How to Apply Category Theory: from Physics to Epidemiology

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A very long time ago, I came to Paris to study psychoanalysis. While here I fell in love with category theory.

Naturally then I was interested to read last year this claim of Alain Connes:

> L'inconscient est structuré comme un topos.

À l'ombre de Grothendieck et de Lacan (2022)

The Two Cultures of Mathematics

Around 20 years ago, Tim introduced a version of Snow's *Two Cultures*, this time for mathematicians:

- **Theory-builders**: Grothendieck's algebraic geometry, Langlands Program, derived geometry, elliptic cohomology,...
- **Problem-solvers**: Combinatorial graph theory, e.g, Ramsey's theorem, Szemerédi's theorem, arithmetic progressions among the primes,...

> *"One can almost imagine a gathering of highly educated mathematicians expressing their incredulity at the ignorance of combinatorialists, most of whom could say nothing intelligent about quantum groups, mirror symmetry, Calabi-Yau manifolds, the Yang-Mills equation, solitons or even* **cohomology***."*

Category theory, today's topic, is very much associated with the **theory-building** culture.

It owes its origins precisely to that concept of **cohomology**, when it was devised by Eilenberg and Mac Lane in the 1940s.

Today category theory is used everywhere in:

- Algebraic topology
- Algebraic geometry
- Langlands program
- ...

Its range is extending...

> *Some time ago I expressed skepticism regarding the possibility of recovering hard analysis categorically. I am still not convinced but it seems soft analysis is already there, in this summer's course by Clausen and Scholze. (Michael Harris, personal communication on Condensed Mathematics and Complex Geometry)*

> *We would like to say that our proofs are proofs by "formal non-sense" and in particular analysis-free. (Clausen and Scholze)*

Topics today

1. What is category theory?
2. How it is applied: Logic, computing, physics,..., epidemiology

Steps in Category Theory

- Collect entities of the same kind and relate them by relevant transformations (or relations) in a **category**.
- These transformations are represented as **arrows** which compose.
- Send one **category** to another systematically via a **functor**.
- Compare **functors** between two categories via a **natural transformation**.
- Look to form approximate inverses to **functors** via **adjoint functors**, on the left and right.
- Understand **dualities** in terms of **adjoint functors**.
- Form **monads** and **comonads** on categories from **adjunctions**

Limits and colimits

Categories may possess additional structure in the form of kinds of limit and colimit:

- Terminal and initial objects
- Products and coproducts
- Equalizers and coequalizers
- Pullbacks and pushouts
- ...

Given an adjunction between two categories:

- The right adjoint preserves limits.
- The left adjoint preserves colimits.

E.g., the underlying set of a product of topological spaces or groups or ... is the product of their underlying sets.

Toposes

Some categories are very nice, havings lots of limits and colimits.

Set, the category of sets and functions is a good example.

But then there are more general categories which resemble *Set*, known as toposes.

We might look to 'improve' a category into a topos by means of the Yoneda embedding:

$$\mathcal{C} \hookrightarrow [\mathcal{C}^{op}, Set]$$

Better a 'nice' category of objects than a category of 'nice' objects.

Infinity-category theory

An even nicer world is presented by ∞-categories.

Here instead of hom-*sets* between objects we find hom-*spaces*.

Limits and colimits there behave more flexibly, just as one would want in many geometric situations.

∞-toposes provide excellent places in which to do modern geometry (and modern physics).

(This last link shows how to 'improve' ∞-categories of spaces into ∞-toposes.)

Logic

Category theory makes sense of constructions in logic.

E.g., we can form a category of propositions and entailments, where we find adjunctions such as:

$$C \& A \vdash B \text{ if and only if } C \vdash A \to B$$

B & C is the product of B and C in this category, therefore since $A \to (-)$ is a right adjoint.

Hence, $A \to (B \& C)$ must be logically equivalent to $(A \to B) \& (A \to C)$.

Quantifiers

Now consider first-order logic where we find the adjunction:

$$P(x) \vdash Q \text{ if and only if } \exists x P(x) \vdash Q$$

Existential quantification is left adjoint and so preserves coproducts (as a colimit).

Hence, $\exists x(Rx \vee Sx)$ is logically equivalent to $\exists x Rx \vee \exists x Sx$.

There is a corresponding result for the universal quantifier.

Into the 21st century

Homotopy type theory is the internal language of an ∞-topos.

(An introductory account of this language for philosophers and others is in my *Modal Homotopy Type Theory: The prospect of a new logic for philosophy*, OUP 2020)

Category theory applied to computer science

This tight connection between category theory and type theory has been strongly influenced by people working in computer science from the 1970s onwards.

In particular, through work on **functional programming languages**.

Monads and comonads model side effects and context dependence.

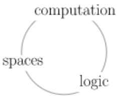
Computational Trinitarianism/Trilogy

1. Idea

A profound cross-disciplinary insight has emerged – starting in the late 1970s, with core refinements in recent years – observing that three superficially different-looking fields of mathematics,

- computation/programming languages
- formal logic/type theory
- ∞-category theory/∞-topos theory (algebraic topology)

are but three different perspectives on a single underlying phenomenon at the foundations of mathematics:



(nLab: computational trilogy)

Physics

From the 1980s there appeared the concept of a **Topological Quantum Field Theory**.

A functor from some spatial category, *n*Cob of manifolds and cobordisms, to something like *Hilb*, Hilbert spaces and linear maps (+ lots of variations).

These categories share a non-Cartesian 'monoidal' product – quantum entanglement.

String diagrammatic presentation of quantum mechanics as processes in monoidal categories.

Cf. Bob Coecke and colleagues' books:

- Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning (2017)
- Quantum in Pictures: A New Way to Understand the Quantum World (2023)

Applications of category theory

(With approximate dates)

- Mathematics (from the 1940s)
- Logic/Foundations (from the 1960s)
- Computer science (from the 1970s)
- Physics (from the 1980s)

Category Theory	Physics	Topology	Logic	Computation
object X	Hilbert space X	manifold X	proposition X	data type X
morphism $f: X \to Y$	operator $f: X \to Y$	cobordism $f: X \to Y$	proof $f: X \to Y$	program $f: X \to Y$
tensor product of objects: $X \otimes Y$	Hilbert space of joint system: $X \otimes Y$	disjoint union of manifolds: $X \otimes Y$	conjunction of propositions: $X \otimes Y$	product of data types: $X \otimes Y$
tensor product of morphisms: $f \otimes g$	parallel processes: $f \otimes g$	disjoint union of cobordisms: $f \otimes g$	proofs carried out in parallel: $f \otimes g$	programs executing in parallel: $f \otimes g$
internal hom: $X \multimap Y$	Hilbert space of 'anti-X and Y': $X^* \otimes Y$	disjoint union of orientation-reversed X and Y: $X^* \otimes Y$	conditional proposition: $X \multimap Y$	function type: $X \multimap Y$

Table 4: The Rosetta Stone (larger version)

(Baez and Stay 2009, Physics, Topology, Logic and Computation: A Rosetta Stone)

Good things continue to happen interrelating these subjects:

- See the work of Hisham Sati, Urs Schreiber et al. on Hypothesis H and Quantum Certification via Linear Homotopy Types, and some slides of mine.
- Note how *modal HoTT* contributes to the construction of useful mathematics, *twisted equivariant differential non-abelian cohomology*.
- Cf. my *Thomas Kuhn, Modern Mathematics and the Dynamics of Reason*, (PhilSci): **New logic for new mathematics for new physics**.

Applications of category theory

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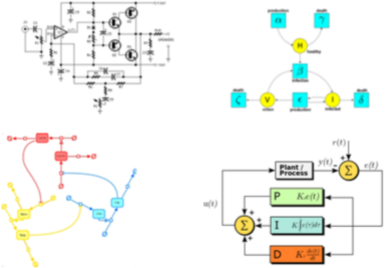
- Mathematics (from the 1940s)
- Logic/Foundations (from the 1960s)
- Computer science (from the 1970s)
- Physics (from the 1980s)
- *'Applied Category Theory'* (from the 2010s)

n-Category Café

This blog starts in 2006 with the three of us interested in higher geometry for a higher physics.

I've mentioned the high-end stuff this leads to, culminating in an approach to M-theory.

Meanwhile John Baez became intrigued by the common use of diagrams in engineering, biology, chemistry,...



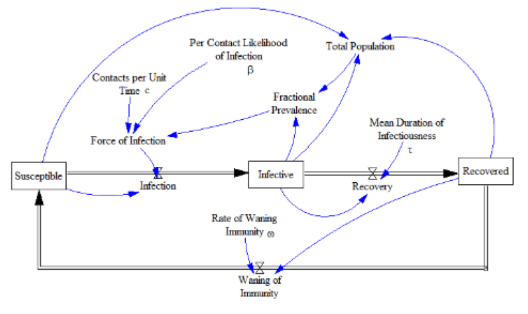
A key idea here is Compositionality.

- Plug together systems in parallel.
- Plug together systems in series.
- Plug one system inside another.

Allows the representation of: open graphs, open Petri nets, open chemical reaction with rates, open electrical circuits, open Markov processes, open dynamical systems,... (c.f. Baez)

A cutting edge development along this line is the applied category-theoretic treatment of Stock and Flow diagrams used to model disease spread and control.

- John C. Baez, Xiaoyan Li, Sophie Libkind, Nathaniel D. Osgood and Eric Redekopp, **A categorical framework for modeling with stock and flow diagrams**, to appear in *Mathematics for Public Health*, Springer.



Applied Category Theory today

We see people applying category theory for:

> *Causality, probabilistic reasoning, statistics, learning theory, deep neural networks, dynamical systems, information theory, database theory, natural language processing, cognition, consciousness, active inference, systems biology, genomics, epidemiology, chemical reaction networks, neuroscience, complex networks, game theory, robotics, quantum computing,...*

With this final entry, *quantum computing*, note that the work on *linear homotopy type theory* mentioned on slide 20 is also relevant here.

Psychoanalysis?

An exciting time to be an applied category theorist, but are we ready for psychoanalysis yet?

Alain Connes relates the function of a topos as a classifier of a geometric theory to the *fundamental fantasy*.

Another possible path:

- There has been work relating *psychoanalysis* to *active inference*.
- There has been work relating *active inference* to *category theory*.

We may conclude

- The future for the application of category theory is bright!