

Getting to Grips with Sweedler Runners for Systems

A conversational tutorial, Part XB: Analytic runners
condensed mathematics and infinite-dimensional systems

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Aim. Part X moved the runner calculus from discrete dynamics to smooth and infinitesimal systems. Part XA asked what happens when probability is moved into the ambient category itself rather than being carried only by a probabilistic residual theory. This part explores a third change of ambient world. Infinite-dimensional systems force us to confront topology, completion, unbounded operators, distributional states and analytic duality. Condensed mathematics and its solid, liquid and more recent light variants were developed precisely to make many such constructions behave more algebraically. We ask whether Sweedler theory can exploit that improvement.

The guiding example is deliberately classical: the one-dimensional heat equation with boundary control and point observation. The point is not to redevelop PDE control theory. It is to discover what a well-posed infinite-dimensional input/output system looks like from the standpoint of theories, comodels, runners and measuring objects. The main lesson will be that the appropriate operational data are not the possibly unbounded infinitesimal operators A, B, C themselves, but the finite-time maps

$$P_t, \quad \Phi_t, \quad \Psi_t, \quad \mathbb{F}_t$$

of a well-posed linear system. Their concatenation laws already have the form of a continuous-time algebraic theory.

This part is therefore partly established mathematics and partly programme. The semigroup and well-posed-systems results are classical. Their reconstruction as coalgebras for Sweedler measuring objects in a condensed or liquid duoidal category remains conjectural.

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1 Why another ambient category?

Our original examples lived in

$$\mathcal{D}_{\mathbf{Set}} = [\mathbf{Set}, \mathbf{Set}]_{\text{acc}},$$

where theories are accessible monads and stateful implementations are organised by the Sweedler operations of the duoidal endofunctor category. Part X changed the base in order to see smooth and infinitesimal structure. Infinite-dimensional analysis presents a different obstruction.

Even a linear system may naturally involve a scale

$$X_1 \subseteq X \subseteq X_{-1},$$

where X_1 is a regularity domain and X_{-1} an extrapolation space. A control operator may have the form

$$B : U \rightarrow X_{-1},$$

while an observation operator has the opposite variance in regularity,

$$C : X_1 \rightarrow Y.$$

Tensor products, continuous duals, closures, quotients and completions are no longer optional decoration. They determine what the system means.

One possible response is simply to choose a familiar category of Banach or locally convex spaces. But the categories used in classical functional analysis are often inconvenient for categorical algebra: exactness, quotients, tensor products, internal Homs and completion do not coexist as smoothly as one would like.

Condensed mathematics suggests a different strategy. Rather than force topology into an inadequately behaved category, one embeds analytic objects in larger sheaf-theoretic categories with stronger algebraic properties.

For the present programme the most relevant sequence is schematically

$$\mathbf{Set} \longrightarrow \mathbf{Cond}(\mathbf{Set}) \longrightarrow \mathbf{Cond}(\mathbf{Ab}),$$

followed by solid and liquid refinements. For size-sensitive endofunctor constructions it is likely better to work with a bounded or light version. Light condensed sets form a genuine Grothendieck topos and avoid some of the set-theoretic difficulties of the unrestricted condensed category.

Accordingly we will write

$$\mathcal{A}_{\text{an}}$$

for an as-yet-undetermined presentable analytic category: perhaps light condensed abelian groups, a suitable light solid module category, a liquid category over $\mathbb{R}$, or a closely related refinement. The aim is not merely to encode familiar Banach spaces inside $\mathcal{A}_{\text{an}}$. We want the objects

$$X, \quad X_1, \quad X_{-1}, \quad U, \quad Y, \quad \mathcal{U}_t, \quad \mathcal{Y}_t$$

and the images, quotients, tensor products and internal Homs attached to them to inhabit one categorical universe.

2 Choosing the test case carefully

Consider

$$\partial_t z(\xi, t) = \partial_\xi^2 z(\xi, t), \quad 0 < \xi < 1.$$

For the first prototype take Neumann boundary control

$$\partial_\xi z(0, t) = 0, \quad \partial_\xi z(1, t) = u(t),$$

and observe the temperature at an interior point:

$$y(t) = z(\xi_*, t).$$

This choice is deliberate. It permits us to work naturally with the base state space

$$X = L^2(0, 1)$$

while still exhibiting unbounded control and observation operators. The corresponding homogeneous Neumann Laplacian generates an analytic C_0 -semigroup

$$P_t = e^{tA}.$$

Dirichlet boundary control is a more delicate and ultimately more interesting second example. For L^2 -boundary controls the natural well-posed state space is weaker, of H^{-1} -type rather than L^2 . Thus the choice of boundary operator can force a change of state space. This will later prevent us from treating movement up and down a Sobolev scale as merely a harmless change of presentation.

The abstract form of either system is

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

but the notation conceals the analytic issue:

$$U \xrightarrow{B} X_{-1} \supseteq X \supseteq X_1 \xrightarrow{C} Y.$$

The control operator can leave the state space, while the observation operator may only be defined on a more regular subspace.

This is the first signal that the finite-dimensional formula

$$U \longrightarrow X \longrightarrow Y$$

is not the correct primitive object.

3 Free evolution and continuous time

First ignore the input and output. The free system

$$\dot{x} = Ax$$

has solutions

$$x(t) = P_t x_0,$$

where

$$P_0 = 1, \quad P_{s+t} = P_t P_s.$$

Thus the additive monoid

$$(\mathbb{R}_{\geq 0}, +, 0)$$

acts on the state.

This is the continuous-time analogue of the unary dynamics of Part I. There the theory

$$\mathbb{N}y$$

encoded an action of the free discrete time monoid $\mathbb{N}$. Here the time monoid is already given:

$$\mathbb{R}_{\geq 0}.$$

We will write provisionally

$$T_{\text{flow}}$$

for the corresponding continuous-time action theory.

In **Set**, an action of the underlying monoid forgets all continuity. In an analytic ambient category we would want the dependence on time and state to be internal to the semantics. The first question for Part XB is therefore already nontrivial:

what is the correct internal theory of a strongly continuous semigroup in $\mathcal{A}_{\text{an}}$?

For the heat equation there is additional parabolic regularisation. For $t > 0$, the operator P_t improves spatial regularity. That feature will matter for point observation, but it should not be confused with admissibility itself: admissibility of input and output histories also makes sense for systems, such as wave equations, without parabolic smoothing.

4 Timed input histories

We can make the operational theory more concrete.

For each $t \geq 0$, let

$$\mathcal{U}_t$$

be the chosen space of input histories on $[0, t]$, for example $L^2(0, t; U)$. Form the disjoint union

$$M_U = \coprod_{t \geq 0} \mathcal{U}_t.$$

An element records both a duration and an input history.

Histories concatenate:

$$\star : \mathcal{U}_s \times \mathcal{U}_t \longrightarrow \mathcal{U}_{s+t}.$$

With the empty history as unit, M_U is a monoid of timed input histories. This is the continuous-time replacement for the free word monoid A^* used for ordinary automata. The important difference is that M_U is not free. Its algebra already remembers the analytic notion of input history and time concatenation.

For a bounded control operator one has

$$\Phi_t u = \int_0^t P_{t-r} B u(r) dr.$$

For an unbounded boundary operator the same formula is interpreted through the extrapolated semigroup:

$$\Phi_t u = \int_0^t P_{t-r}^{(-1)} B u(r) dr.$$

If B is admissible, then

$$\Phi_t : \mathcal{U}_t \rightarrow X$$

is a continuous finite-time input-to-state map. Define

$$E_t(x, u) = P_t x + \Phi_t u.$$

Then running u_1 for time s and u_2 for time t gives

$$E_t(E_s(x, u_1), u_2) = E_{s+t}(x, u_1 \star u_2).$$

Equivalently,

$$\Phi_{s+t}(u_1 \star u_2) = P_t \Phi_s u_1 + \Phi_t u_2.$$

This is precisely a 1-cocycle identity for Φ relative to the semigroup action P . Thus the controlled evolution is an affine action of the timed-history monoid:

$$M_U \curvearrowright X.$$

This is our first completely explicit candidate for the equations of the controlled-evolution theory.

5 The well-posed system matrix

Control alone is not enough. In infinite dimensions, admissibility of B and C separately does not by itself determine a well-posed input/output system. One also needs a compatible input-to-output map.

A well-posed linear system is described at finite time by a block operator

$$\Sigma_t = \begin{bmatrix} P_t & \Phi_t \\ \Psi_t & \mathbb{F}_t \end{bmatrix}.$$

Here

$$P_t : X \rightarrow X$$

is free evolution,

$$\Phi_t : \mathcal{U}_t \rightarrow X$$

is the input-to-state map,

$$\Psi_t : X \rightarrow \mathcal{Y}_t$$

is the state-to-output-history map, and

$$\mathbb{F}_t : \mathcal{U}_t \rightarrow \mathcal{Y}_t$$

is the input-to-output map.

The full finite-time behaviour is therefore

$$E_t(x, u) = P_t x + \Phi_t u,$$

$$\Omega_t(x, u) = \Psi_t x + \mathbb{F}_t u.$$

This formulation is preferable to writing pointwise

$$y(s) = C E_s(x, u),$$

because C need not be defined on arbitrary elements of X . The history map Ψ_t is part of the well-posed system even when instantaneous evaluation through C is analytically singular.

The block family satisfies concatenation laws. With the evident reparametrisation of histories,

$$\Phi_{s+t}(u_1 \star u_2) = P_t \Phi_s u_1 + \Phi_t u_2,$$

$$\Psi_{s+t} x = \Psi_s x \star \Psi_t P_s x,$$

and

$$\mathbb{F}_{s+t}(u_1 \star u_2) = \mathbb{F}_s u_1 \star (\Psi_t \Phi_s u_1 + \mathbb{F}_t u_2).$$

These three equations are the natural starting point for T_{io} . We no longer need to leave the theory vague. A continuous-time input/output theory should have precisely enough operations to encode timed histories and precisely these equations of concatenation.

6 The first Sweedler design: a comodel

There are two possible places to put the dynamics in a runner construction, and they should not be conflated.

One possibility is to place the finite-time maps

$$P, \Phi, \Psi, \mathbb{F}$$

in the state coalgebra itself. Then the residual theory is trivial, or merely records the ambient additive structure. This is the direct analogue of the ordinary Moore comodel

$$T_{\text{Moore}} \rightsquigarrow \text{Id}.$$

The alternative is to choose a genuine residual flow theory

$$R_{\text{flow}}$$

and let a runner implement abstract input/output operations by residual time-passing operations. That is also legitimate, but it splits the dynamics between the residual theory and the state.

For the prototype we choose the first design. Thus the initial target is not

$$\{T_{\text{io}}, R_{\text{flow}}\}$$

but

$$\boxed{\{T_{\text{io}}, \text{Id}\}}.$$

Its coalgebras should be analytic comodels of the input/output theory. The state object is X , and its co-operations are the finite-time maps represented by the system family Σ_t .

Only after this case is understood should we ask whether passing time itself deserves to become a nontrivial residual effect. That second construction would produce a genuinely different class of runners.

7 Analytic admissibility

The infinitesimal data still matter because they explain which finite-time comodels arise from evolution equations.

A control operator

$$B : U \rightarrow X_{-1}$$

is admissible when convolution against the extrapolated semigroup produces a bounded map

$$\Phi_t : \mathcal{U}_t \rightarrow X.$$

Thus an infinitesimal control operation may leave the state object while its integrated operational effect returns to it.

An observation operator

$$C : X_1 \rightarrow Y$$

is admissible when it determines a bounded history map

$$\Psi_t : X \rightarrow \mathcal{Y}_t.$$

Crucially, this need not mean that

$$CP_s : X \rightarrow Y$$

exists as a bounded map for each individual time s . For hyperbolic systems, for example, the observation history may be well defined in L^2 even when pointwise observation is not.

For the heat equation there is an additional mechanism. Parabolic smoothing can improve regularity for each positive time, making expressions such as

$$CP_tx$$

meaningful for $t > 0$. This is stronger than mere history admissibility and should be kept conceptually separate from it.

We can therefore sharpen the provisional notion introduced earlier.

Provisional definition. An *analytic comodel* of T_{io} is a state object X equipped with a well-posed finite-time family

$$\Sigma_t = \begin{bmatrix} P_t & \Phi_t \\ \Psi_t & \mathbb{F}_t \end{bmatrix}$$

satisfying the three concatenation equations.

An *infinitesimal presentation* of such a comodel consists of operators $A, B, C, \dots$ from which the finite-time data are generated. Admissibility is the analytic condition ensuring that the infinitesimal presentation produces a genuine finite-time comodel.

This gives a clean division:

infinitesimal presentation $\xrightarrow{\text{admissibility}}$ analytic comodel.

8 The Lax–Phillips hint

There is a further clue that finite-time structure is the correct primitive.

In the Lax–Phillips style of representation one enlarges the state by adjoining input and output history spaces. A well-posed input/output system can then be represented by a genuine C_0 -semigroup on an extended space of the schematic form

$$\mathcal{U}_- \oplus X \oplus \mathcal{Y}_+.$$

From the present perspective this suggests that the controlled-evolution theory may ultimately reduce to an ordinary flow theory after the correct enlargement of state:

$$T_{\text{io}}\text{-comodel on } X \quad \rightsquigarrow \quad T_{\text{flow}}\text{-comodel on } \mathcal{U}_- \oplus X \oplus \mathcal{Y}_+.$$

We record this as a first structural conjecture.

Conjecture XB.1: history-state representation. There is a categorical form of the Lax–Phillips construction in which analytic T_{io} -comodels correspond, under suitable hypotheses, to T_{flow} -comodels on enlarged state objects carrying input and output history.

If true, this would make the theory of flow substantially more fundamental than a separate theory for each kind of input/output system.

9 Which duoidal structure?

At this point a subtlety becomes unavoidable.

For **Set**, the pointwise monoidal structure used in the endofunctor duoidal category is cartesian:

$$(F \times G)(X) = FX \times GX.$$

The interchange map with substitution uses the projections available from cartesian structure.

An analytic category $\mathcal{A}_{\text{an}}$ may possess a completed tensor product

$$\widehat{\otimes}$$

and internal Hom, but this does not imply that arbitrary accessible endofunctors admit the interchange map required to form the same duoidal structure with $\widehat{\otimes}$.

In an additive category there is, however, a safer first choice:

$$\oplus.$$

Finite products and coproducts agree, and the dynamics itself is linear on

$$X \oplus \mathcal{U}_t.$$

Thus a first analytic Sweedler category should probably use

$$(F \oplus G)(X) = FX \oplus GX$$

as its pointwise monoidal operation, together with composition. This gives a conservative programme:

$$\mathcal{D}_{\text{an}}^{\oplus} = [\mathcal{A}_{\text{an}}, \mathcal{A}_{\text{an}}]_{\text{acc}}$$

with substitution and pointwise biproduct.

The completed analytic tensor

$$\widehat{\otimes}$$

should not be silently identified with this structure. There is a second, more ambitious possibility: restrict to endofunctors carrying suitable oplax-monoidal structure for $\widehat{\otimes}$, and seek a genuinely tensor-analytic duoidal category. That would be the setting in which completed tensor products and analytic internal Homs could enter Sweedler induction and delegation directly.

We therefore separate two questions:

additive Sweedler theory:	$\oplus$ and substitution,
tensor-analytic Sweedler theory:	$\widehat{\otimes}$ with restricted endofunctors.

The first is the prototype. The second is a further conjectural layer.

10 Light condensed and liquid bases

This distinction also sharpens the choice of ambient category.

The unrestricted category of condensed sets is too large for a naive application of the locally-presentable endofunctor technology used earlier. A bounded version, and particularly the newer

category of light condensed sets, is a more natural candidate when we want a Grothendieck topos and accessible endofunctors. On the linear side we can therefore investigate categories such as

$$\text{LightCond}(\mathbf{Ab})$$

and suitable light solid module categories.

For real functional analysis another issue appears. Solid abelian groups arose largely from nonarchimedean and integral considerations; liquid real vector spaces were introduced to obtain a better archimedean analytic setting. Thus there is no reason to assume in advance that “solidification” is the correct completion operation for real control systems. The right question is instead:

which presentable analytic category contains the state, history and duality constructions required by well-posed

The heat equation is intended to answer that question experimentally rather than by terminology.

11 Reachability without throwing away the dense image

The finite-time reachable subobject is simply

$$\text{im } \Phi_t \hookrightarrow X.$$

Because the timed-history spaces and their images are compatible under extension of the time interval, one can form the total reachable subobject

$$X_{\text{reach}} = \bigcup_{t>0} \text{im } \Phi_t$$

in an appropriate sense. In finite-dimensional systems the distinction between the image and its closure disappears. In infinite dimensions it is fundamental.

The one-dimensional heat equation illustrates this dramatically. For boundary control, the exactly reachable states form a proper space of highly regular, indeed analytically extendable, functions. In the Dirichlet-controlled problem this reachable space has been identified with a Bergman-type space of holomorphic functions associated with the complex square having the interval as a diagonal. The system is nevertheless approximately controllable in the relevant state topology.

Thus

$$\text{im } \Phi \neq \overline{\text{im } \Phi}$$

is not a technical possibility but part of the running example itself. This suggests a more interesting role for condensed algebra than simply “complete the reachable space.” In an abelian condensed setting the nonclosed image can remain a genuine subobject, and the quotient

$$X / \text{im } \Phi$$

can retain information that ordinary separated topological quotients tend to erase.

One should therefore distinguish:

$$\text{exact controllability} \quad \leftrightarrow \quad \text{im } \Phi = X,$$

from

$$\text{approximate controllability} \quad \leftrightarrow \quad \overline{\text{im } \Phi} = X.$$

A condensed formulation may allow the difference between these two statements to be represented internally by the failure of the quotient

$$X / \text{im } \Phi$$

to be separated, rather than by immediately replacing the image by its closure. This is more promising than treating closure simply as an abstract reflector on the algebraically reachable subspace. The topology of the embedding

$$\text{im } \Phi \hookrightarrow X$$

matters.

There is also a specifically parabolic variant. Null controllability asks whether

$$P_t X \subseteq \text{im } \Phi_t.$$

It therefore compares two images rather than merely asking whether every state is reachable from the distinguished zero state. Even in a linear setting, where 0 supplies a canonical basepoint, the reachability problem is not exhausted by pointed reachability.

12 Observability as behaviour

The output-history operator is

$$\Psi_t : X \rightarrow \mathcal{Y}_t.$$

Passing to arbitrarily long histories gives schematically

$$\Psi_\infty : X \longrightarrow \mathcal{Y}_\infty.$$

Two states are observationally equivalent when they induce the same output history. In the linear case the unobservable subobject is therefore

$$X_{\text{unobs}} = \ker \Psi_\infty.$$

The observable quotient is

$$X / X_{\text{unobs}}.$$

This is precisely the behaviour map of Part IX in analytic dress:

$$x \longmapsto \text{its entire future observable history}.$$

The analogy is stronger than in the finite-state examples. The classical systems literature already treats the output trajectory, rather than an instantaneous observation alone, as the natural object of observability theory.

The basic pair is therefore

$$\text{im } \Phi \quad \text{and} \quad \ker \Psi_\infty.$$

Reachability is generated forward from inputs; observability quotients away states invisible to future behaviour. This is exactly the reachability–observability architecture of Part IX.

13 Duality and the master asymmetry

Finite-dimensional linear systems are unusually symmetric. For a finite-dimensional vector space,

$$X \cong X^{\vee\vee},$$

and duality exchanges controllability and observability. Kalman theory consequently makes the two sides look almost perfectly dual. This can obscure a structural asymmetry that became visible earlier in the tutorial. In **Set**, algebra and coalgebra are not simply exchanged by a self-duality of the base. The asymmetry between reachability and observability is therefore structural.

Infinite-dimensional analysis gives us a new experiment. In a Hilbert space one still has a strong adjoint theory. In reflexive Banach spaces much of the dual picture survives. In nonreflexive spaces the relationship weakens. The choice of ambient analytic category can therefore determine how much of the apparent finite-dimensional symmetry remains. This makes analytic systems an especially interesting test of the *master asymmetry*. Instead of merely observing that reachability and observability are not perfectly dual, we may be able to tune the degree of duality by changing the ambient category.

The Sweedler structure adds another layer. Its basic adjunction has the form

$$(-) \triangleright T \dashv \{T, -\}.$$

We should therefore ask whether classical adjoint-system dualities can be reconstructed from, or at least compared with, the same tensor-hom structure that produces the measuring object.

Problem XB.2. Determine how controllability–observability duality interacts with Sweedler measuring objects, and how the answer changes between Hilbert, reflexive, liquid and more general condensed settings.

The point is not to assume that Sweedler duality *is* Kalman duality. It is to understand why both are controlled by the available notion of analytic dual.

14 Feedback as analytic guardedness

A well-posed system has output

$$y = \Psi x + \mathbb{F}u$$

at the history level. Suppose we impose feedback

$$u = Ky.$$

Then the loop equation is

$$u = K(\Psi x + \mathbb{F}u).$$

Thus feedback requires invertibility of an operator of the form

$$I - K\mathbb{F}.$$

In a static finite-dimensional presentation

$$y = Cx + Du,$$

this reduces to the familiar algebraic-loop condition

$$I - KD$$

being invertible. Only when $D = 0$ does the formal expression collapse to

$$A + BKC.$$

For well-posed infinite-dimensional systems one formulates the feedback criterion on finite-time input histories. A sufficient small-time mechanism is that

$$\|K\mathbb{F}_t\| < 1$$

for sufficiently small t , so that

$$(I - K\mathbb{F}_t)^{-1}$$

exists by a Neumann series.

This has a striking resemblance to the guardedness conditions that appear when traces are only partially defined. We should therefore refine the earlier division between “formal wiring” and “analytic well-posedness.”

The more accurate slogan is

feedback is trace; analytic admissibility supplies a candidate notion of guardedness.

The identification is at present only programmatic. A future theorem would have to compare the analytic condition ensuring invertibility of

$$I - K\mathbb{F}_t$$

with the categorical guardedness required for the trace in the corresponding runner category.

If they coincide, well-posed feedback would become one of the clearest analytic interpretations of partial trace in $\mathbf{Run}(\mathcal{D})$.

15 Induction and delegation: two versions

The monoidal distinction of Section 9 changes the interpretation of induction and delegation.

In the conservative additive theory, the pointwise monoidal product is

$$\oplus.$$

The corresponding state extension therefore looks schematically like

$$X \oplus -$$

rather than a completed analytic tensor product. This is already useful: linear systems themselves compose states and input histories through direct sums. But there is a stronger possibility. Suppose one can construct a restricted duoidal category whose first monoidal product is an analytic completed tensor

$$\hat{\otimes}$$

and whose endofunctors carry the required oplax structure. Then induction and delegation could take the genuinely analytic form

$$X \hat{\otimes} - \dashv [X, -].$$

This would recover the attractive tensor/internal-Hom interpretation suggested at the beginning of the programme. We should therefore not claim this structure prematurely. Instead we record it as a separate target.

Conjecture XB.3: tensor-analytic Sweedler structure. For a suitable liquid or related analytic category $\mathcal{A}_{\text{an}}$, there is a restricted duoidal category of analytic endofunctors in which a completed tensor product and its internal Hom support the Sweedler measuring construction.

If this holds, the additive theory developed first should embed into a richer analytic runner calculus.

16 Different state spaces are not automatically equivalent

It is tempting to regard

$$H^2 \subset H^1 \subset L^2 \subset H^{-1} \subset \dots$$

as merely different presentations of the same physical system. The boundary-control example shows that this is too quick.

Changing the state space can determine whether B is admissible at all. Dirichlet boundary control is a particularly clear warning: moving from L^2 to a weaker H^{-1} -type state space is not simply aesthetic. It changes well-posedness. Nevertheless, distinct well-posed realisations can induce the same external input/output map. This is the correct point at which to ask about independence of state presentation.

Infinite-dimensional realisation theory reveals an important difference from finite dimensions. Minimal realisations need not be related by a bounded state-space isomorphism. The appropriate comparison can involve a closed densely defined intertwining map—a form of pseudo-similarity. This is exactly the analytic analogue of the minimisation problem of Part IX. In finite automata one expects a minimal behavioural realisation to be unique up to isomorphism. In infinite-dimensional systems, the correct equivalence is weaker.

This gives a particularly attractive condensed question:

Problem XB.4. Does pseudo-similarity of analytic realisations become an ordinary isomorphism, a span of morphisms, or another natural equivalence when the realisations are embedded in a suitable condensed or liquid category?

A positive answer would show that the enlarged categorical universe does not merely accommodate analytic systems; it improves their realisation theory.

17 The corrected prototype theorem

We can now state the established finite-time structure accurately.

$$\mathbb{F}_s u_1 \star (\Psi_t \Phi_s u_1 + \mathbb{F}_t u_2).$$

Admissibility of B and C separately is not by itself enough; the compatible input/output map $\mathbb{F}$ is additional well-posed-system data. These identities give the algebraic skeleton required for T_{io} . The genuinely new theorem would then be:

Target theorem XB.6. There is a presentable analytic category $\mathcal{A}_{\text{an}}$ and an associated duoidal category

$$\mathcal{D}_{\text{an}}$$

in which a continuous-time input/output theory T_{io} exists and well-posed linear systems determine coalgebras for

$$T_{\text{io}}, \text{Id}.$$

The measuring-coalgebra equations reproduce the finite-time concatenation laws above.

This is the first central mathematical target of Part XB. Only after proving it should one attempt the stronger residual-flow version

$$T_{\text{io}}, R_{\text{flow}},$$

where time evolution itself is moved into the residual theory.

18 The sequence of tests

The programme now has a clear order.

Test 1: continuous flow. Construct T_{flow} internally and verify that strongly continuous semigroups give the intended comodels.

Test 2: bounded input/output systems. Take bounded

$$B \rightarrow X, \quad C \rightarrow Y$$

and derive explicitly the block family

$$\Sigma_t.$$

Test 3: timed-history theory. Construct T_{io} from the monoid

$$M_U = \coprod_t \mathcal{U}_t$$

and the three concatenation laws.

Test 4: Neumann-controlled heat flow. Work on

$$X = L^2(0, 1)$$

with boundary control and interior point observation, and identify its analytic T_{io} -comodel.

Test 5: Dirichlet control and state-scale dependence. Move to the appropriate H^{-1} -type state space and study how the comodel changes when well-posedness forces a different analytic state object.

Test 6: exact and approximate reachability. Recover both

$$\text{im } \Phi$$

and its closure without identifying them.

Test 7: behavioural observability. Recover

$$\ker \Psi_\infty$$

and the observable quotient.

Test 8: feedback. Compare invertibility of

$$I - K\mathbb{F}_t$$

with categorical guardedness for trace.

Test 9: Lax–Phillips enlargement. Determine whether T_{io} -comodels become T_{flow} -comodels after adjoining past-input and future-output state.

This sequence moves from established mathematics to increasingly structural Sweedler questions.

19 Relation to Parts IX, X and XA

Part XB brings together several earlier themes.

Part IX studied reachability and observability abstractly. Here they become

$$\text{im } \Phi \quad \text{and} \quad \ker \Psi_\infty.$$

Unlike the finite examples, however, exact and approximate reachability need not agree, and behavioural minimisation may lead only to pseudo-similarity rather than state-space isomorphism.

Part X studied infinitesimal smooth dynamics. Here the generator A again supplies infinitesimal evolution, but unbounded B and C show that the infinitesimal presentation need not itself consist entirely of morphisms of the state object. The finite-time well-posed system is more fundamental operationally.

Part XA moved probability into the ambient category. Part XB asks the analogous question for analysis:

what becomes internal when the Sweedler universe is changed?

In XA, probabilistic structure ceases to be merely a residual effect. In XB, topology, nonclosed images, analytic history spaces and continuous duality cease to be external annotations on a set-based runner.

The progression is therefore

Set	: discrete and algebraic settings,
SmoothSet	: smooth and infinitesimal settings,
QBS-like bases	: measurable and probabilistic settings,
LightCond/Liq/...	: functional-analytic settings.

The architecture is intended to persist:

$$\text{theory} \longrightarrow \text{comodel/runner} \longrightarrow \text{behaviour},$$

while the meaning of state and morphism changes with the ambient category.

20 Beyond parabolic equations

The heat equation is intentionally only the first test.

- A wave equation would remove the parabolic smoothing used in pointwise observation. One would then see clearly that admissibility belongs fundamentally to history maps

$$\Phi_t, \quad \Psi_t,$$

rather than to pointwise composites such as CP_t .

- Delay equations would test state objects which are themselves history spaces.
- Port-Hamiltonian and passive systems would connect analytic runners with the resource and certification structures of Part V.
- Stochastic PDEs would connect XB with the probabilistic ambient categories of XA.
- Infinite-dimensional quantum systems would connect the same analytic programme with the quantum appendix.

The heat equation is therefore not privileged. It is simply the smallest example in which unbounded operators, analytic regularisation, nonclosed reachability and infinite-dimensional duality already appear together.

21 Outlook

The original temptation was to say that condensed mathematics provides “better state spaces” for Sweedler theory. The heat equation suggests something more precise.

$$\begin{bmatrix} P_t & \Phi_t & \Psi_t & \mathbb{F}_t \end{bmatrix}.$$

The passage

$$(A, B, C) \longrightarrow (P, \Phi, \Psi, \mathbb{F})$$

is governed by semigroup generation, admissibility and well-posedness. Once it has been made, the resulting data satisfy algebraic concatenation equations.

This leads to the revised central diagram:

$\begin{array}{ccc} (A, B, C) & \longrightarrow & (P, \Phi, \Psi, \mathbb{F}) \longrightarrow X \rightarrow \{T_{\text{io}}, \text{Id}\}(X) \\ \text{infinitesimal} & \text{admissible finite-time} & \text{Sweedler comodel} \end{array}$
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The first arrow belongs to analysis. The second is the proposed categorical reconstruction.

Condensed and liquid mathematics enter because infinite-dimensional behaviour remembers distinctions that ordinary algebra or separated topology easily destroys:

$$\text{im } \Phi \quad \text{versus} \quad \overline{\text{im } \Phi},$$

bounded versus densely defined state transformations, strong versus weak realisations, and several inequivalent notions of dual.

The resulting programme can therefore be stated more sharply than before:

$\text{Sweedler theory} + \text{condensed analysis} \rightsquigarrow \text{compositional analytic systems}.$
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The first decisive test is no longer merely whether the heat equation can be represented. It is whether its finite-time well-posed system can be recognised as a measuring coalgebra in a suitable analytic Sweedler universe, while retaining the distinctions between exact and approximate reachability, analytic and behavioural equivalence, and formal feedback and analytic guardedness.

Further reading

For well-posed linear systems, admissible control and observation operators, feedback and system nodes, see O. J. Staffans, *Well-Posed Linear Systems*, Cambridge University Press, 2005, and M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser, 2009. See also G. Weiss, “Admissibility of unbounded control operators,” *SIAM Journal on Control and Optimization*, 1989.

For boundary control and reachable states of the one-dimensional heat equation, see A. Hartmann, K. Kellay and M. Tucsnak, From the reachable space of the heat equation to Hilbert spaces of holomorphic functions,” *Journal of the European Mathematical Society*, 2020; K. Kellay, T. Normand and M. Tucsnak, Sharp reachability results for the heat equation in one space dimension,” *Analysis PDE*, 2022; and A. Hartmann and M.-A. Orsoni, “Separation of singularities for the Bergman space and application to control theory,” *Journal de Mathématiques Pures et Appliquées*, 2021.

For condensed and solid mathematics, see P. Scholze, *Lectures on Condensed Mathematics*. For liquid real vector spaces see P. Scholze, *Lectures on Analytic Geometry*. The more recent Clausen–Scholze theory of light condensed sets and analytic stacks is likely the appropriate starting point when presentability and size control are essential.

The reconstruction of well-posed systems as Sweedler measuring coalgebras, the comparison between analytic feedback and guarded trace, and the proposed use of condensed quotients to retain the distinction between exact and approximate reachability are research questions of Part XB rather than established results.