure and/or additional nice properties: 'Extra structure simplifies the underlying object'. . .

(2) *Universal objects*. That is mathematical objects which satisfy a universality property. They tend to have:

   (a) a nice regular underlying structure
   (b) additional universal properties (sometimes seemingly completely unrelated to the defining universal property)

(3) Nice objects tend to be large and inversely large objects of one kind or another tend to have additional nice properties. For instance, large projective modules are free.
(4) Extremal objects tend to be nice and regular. ([That] [t]he symmetry of a problem tends to survive in its extremal solutions is one of the aspects of this phenomenon; even when (if properly looked at) there is bifurcation (symmetry breaking) going on.)
(5) Uniqueness theorems and rigidity theorems often yield nice objects (and inversely). They tend to be unreasonably well behaved. I.e. if one asks for an object with such and such properties and the answer is unique the object involved tends to be very regular. This is not unrelated to 4. [2009a, p. 108]

Indeed, 5 is not unrelated to 4. In fact, we may say of all of 1–5 that they are 'not unrelated'. In sum, we may say that universally defined entities tend to be regular, large (generally infinitely large), and have more compatible structure and properties than we would expect from their definition.

Consider, for instance, Hazewinkel's 'star example'—**Symm**, the ring of symmetric functions in a countably infinite number of indeterminates. I shall quote here at length from another of Hazewinkel's papers 'Witt Vectors Part 1'. There he says:

> **Symm**, the Hopf algebra of the symmetric functions is a truly amazing and rich object. It turns up everywhere and carries more extra structure than one would believe possible. For instance it turns up as the homology of the classifying space **BU** and also as the cohomology of that space, illustrating its self-duality. It turns up as the direct sum of the representation spaces of the symmetric group and as the ring of rational representations of the infinite general linear group. This time it is Schur duality that is involved. It is the free λ-ring on one generator. It has a nondegenerate inner product which makes it self-dual and the associated orthonormal basis of the Schur symmetric functions is such that coproduct and product are positive with respect to these basis functions . . . **Symm** is also the representing ring of the big Witt vectors and the covariant bialgebra of the formal group of the big Witt vectors (another manifestation of its auto-duality) . . .
>
> As the free λ-ring on one generator it of course carries a λ-ring structure. In addition it carries ring endomorphisms which define a

> functorial λ-ring structure on the rings $W(A) = \mathbf{CRing}(\mathbf{Symm}, A)$ for all unital commutative rings A. A sort of higher λ-ring structure. Being self-dual there are also co-λ-ring structures and higher co-λ-ring structures (whatever those may be).
>
> Of course, **Symm** carries still more structure: it has a second multiplication and a second comultiplication (dual to each other) that make it a co-ring object in the category of algebras and, dually, (almost) a ring object in the category of coalgebras.
>
> The functor represented by **Symm**, i.e. the big Witt vector functor, has a comonad structure and the associated coalgebras are precisely the λ-rings.
>
> All this by no means exhausts the manifestations of and structures carried by **Symm**. It seems unlikely that there is any object in mathematics richer and/or more beautiful than this one, and many more uniqueness theorems are needed. [2009b, pp. 327–328]

Hazewinkel spends a large part of his paper [2009a] making sense of the connections between these many varied characterisations of **Symm**. I think there is little point even beginning to sketch the mathematical concepts mentioned here.[3] What we can say is that while more work needs to be done to systematise these findings mathematically, the kinds of construction at stake are of a piece with the simpler case explained above of the ring structure on the integers. The important thing to focus on here is that it is possible to give a *mathematical* explanation of why we will find extra structure in some situations. In the next section we will try to make sense of this style of explanation by contrast with other styles, but first I would like to extend this discussion of universal entities to take in whole categories.

As well as locating nice objects within a particular category, we can also find nice categories, that is, categories with nice properties. There is an interesting tale to tell in this regard about how, after Grothendieck in the 1960s, mathematicians have opted to work with nice categories of objects rather than categories of nice objects when required to make the choice. We can understand sometimes why a collection of nice objects does not form a nice category. The former will have properties less likely to be preserved by category-theoretic constructions. Better then to embed them in something larger on which these constructions can be made. A classic case is that of manifolds. These are typically the objects of study of differential geometers. However the category of manifolds lacks many nice

[3] For a reasonably gentle treatment of the related category of Schur functors, see http://ncatlab.org/nlab/show/Schur+functor.

properties, including the possession of equalizers and coequalizers. For example, two diffeomorphic submanifolds of a manifold need not intersect in another submanifold. There are many ways to rectify this problem by expanding what we take to be a smooth space. Again, mathematicians would rather have a definition of smooth space that allowed the collection of them to possess what are taken to be good properties, even if under this definition we include spaces that might not otherwise have been considered.

All the same, sometimes it happens that a category of ordinary objects already possesses nice properties. Take for example the category of sets and functions. It has a universal characterisation as the free cocomplete category on one object. On top of this it has many other properties, including being complete. This means it has products and equalizers. But it has much more, including a form of *exponentiation*. The set of functions from B to A, designated A^B, allows for an adjunction $Hom_{Set}(B \times C, A) \cong Hom_{Set}(C, A^B)$. Set also has a *subobject classifier*, allowing us to characterise power sets $P(A) = 2^A$. In sum, Set is a special category known as a *topos*. This means it supports a form of higher-order logic [Lambek and Scott, 1988].

For a pair of more geometric examples, let us now consider the category of braids and the category of tangles. Braids are arrows in a category in which an object is a finite set of points in a plane. A braid is a collection of possibly interweaving threads joining one set of such points to a second, necessarily equal-sized set. This category is the free braided monoidal category, and it possesses extra structure represented by the binary operation where, given two braids, we replace each strand of the first braid by a copy of the second in a process known as cabling. In the case of tangles, again we take an object of the category to be a finite collection of points sprinkled on a plane. But now an arrow going from a first plane to a second plane of points is a collection of threads each linking two points either in the same plane or in different planes, along with a collection of knots sitting between the two planes. They can be tangled up with each other anyhow, as the name suggests. The category of tangles may be described universally. In this case, we are dealing with the free braided monoidal category with duals on one object [Shum, 1994]. The freeness of this entity, and the ensuing mapping from it to similarly structured categories, is part of what is called quantum topology (see chap. 10 of [Corfield, 2003]). Again at the level of categories there are mathematical reasons for the coincidence of many properties in a single category when it is definable by a universal property amongst categories of the same kind. In these cases, it is usually preferable to take the categories of the same kind to form a higher-dimensional category, here a 2-category or bicategory.

5. Arguing for Contingency

So with Hazewinkel we see the emergence of a type of mathematical explanation for the niceness of some mathematical entities, and consequently of their pervasiveness. They have more properties than we would expect from their initial characterisation, which makes it more probable that mathematicians will come across them in their work. For instance, in view of the very many properties which coincide in **Symm**, it is hardly surprising that this structure has been repeatedly encountered. Now let us consider how these observations might be seen to bear on the spectrum of opinion concerning the 'internal reality' of mathematical entities. I shall designate opposing wings of this spectrum as 'realist' and 'nominalist' as in [Hacking, 1999, p. 82], who scores positions from 1 to 5 (see [Corfield, 2003, pp. 12–14]). The very extremes of this spectrum are perhaps held by nobody, but it should be helpful to sketch them.

Extreme realist: There is very little freedom as to how to develop mathematical concepts. We should not be surprised that mathematics developed for internal reasons finds application in the physical sciences. We expect different cultures to arrive at the same concepts. Were we able to communicate with other civilisations in the universe, we would be able to understand their mathematics.

Extreme nominalism: There is a huge amount of freedom as to how to develop mathematical concepts. The reason concepts are selected is down to who happens to be influential at the time. A huge effort is expended making these choices look natural retrospectively, but the very large number of choices made as we extend our theories means we might have had a very different mathematics. We underestimate the otherness of the mathematics of different times and different cultures by rewriting it in our own terms.

Explicit positioning of oneself on the internal realist spectrum is fairly rare, and yet writings reflecting such positioning are reasonably common. For some examples of work towards the nominalist end other than the ones I shall treat here, see [Bloor 1976; 1994] and [Ruelle 1988; 2000]. Possible strategies for different wings of the internal realism spectrum run as follows.

Diachronic—Look at the historical development of the field to assess the choices made along the way:

- Nominalism: Contingent choices based on idiosyncracies of the mathematicians concerned set the future conditions of the use of a concept.
- Realism: Independent paths unexpectedly leading to the same construction suggest that details of the origins of a concept are of little

importance, and that this construction would have been found come what may.

Synchronic—Consider the field as we know it to be now:

- Nominalism: A specific current construction seems baroque and arbitrary.
- Realism: That constructions fit together with other constuctions into a larger scheme, possessing many nice properties, is part of a family, and explains and is explained by other things.

Let us see an example of a synchronic nominalist claim:

> Stated in realist terms, the extended number system [of the complex numbers—DC] is presumed in effect to stake out a 'natural kind' of reality. Far from 'carving reality at the joints', however, the system can be shown to feature a flagrantly gerrymandered fragment of heterogeneous reality that is hardly suited to enshrinement at the centre of a serious science like physics, not to mention a rigorous one like pure mathematics. Couched in these ultra-realist terms, the puzzle might be thought to be one that someone with more pragmatic leanings—the system works, doesn't it?—need not fret over; and in fact such a one might even look forward to exploiting it to the discomfort of the realist. Fair enough. I should be happy to have my discussion of this Rube Goldberg contraption (as the extended number system pretty much turns out to be) serve as a contribution to the quarrel between anti-realist and realist that is being waged on a broad front today [Benardete, 1989, p. 106].

The claim that the system of complex numbers—the algebraic completion of the reals, a field extension of degree 2, with its accompanying theory of complex analysis and Riemann surfaces—is 'flagrantly gerrymandered' seems to me to be a very difficult one to maintain. Certainly constructions of the mathematics of the past couple of centuries may look convoluted, but one must be careful not to mistake this appearance for reality. Never was there a more integral part of mathematics than that surrounding the complex numbers.

A diachronic approach seems more likely to work for the nominalist as with [Pickering, 1997], which tells the story of Hamilton's work on quaternions. Histories of a practice of this type delight in bringing contingency to centre stage—things could have been so very different. What is very noticeable in such histories is that often the very early days of a practice are treated. This gives the advantage of only needing to study a handful of people with all their idiosyncracies. The underlying thought is that if so much could have been so different while the course of a practice was being set, how different things could be decades later. And, if we can find a sharp change of direction away from the original pioneer's intentions late on our story, so much the better. Most of the original thinking guiding the practice will be revealed to be 'just a story'. Any number of such stories might have governed at that time, leading mathematics in very different directions.

So, in Pickering's paper, with the pace of research so slow, we can dwell on Hamilton's idiosyncratic Coleridgean and Kantian metaphysical views, and we can tell the story of the quaternions as having 'mutated over time into the vector analysis central to modern physics' (p. 45). Hamilton had failed to reach his original goals, only achieving 'a local association of calculation with geometry rather than a global one. He had contructed a one-to-one correspondence between a particular algebraic system and a particular geometric system, not an all purpose link between algebra and geometry, considered as abstract, all-encompassing entities' (p. 59). The quaternions could not form the required calculus for reasoning about entities in three-dimensional space. Even after Hamilton had considered multiplication on just the imaginary part, where the product of two lines could be an ordinary number or another imaginary, '...the association of algebra with geometry remained local. No contemporary physical theories, for example, spoke of entities in three-dimensional space obeying Hamilton's rules' (p. 60). 'It was only in the 1880s, after Hamilton's death, that Josiah Willard Gibbs and Oliver Heaviside laid out the fundamentals of vector analysis, dismembering the quaternion system into more useful parts in the process. This key moment in the delocalization of quaternions was also the moment of their disintegration' (p. 60).

From this an innocent reader might take it that, by and large, that was that as far as the quaternions were concerned, and that from the 1880s they fell into disuse. Such a reader would be surprised then to learn of a paper by Gsponer and Hurni [2005] which documents the study of the quaternions and allied algebras in mathematics and physics, the vast majority of which has taken place since 1900, in the form of an analytic bibliography of 1430 references. This raises the question of whether, with so many man-hours devoted to the extraction of whatever can be found to be useful about quaternions and their relationships with other mathematical structures, the first few decades of their use tell us very much. Although it makes for engaging history, do we learn so much about the ways in which mathematics operates at its highest level of organisation from the quixotic quests of individuals, rather than from an account of the work carried out by droves of workers, most of whom have sunk into obscurity?

Indeed, there is a danger in what is called 'one-pass history'. If we follow up the story of the quaternions we can find substantial reworking since the nineteenth century. We now know that they form one of four normed division algebras: real numbers, complex numbers, quaternions, octonions.

> The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on. The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete. The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings. But the octonions are the crazy old uncle nobody lets out of the attic: they are nonassociative. [Baez, 2002, p. 145]

There is a 'Cayley-Dickson' construction which allows us to pass along the line, doubling the dimension but systematically losing a property at each step so that as we pass from the reals to the complex numbers to the quaternions to the sedenions we successivly lose the properties: elements being identical to their conjugates, commutativity, associativity, the division-algebra property [Baez, 2002, p. 154]. So there is a 16-dimensional algebra, but within it we have lost the ability to divide by a nonzero element. For most purposes, mathematicians take this to be a step too far, and so decide to take the 1, 2, 4, and 8-dimensional members to compose the family.

Now, one can of course respond to this effort to find a place for the quaternions by saying that the only reason anyone ever came to the notion of normed division algebra in the first place was a somewhat arbitrary choice, part of whose motivation was to make sense of the quaternions themselves. Small wonder then that the quaternions seem so natural. Had we achieved a different extension of the complex numbers we would have found an overarching concept which would have made *that* extension appear to fit nicely with the complex numbers and reals. The worry being expressed here is that we fall into calling concepts 'natural' because of a lack of imagination as to how things might have gone differently. But then how easy is it to find appropriate extensions? The realist tends to believe that the options are usually very limited, at least options which make good mathematical sense, according to prevailing views on the ends of mathematics. In the case of normed division algebras, we learn that 'the octonions are important because they tie together some algebraic structures that otherwise appear as isolated and inexplicable exceptions' [Baez, 2002, p. 147]. As Baez goes on to explain, there are three infinite families of classical simple Lie algebras associated with the reals, complex numbers, and quaternions, and then five 'exceptional' such algebras. These five have all been found to be related to the octonions. Thus, normed division algebras are intimately related to simple Lie algebras, providing evidence for their naturalness.

Realist strategies for demonstrating robustness are:

1. *Diachronic*: Independent discovery means the idiosyncracy of originators is irrelevant. It would be a miracle if many people came across

the same construction and there were not some good mathematical reason for this.

2. *Synchronic*: Mathematical demonstrations show how constructions fit snugly into the larger scheme of things.

Both strategies trade on an independence, the first blatantly so, the second via independent links to the existing body of mathematics. But, in the first case, how can we establish whether two paths to the same or similar construction are independent? We may not know of the communication taking place at the time. And if the supposedly independent discovery happens years later, it will be hard to know whether it was truly made without knowledge of the first. As for synchronic accounts, on the other hand, perhaps the 'larger scheme of things' itself arises from attempts to accommodate the discovery, to make it appear natural. Where Vladimir Arnold in 'Polymathematics' [2001] finds many triples or 'trinities' in mathematics modelled on that of the triple ⟨real, complex, quaternion⟩, one might say that this is just to fish for analogies of an already accepted construction. Fishing is hard, says the realist, and so the debate would continue.

6. Conclusion

The notion of multiple possible characterisations and hence routes to the study of **Symm** and other universally defined structures, is a new weapon in the arsenal of the internal realist, although one might concede that there is work to be done to resist the charge that the very language used in this justification is the product of the victory of a certain body of thought which need not have occurred. We would need to do further work to support the idea of a certain independence of construction in the metaposition which Hazewinkel has begun to describe in terms of *niceness*, where structures can be shown to possess more properties than are apparent from their initial characterisation. It may seem unlikely that disagreements of this kind will be decided to each side's satisfaction. But even if the question of internal realism is unresolvable, still the debate is worth pursuing since it would help to bring to light some less noticed features of mathematics. In particular, we could expect to acquire a much clearer picture of how existing mathematics may best be described as fitting together, and of how the perception of new opportunities to make pieces of mathematical theory fit together drives research. It would be very interesting to see how mathematicians assess the power of methods which allow new forms of description of this fitting together to emerge.

We set out from Wimsatt's interesting challenge to the philosophy of the natural sciences to reconsider the reality of what science deals with through the lens of robustness:

> Things are robust if they are accessible (detectable, measurable, derivable, definable, producible, or the like) in a variety of independent ways... [T]he independent means of access are not limited to experimental manipulations but can range all the way from non-interventive observation or measurement to mathematical or logical derivation, with many stops in between. [Wimsatt, 2007, p. 196].

We have seen that a form of multiple determination also occurs in mathematics, giving support to the thought expressed by the French mathematician, Alain Connes, that

> The scientific life of mathematicians can be pictured as a trip inside the geography of the 'mathematical reality' which they unveil gradually in their own private mental frame... The really *fundamental point* in that respect is that while so many mathematicians have been spending their entire life exploring that world they all agree on its contours and on its connexity: whatever the origin of one's itinerary, one day or another if one walks long enough, one is bound to reach a well known town *i.e.* for instance to meet elliptic functions, modular forms, zeta functions. 'All roads lead to Rome' and the mathematical world is 'connected'. [Connes unpublished, pp. 2, 3]

What I have concentrated on in this paper are phenomena where we can understand *mathematically* why certain infinitely large 'well known towns' lie on the confluence of many routes, displaying a form of mathematical robustness. It is quite possible there will be such towns which we will not be able to explain via universal characterisation. Here I hope to have conveyed some insight, via Hazewinkel's idea of niceness, into the reasons why in many entities a multitude of interesting properties are forced to coincide. I believe that a thorough study of such phenomena will amply reward philosophers of mathematics.

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