The ubiquity of modal types

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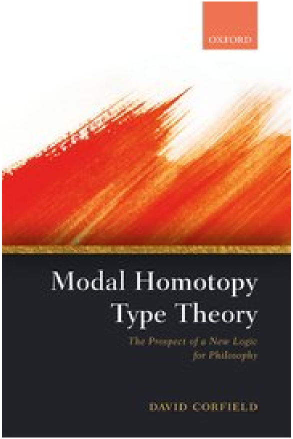
New tools for philosophy



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- Other disciplines develop these theories and formalisms.
- Philosophers continue along their own path without paying much attention to offspring theories.

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- Goal is to explore alethic, epistemic, doxastic, deontological, temporal,... modalities.
- They might consider the differences, if any, between *physical*, *metaphysical* and *logical* necessity and possibility.
- Technically, still largely in the era of modal logics (K, S4, S5, etc.) and Kripke models for semantics.

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Computer scientists' modal logic

- New modalities to represent security levels, resources, and generally, effects and coeffects.
- Philosophers' modalities but with different uses: Model-checking (temporal). Multi-agent systems (epistemic). Graded modalities.
- Technically, use of sub-structural logics, (idempotent) monads, coalgebra, labelled transition systems, bisimulations, ...

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- Kripke models, 1959 (presheaves over states).
- Metaphysical phase - possible worlds, e.g., Kripke, *Naming and Necessity* (1970/80), David Lewis, *On the Plurality of Worlds* (1986).

Naturally there were efforts to develop a first-order modal logic, leading to questions about, say, the relationship between $\exists\Diamond$ and $\Diamond\exists$.

- Something is possibly P.
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A different solution has the relationship made trivial by allowing quantification over all possible things.

Category theory to the rescue

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Category theory to the rescue

Modal logicians have devoted the overwhelming majority of their inquiries to propositional modal logic and achieved a great advancement. In contrast, the subfield of quantified modal logic has been arguably much less successful... frameworks seem to keep ramifying rather than to converge. This is probably because building a system and semantics of quantified modal logic involves too many choices of technical and conceptual parameters, and perhaps because the field is lacking in a good methodology for tackling these choices in a unifying manner. (Kishida 2017, p. 192)

Or jump to modal dependent type theory?

$$\begin{array}{ccc} \text{Logic} & \rightarrow & \text{Modal Logic} \\ \downarrow & & \downarrow \\ \text{DTT} & \rightarrow & \end{array}$$

Modal dependent type theory

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Disruptive technology

With such a formalism available, we might then look to shake up the ways in which philosophy can interact with other disciplines.

Lawvere on quantifiers

For **H** a topos (or ∞-topos) and $f : X \to Y$ an arrow in **H** induces a 'base change', f^*, between slices (categories of dependent types):

$$(\sum_f \dashv f^* \dashv \prod_f) : \mathbf{H}/X \underset{f_*}{\overset{\overset{f_!}{\longrightarrow}}{\overset{\overset{f^*}{\longleftarrow}}{\longrightarrow}}} \mathbf{H}/Y$$

This base change has dependent sum and product as left and right adjoint.

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- The unit of the monad is the injection of a world where P holds into all such worlds.
- The counit of the comonad applies a function proving P at each world to this world.

Accessible worlds

More generally, we might consider an equivalence relation: $W \to V$, then

- Necessarily P holds at a world if P holds at all related worlds.
- Possibly P holds at a world if it holds at some related world.

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$$spec : Animal \to Species$$

Then for an *Animal*-dependent type, $Leg(x)$:

- $\bigcirc_{spec} Leg(Fido)$ is the set of legs of dogs
- $\Box_{spec} Leg(Fido)$ is the set of choices of a leg for each dog.

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- front right leg: $BodyPart(Dog)$
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‘Front right leg’ is acting as a *rigid designator* over the animals which are dogs.

Recall that generally we have a map $\Box A \to A$, but not one from $A \to \Box A$.

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Given an element in $spec^* BodyPart(Fido)$, such as canine front right leg, we can name a similar body part for Fido's conspecifics, i.e., an element of $\Box_{spec} spec^* BodyPart(Fido)$.

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It’s all about knowing how to continue to counterparts in neighbouring worlds/fibres/dogs. If I point to the front right leg of a dog and show you another dog, you probably choose the same leg.

There's a short route from this construction to partial differential equations, being told how behaviour carries over to infinitesimally neighbouring points.

Here we are in a differentiable context with a map $X \to \Im(X)$, identification of infinitesimal neighbourhoods.

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I. Khavkine, U. Schreiber (2017), Synthetic geometry of differential equations: I. Jets and comonad structure

Actualism and Higher-Order Worlds

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Hayaki arranges things through nested trees. The first sentence presents a level 1 world, the second a level 2 world.

In modal type theory we could imagine an approach via changes to the context.

$$\Gamma \ = \ x_0 : A_0, \ x_1 : A_1(x_0), \ x_2 : A_2(x_0, x_1), \ \ldots x_n : A_n(x_0, \ldots, x_{n-1}),$$

We could base change, etc., relative to an initial segment of the context.

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Counterfactuals could work by stripping back a context until the counterfactual antecedent can hold, an idea first developed by Aarne Ranta.

Further monads

$$(\sum_f \dashv f^* \dashv \prod_f) : \mathbf{H}/X \underset{f_*}{\overset{\overset{f_!}{\rightarrow}}{\overset{f^*}{\leftarrow}\atop\rightarrow}} \mathbf{H}/Y$$

Returning to the possibility/necessity situation ($f : W \to *$), compositions may be made in a different order, generating

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Not idempotent, but still interesting.

Reader monad

Asudeh, A. and Giorgolo, G. 2016. ‘Perspectives’, Semantics and Pragmatics, 9(21).

We have a nice category, $\mathcal{C}$, and an object, I, so that the slice category, $\mathcal{C}/I$ is sufficiently nice.

Then the lower adjunction generates the reader monad.

$$\mathcal{C}_{/I} \underset{\prod_I}{\overset{\overset{\sum_I}{\longrightarrow}}{\overset{\xleftarrow{I^*}}{\longrightarrow}}} \mathcal{C}$$

The monad sends an object/type A to A^I.

We can read its elements as terms ‘such-and-such according to i’.

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Not only is 'the person downstairs' perspective-relative, but so is 'who i believes is the person downstairs'.

The reader monad, according to Asudeh,

> *"allows us to consider more complex types of meaning only when truly necessary, avoiding the notorious problem of generalizing our lexical entries to the worst case."*

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We could have y as variable too,

$X : I, x, y : A^I \vdash x(X) = y(X) : Prop$

de re/de dicto

X believes that *the person downstairs* is his father.

Substitution is interesting when perspectives differ:
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Y says ‘X believes that x is a’, so also ‘X believes that b is a’,

This might result in a sentence like, ‘X believes that his mother is his father’.

Asudeh uses this to show how to make sense of

- She loves Peter Parker but not Spiderman.
- He believes π is not π.

Intermodalities

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One form of this comes from making such a span represent time.

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We'll have at least $b, e : Time_1 \to Time_0$ generating two adjoint triples to express the temporal operators - F, G, H, P.

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Composition between matching intervals allows for the expressivity of *until* and *since* by quantifying over ways to chop up intervals.

Invariance

With the availability of the homotopic aspect of HoTT, we can also consider slices over BG, the one object pointed groupoid associated to a group.

This yields orbits and fixed points of group actions, etc.

But let's now briefly consider the bigger picture.

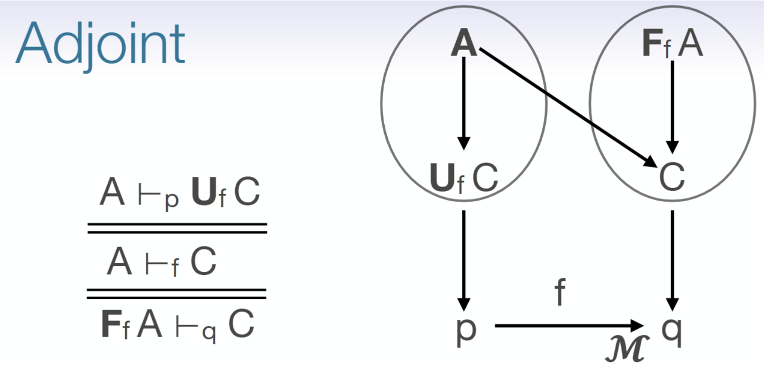
Internalisation of judgements

Curry's proposal was to take $\bigcirc\phi$ as the statement "in some stronger (outer) theory, ϕ holds". As examples of such nested systems of reasoning (with two levels) he suggested Mathematics as the inner and Physics as the outer system, or Physics as the inner system and Biology as the Outer. In both examples the outer system is more encompassing than the inner system where reasoning follows a more rigid notion of truth and deduction. The modality $\bigcirc$, which Curry conceived of as a modality of possibility, is a way of reflecting the relaxed, outer notion of truth within the inner system. (Fairtlough and Mendler, On the Logical Content of Computational Type Theory: A Solution to Curry's Problem)

In the judgmental approach to modal type theory, we view modal types $\Diamond A$ as internalizations of categories of judgment.

That is, in Martin-Löf's judgmental methodology, we take the assertion "P is true", and then introduce a judgement of "P is true", which explains what constitutes evidence for P (the introduction rules), and how to use a P (the elimination rules). We can extend this to modalities by introducing new judgements to represent new categories of assertion. So in addition to "P is true", we might also have categories of judgment such as "P is known to X", "P will eventually be true", "P is possible", and so on. Then, a modal type like $\Diamond A$ is an internalization of a judgement. That is, we can say that the introduction rule for the judgement "$\Diamond A$ is true", is actually evidence for the judgement "A is possible". (Neel Krishnaswami)

Reflections of objects and morphisms across adjunctions



Dan Licata and Felix Wellen, Synthetic Mathematics in Modal Dependent Type Theories.

Licata-Shulman-Riley project

2-Bifibrations

$$\begin{array}{ccc} \mathcal{C} & & \\ \downarrow & & \\ \mathcal{M} & \rightarrow & \mathit{Adj} \end{array}$$

- Unary: "syntax for adjunctions"
- Simple: "syntax for multivariable adjunctions"
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There is considerable overlap with Melliès-Zeilberger on type refinement and the unification of intrinsic and extrinsic types. Their "functor as a type refinement system" is the vertical view.

What is an *n*-theory?

In a syntactic 2-theory with multiple generating types, the objects of the resulting semantic 2-category are not single structured categories, but diagrams of several categories with functors and natural transformations between them. Thus, the corresponding syntactic 1-theories have several "classes" of types, one for each category. These classes of types are generally called "modes", type theory or logic with multiple modes is called "modal", and the functors between these categories are called "modalities". Thus, modal logics are particular 2-theories, to which our framework applies. (Mike Shulman)

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- Applications in computer science and in mathematics are already happening.
- What philosophy will make of it all is much harder to predict, but there are opportunities all around.

Temporal logic

Then we find two adjunctions $\sum_b e^* \dashv \prod_e b^*$ and $\sum_e b^* \dashv \prod_b e^*$.

Now consider for the moment that $C(t)$ and $D(t)$ are propositions. Then

- $\sum_b e^* C(t)$ means "there is some interval beginning at t and such that C is true at its end", i.e. $FC(t)$.
- $\prod_e b^* D(t)$ means "for all intervals ending at t, D is true at their beginning", i.e. $HD(t)$
- Hence our adjunction is $F \dashv H$.
- Similarly, interchanging b and e, we find $P \dashv G$.
- Note that we don't have to assume the classical $G\phi = \neg F \neg \phi$ and $H\phi = \neg P \neg \phi$.

[F, H, P, G are the standard temporal modalities.]

The various units and counits

- $\phi \to GP\phi$ "What is, will always have been"
- $PG\phi \to \phi$ "What came to be always so, is"
- $\phi \to HF\phi$ "What is, has always been to come"
- $FH\phi \to \phi$ "What always will have been, is"

With maps $p, q, c : Time_1 \times_{Time_0} Time_1 \to Time_1$, we can be more expressive, e.g., to capture *since* and *until*.

- $\phi S \psi := \Sigma_e(b^* \psi \times \Pi_c(ep)^* \phi)$
- $\phi U \psi := \Sigma_b(e^* \psi \times \Pi_c(ep)^* \phi)$

Such a map of predicates on $Time_0$ as $F := \sum_b e^*$, is a form of integral transform. Because base change has a right adjoint, it shares the property with dependent sum of preserving sums. This allows the expression of a kind of linearity.

Whenever we have a span between two objects, we can transfer an object indexed over the first to one indexed over the second. Think of matrix multiplication. Here, for a linear time, imagine multiplying a *Time*-indexed column by a $Time \times Time$-indexed matrix, where all entries are 1 on and above the NW-SE diagonal, and 0 elsewhere.

Of course we may also consider types beyond propositions. $FLightningFlash(t)$ at a moment in time is the type of ‘Lightning Flashes occurring at a time in the future of t’. We are summing here rather than applying ‘or’.

We can also look at states represented by branching trees or partially ordered sets. This brings more modalities.

- Some path always...
- Some path at some point..
- Every path always...
- Every path at some point...