Getting to Grips with Sweedler Runners for Systems

A conversational tutorial, Part XC: Differential Sweedler Theory
Vector Fields, Jets and PDEs

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1 From polynomial machines to differential systems

The Polynomial Interlude gave a particularly concrete form to the Sweedler picture. In **Poly**, a comonoid is a category, so it may be read as a machine with states and elementary transitions. A monoid is a polynomial monad, hence a language of commands. A runner from a theory T to a residual theory R is then a stateful interpreter: it receives commands from T, changes its internal state, and may hand residual commands on to R.

The Interlude also uncovered a useful special case of this picture. If

$$c = \sum_{x \in C_0} y^{C[x]}$$

is the polynomial comonoid associated to a category C, then its linear dual

$$[c, y] = c \multimap y$$

is a polynomial monad. Its commands are categorical vector fields

$$v = (v_x)_{x \in C_0}, \qquad v_x : x \longrightarrow t(v_x),$$

one outgoing transition chosen at every state. But the dual remembers more than the bare set of such fields: over each field there is still one direction for every object $x \in C_0$. Thus

$$[c, y] \cong \sum_{v \in \Gamma(c)} y^{C_0}.$$

Those directions are what allow a command v to be evaluated at a particular state x.

The Interlude then considered the second dual and found

$$\boxed{c \cong M([c, y], y).}$$

So the machine is recovered from the full polynomial theory of its vector fields: not just from the list of commands, but from their statewise evaluation and substitution structure.

This part starts from that observation and changes the ambient geometry. The question is deliberately simple:

$$\boxed{\text{what becomes of this picture when the states form a smooth space?}}$$

The answer is that "vector field" stops being an analogy. The commands in the linear dual become literal smooth vector fields, and the same progression eventually leads from infinitesimal commands to jets and formally integrable PDEs.

Thus the operational reading from the earlier parts remains in force. A program still drives a machine through a runner. What changes is the kind of state the machine carries and the kind of command available to the program.

2 The tangent machine

Let $\mathbf{H}$ be a well-adapted model of synthetic differential geometry, such as the Cahiers topos, and let $X \in \mathbf{H}$ be microlinear. Write

$$TX := X^D$$

for the synthetic tangent bundle and

$$\pi : TX \longrightarrow X$$

for evaluation at $0 \in D$.

The fibres $T_x X$ carry their usual synthetic linear structure. Fibrewise addition and the zero section therefore make

$$TX \rightrightarrows X$$

an internal category by taking

$$s = t = \pi,$$

the zero vector as identity, and

$$u \circ v := u + v$$

for tangent vectors over the same point. One may think of this as a very local machine: its states are points of X, and its elementary transitions are infinitesimal displacements based at those points.

Shapiro and Spivak show that polynomial comonoids in a finite-limit base are internal categories whose source map is exponentiable [3]. Since a topos is locally cartesian closed, the tangent projection is exponentiable. Thus the tangent group bundle determines a polynomial comonoid

$$c_X = \sum_{x:X} y^{T_x X}$$

in the internal polynomial category $\mathbf{Poly}_{\mathbf{H}}$.

This is the smooth analogue of the machine c from the Polynomial Interlude. We can therefore ask the same question as before: what command language is obtained by taking its linear dual?

3 The linear dual of the tangent comonoid

Now take the linear dual of the tangent machine. Internally, the position object of

$$[c_X, y]$$

is the dependent product

$$\prod_{x:X} T_x X = \Gamma_X(TX),$$

the internal object of sections of π. Its global elements are the smooth vector fields on X. Write

$$\mathfrak{X}(X) := \Gamma_X(TX).$$

The directions over each position are still the points of X. Hence, internally,

$$\boxed{[c_X, y] \cong \sum_{v \in \mathfrak{X}(X)} y^X.}$$

So the command language of the tangent machine has literal smooth vector fields as its operations, and the polynomial directions record where a field is used: a command v is evaluated at a current state $x \in X$. The primitive interaction is evaluation,

$$(v, x) \longmapsto v(x) \in T_x X.$$

Before computing the monad structure on $[c_X, y]$, it pays to ask where that structure comes from. A polynomial over the terminal object forgets which state a transition starts from; the directions y^X have to put that information back. Working instead over the object of states keeps it throughout. The next section sets up this indexed picture for an arbitrary internal category, and then recovers both c and $[c, y]$ from it.

4 Indexed machines: moves and responses

Let $\mathbb{C} = (s, t \colon C_1 \rightrightarrows C_0)$ be an internal category in a topos $\mathcal{E}$, which may be **Set** or **H**. Work in the slice $\mathcal{E}_{/C_0}$, whose objects are families $Z = (Z_x)_{x : C_0}$ indexed by states. Base change along s and t gives adjoint triples $s_! \dashv s^* \dashv s_*$ and $t_! \dashv t^* \dashv t_*$. Define

$$\boxed{\mathsf{F}_{\mathbb{C}} := s_!\, t^*, \qquad \mathsf{G}_{\mathbb{C}} := s_*\, t^*.}$$

In elements,

$$(\mathsf{F}_{\mathbb{C}} Z)_x = \coprod_{i \colon x \to u} Z_u, \qquad (\mathsf{G}_{\mathbb{C}} Y)_x = \prod_{i \colon x \to u} Y_u.$$

An element of $\mathsf{F}_{\mathbb{C}} Z$ at x is a *move*: an outgoing transition i together with a value at its target. An element of $\mathsf{G}_{\mathbb{C}} Y$ at x is a *response table*: a value prepared at the target of every outgoing transition.

Proposition 1. *$\mathsf{F}_{\mathbb{C}}$ is a monad and $\mathsf{G}_{\mathbb{C}}$ is a comonad on $\mathcal{E}_{/C_0}$. The unit of $\mathsf{F}_{\mathbb{C}}$ chooses identity transitions, and its multiplication composes,*

$$(i, (j, z)) \longmapsto (j \circ i,\ z).$$

The counit of $\mathsf{G}_{\mathbb{C}}$ reads off the identity entry, and its comultiplication is

$$(y_k)_k \longmapsto \big((y_{j \circ i})_j\big)_i.$$

Proof. The span $C_0 \xleftarrow{s} C_1 \xrightarrow{t} C_0$ is a monad in the bicategory of spans, and $s_! t^*$ is its image under the pseudofunctor from spans to categories. For $\mathsf{G}_{\mathbb{C}}$, apply the same argument to the opposite category: $t_! s^*$ is a monad, and its right adjoint $s_* t^*$ is therefore a comonad. □

The two structures interact through a canonical pairing. Define

$$\psi_{Z,Y} \colon \mathsf{F}_{\mathbb{C}} Z \times \mathsf{G}_{\mathbb{C}} Y \longrightarrow \mathsf{F}_{\mathbb{C}}(Z \times Y), \qquad \big((i, z), (y_k)_k\big) \longmapsto \big(i, (z, y_i)\big).$$

A move along i meets a response table and collects the response prepared for i.

Proposition 2. *ψ is a Sweedler measuring of $\mathsf{F}_{\mathbb{C}}$ by $\mathsf{G}_{\mathbb{C}}$ into $\mathsf{F}_{\mathbb{C}}$.*

Proof. The unit law holds because both sides use the identity transition at x. For the multiplication law, take a nested move $(i, (j, z))$. Composing first and then pairing gives $(j \circ i, (z, y_{j \circ i}))$. Duplicating the table first, the outer pairing extracts $(y_{j \circ k})_k$ at i, the inner pairing then extracts its j-entry $y_{j \circ i}$, and multiplication composes the moves. The results agree. □

We call ψ *transparent*: a move along i is answered by the entry for that same i, and the residual move is again i.

Recovering the polynomial machine and its dual. Let $p\colon C_0 \to 1$, with base-change triple $p_! \dashv p^* \dashv p_*$. Conjugating a comonad by $p_! \dashv p^*$ gives a comonad on $\mathcal{E}$, and conjugating a monad by $p^* \dashv p_*$ gives a monad on $\mathcal{E}$.

Proposition 3. *There are isomorphisms*

$$\boxed{c \;\cong\; p_!\,\mathsf{G}_{\mathbb{C}}\,p^*, \qquad [c,y] \;\cong\; p_*\,\mathsf{F}_{\mathbb{C}}\,p^*,}$$

of comonads and monads respectively, where $c = \sum_{x\in C_0} y^{C[x]}$ is the polynomial comonoid of $\mathbb{C}$.

Proof. For the comonad, $(p_!\mathsf{G}_{\mathbb{C}}p^*)(Y) = \sum_x Y^{C[x]} = c(Y)$. The transported comultiplication sends (x,g) to $\big(x,\ i \mapsto (t(i),\ j \mapsto g(j \circ i))\big)$, which is the comonoid structure of $\mathbb{C}$ in **Poly**.

For the monad, an element of $(p_*\mathsf{F}_{\mathbb{C}}p^*)(X) = \prod_x \coprod_{i\colon x\to u} X$ is a pair (v,f): a vector field $v = (v_x)_x$ and a function $f\colon C_0 \to X$. This is $[c,y](X) = \sum_v X^{C_0}$. A nested element consists of an outer field v and, at each x, an inner pair $(W(x), g_x)$. The transported multiplication first evaluates the inner family at the target $t(v_x)$, using the counit $p^*p_* \to 1$, and then composes in $\mathsf{F}_{\mathbb{C}}$. The result is the field

$$\boxed{\mu(v,W)_x = W(x)_{t(v_x)} \circ v_x}$$

with value $g_x(t(v_x))$, so the direction map is $x \mapsto (x, t(v_x))$. The unit is the identity field $\eta(*) = (1_x)_{x\in C_0}$. □

So the multiplication of the vector-field monad, which looks somewhat ad hoc when computed directly in **Poly**, is simply composition of moves, transported along $C_0 \to 1$. The directions y^{C_0} of $[c,y]$ are exactly the state index that p_* would otherwise forget.

For the tangent group category, $t(v(x)) = x$ and composition is fibrewise addition. Hence

$$\boxed{\mu(v,W)(x) = v(x) + W(x)(x).}$$

Writing $\widehat{W}(x) := W(x)(x)$,

$$\boxed{\mu(v,W) = v + \widehat{W}.}$$

Ordinary addition is the special case in which W is constant. Thus the tangent-group linear dual is the monad of vector fields under diagonal addition.

The evaluation pairing is transported in the same way. The composite

$$p_*\mathsf{F}p^*X \times p_!\mathsf{G}p^*Y \;\cong\; p_!\big(p^*p_*\mathsf{F}p^*X \times \mathsf{G}p^*Y\big) \longrightarrow p_!\big(\mathsf{F}p^*X \times \mathsf{G}p^*Y\big) \xrightarrow{p_!\psi} p_!\mathsf{F}p^*(X\times Y) \longrightarrow X \times Y$$

uses Frobenius reciprocity, then the counit, then ψ, and finally forgets the residual transition. It sends $\big((v,f),(x,g)\big)$ to $\big(f(x), g(v_x)\big)$. This is precisely the pairing of $[c,y]$ with c into y: evaluate the field at the current state, and read the response along the chosen transition.

The indexed picture is primary. Over the object of states, an internal category gives a monad of *moves* and a comonad of *response tables*, linked by a transparent measuring. The polynomial comonoid c, the vector-field monad $[c,y]$ and their evaluation pairing are all obtained by transporting this structure along $C_0 \to 1$.

5 Reflexivity in ordinary polynomial categories

The polynomial calculation raises a natural double-dual question. For a polynomial comonoid c, is the canonical map

$$c \longrightarrow M([c, y], y)$$

an isomorphism?

In ordinary **Poly** over **Set**, the answer is yes for categorical comonoids. Equivalently, the linear-dual functor

$$[-, y] : \mathbf{Comon}(\mathbf{Poly})^{\mathrm{op}} \longrightarrow \mathbf{Mon}(\mathbf{Poly})$$

is fully faithful.

Theorem 1 (Full faithfulness in **Set**)**.** *Let c, d be polynomial comonoids corresponding to small categories C, D. Precomposition induces a natural bijection*

$$\mathbf{Cof}(D, C) \cong \mathbf{Mon}([c, y], [d, y]).$$

Naturally in d, this says that c represents the functor defining the Sweedler dual of $[c, y]$. Hence

$$\boxed{M([c, y], y) \cong c.}$$

Proof. Write

$$[c, y] = \sum_{v \in V_C} y^{C_0}, \qquad [d, y] = \sum_{w \in V_D} y^{D_0},$$

where V_C, V_D are the sets of vector fields. A polynomial map

$$\alpha : [c, y] \to [d, y]$$

consists of a map on positions $v \mapsto \alpha v$ and, for each $v \in V_C$, a backward direction map

$$\phi_v : D_0 \to C_0.$$

The direction part of compatibility with monad multiplication gives

$$\phi_{\mu(v, W)} = \phi_v.$$

Taking v to be the identity field and W to be the constant family with value u gives

$$\mu(1, W) = u.$$

Hence

$$\phi_u = \phi_1$$

for every u, without any extension-by-cases argument. Put

$$F := \phi_1 : D_0 \to C_0.$$

The remaining direction equation says

$$t_C(v_{F(d)}) = F\big(t_D((\alpha v)_d)\big).$$

The position part of multiplication shows that $(\alpha v)_d$ depends only on the single arrow $v_{F(d)}$. This is the first place where the Set proof uses a case-split: if v, v' agree at $F(d)$, choose a family of vector fields whose diagonal is v and whose value at $F(d)$ is v'. Given an arrow

$$f : F(d) \to c'$$

in C, extend f to a vector field $\bar{f}$ by identities elsewhere and define

$$f^\sharp_d := (\alpha \bar{f})_d.$$

This is independent of the chosen extension, has source d, and satisfies

$$F(t_D(f^\sharp_d)) = c'.$$

Preservation of the monad unit gives

$$(1_{F(d)})^\sharp_d = 1_d,$$

and the position part of preservation of multiplication gives

$$(g \circ f)^\sharp_d = g^\sharp_{t(f^\sharp_d)} \circ f^\sharp_d.$$

Thus $(F, (-)^\sharp)$ is precisely a cofunctor $D \to C$.

Conversely, a cofunctor $D \to C$ defines a monad map by

$$(\alpha v)_d = (v_{F(d)})^\sharp_d, \qquad \phi_v = F,$$

and the cofunctor identity and composition laws are exactly the unit and multiplication laws for the monad map. The two constructions are inverse. □

The proof uses extension-by-cases in exactly two places: to show that $(\alpha v)_d$ depends only on $v_{F(d)}$, and to extend a single arrow f at $F(d)$ to a global field when defining $f^\sharp_d$. Both steps reconstruct *per-state* information from *global* fields. These are the steps that an internal smooth proof must replace.

The theorem should be read as a statement about the full polynomial theory, not merely its set of positions: a category is reconstructed from the evaluation directions and substitution law of all its vector fields. We have not found this full-faithfulness statement in this explicit form in the sources consulted, so no novelty claim is made here.

The indexed counterpart. The indexed picture of the previous section never forgets the state, so it suggests a version of reflexivity that needs no extension of arrows to global fields. Work in $\mathbf{Set}_{/C_0}$, and let δ_u be the family that is a singleton at u and empty elsewhere.

Proposition 4. *Natural transformations $\mathsf{F}_{\mathbb{C}} Z \times DY \to \mathsf{F}_{\mathbb{C}}(Z \times Y)$ correspond to natural transformations $D \to [\mathsf{F}_{\mathbb{C}}, \mathsf{F}_{\mathbb{C}}]$, where*

$$[\mathsf{F}_{\mathbb{C}}, \mathsf{F}_{\mathbb{C}}](Y)_x \;\cong\; \prod_{i\colon x \to u} \; \coprod_{j\colon x \to u} Y_u.$$

Proof. $(\mathsf{F}_{\mathbb{C}} Z)_x = \coprod_{i\colon x \to u} \mathbf{Set}_{/C_0}(\delta_u, Z)$ is a coproduct of representables, so Yoneda gives the product over i of $\mathsf{F}_{\mathbb{C}}(\delta_u \times Y)_x$. Since $\delta_u \times Y$ is concentrated at u, this is $\coprod_{j\colon x \to u} Y_u$. □

In the formula, i is the move being interpreted and j is the move the interpretation actually makes. When there is at most one transition between any two states, the inner coproduct is trivial.

Proposition 5 (Thin categories)**.** *If $\mathbb{C}$ is thin, then $\mathsf{G}_{\mathbb{C}}$ is the universal measuring comonad of $\mathsf{F}_{\mathbb{C}}$ into itself:*

$$\boxed{\{\mathsf{F}_{\mathbb{C}}, \mathsf{F}_{\mathbb{C}}\} \cong \mathsf{G}_{\mathbb{C}}.}$$

Proof. By Proposition 4, $[\mathsf{F}_{\mathbb{C}}, \mathsf{F}_{\mathbb{C}}] \cong \mathsf{G}_{\mathbb{C}}$ when $\mathbb{C}$ is thin. Let D be a comonad with a measuring ψ_D, and let $\varphi\colon D \to \mathsf{G}_{\mathbb{C}}$ be the classifying map. The unit law for ψ_D, evaluated at identities, gives $\varepsilon\varphi = \varepsilon_D$. The multiplication law, evaluated on $(i,(j,z))$, gives $\varphi(e)_{j\circ i} = \varphi\big(\varphi(\delta_D e)_i\big)_j$, which is the comonad-morphism condition. So φ is the unique comonad map through which ψ_D factors. □

For categories that are not thin, the universal measuring comonad is larger, because a measuring may reinterpret moves.

Example. Let $\mathbb{C}$ be the one-object category of the monoid $(\mathbb{N}, +)$. Then $\mathsf{F}_{\mathbb{C}} = \mathbb{N} \times -$ is the unary tick monad, $\mathsf{G}_{\mathbb{C}} = (-)^{\mathbb{N}}$, and the transparent measuring is $((n,z),f) \mapsto (n,(z,f(n)))$. For each $k \in \mathbb{N}$,

$$\psi_k\big((n,z),y\big) = \big(kn,(z,y)\big)$$

is a measuring with the identity comonad as capacity, because $n \mapsto kn$ is a monoid endomorphism. A factorisation through $\mathsf{G}_{\mathbb{C}}$ would have to return the move n, so ψ_k does not factor for $k \geq 2$. Operationally, ψ_k executes each tick as k ticks.

This is consistent with the theorem above. A monoid map $[c,y] \to [d,y]$ corresponds to a cofunctor, and $n \mapsto kn$ is a cofunctor from the one-object category to itself. The polynomial theorem concerns $[c,y]$ against all comonoids d; the indexed proposition concerns $\mathsf{F}_{\mathbb{C}}$ against itself. The second is sharper and correspondingly more restrictive.

6 The internal reflexivity problem

For an internal category in a topos, the Set proof does not internalize verbatim: its two case-splits require a way to replace a field locally while controlling its diagonal values. Mere epimorphic evaluation

$$\mathrm{ev} : \mathfrak{X}(X) \times X \longrightarrow TX$$

provides extensions of individual tangent vectors, but it does not by itself supply the required interpolating families of vector fields.

For ordinary finite-dimensional manifolds there is a stronger and very concrete replacement.

Finite global-frame hypothesis. Assume that the tangent bundle is a retract of a finite trivial bundle *as a vector bundle*:

$$TX \xrightarrow{j} X \times \mathbb{R}^N \xrightarrow{p} TX, \qquad pj = 1_{TX},$$

with j and p fibrewise linear. Put

$$e_i := p(-, \mathbf{e}_i), \qquad g^i := \mathrm{pr}_i \circ j \circ g$$

for a vector field g. Then

$$g = \sum_i g^i e_i,$$

and, crucially,

$$g(x_0) = 0 \quad \Longrightarrow \quad g^i(x_0) = 0 \quad \text{for every } i,$$

because j is fibrewise linear. Ordinary finite-dimensional manifolds satisfy this hypothesis after embedding in some $\mathbb{R}^N$ and using an orthogonal projection from the trivial ambient tangent bundle onto TM.

This hypothesis replaces both case-splits in the Set proof. First, a tangent vector is extended using the splitting. Second, if two fields u, u' agree at x_0, put $g = u' - u$ and use the coefficients supplied by j:

$$g(z) = \sum_i g^i(z) e_i(z).$$

Then the family

$$W(x)(z) := u(z) + \sum_i (g^i(z) - g^i(x)) e_i(z)$$

has diagonal $W(x)(x) = u(x)$ and satisfies $W(x_0) = u'$, because $g(x_0) = 0$ implies $g^i(x_0) = 0$ for every i. Thus the two local replacements needed by the Set proof are available internally.

This yields the following expected smooth analogue.

Conjecture 1 (Internal reflexivity under a finite global frame)**.** *Let c_X be the tangent-group polynomial comonoid of a smooth object X in a well-adapted smooth topos. If TX is a retract of a finite trivial bundle, then for every internal categorical polynomial comonoid d, precomposition induces*

$$\mathbf{Cof}(d, c_X) \cong \mathbf{Mon}([c_X, y], [d, y]).$$

Consequently, once the corresponding internal measuring object is constructed,

$$M([c_X, y], y) \cong c_X.$$

For ordinary manifolds the finite-frame hypothesis is available, so this gives a concrete route to the internal proof. For arbitrary microlinear objects, however, the needed replacement property remains open. Notice that the hypothesis falls only on the target comonoid c_X; the Set proof uses no corresponding assumption on d.

An indexed route. The indexed picture suggests a second strategy, which separates the two difficulties. In the slice $\mathbf{H}_{/C_0}$ the state is never forgotten, so no field ever has to be extended from a single state. The proof of Proposition 5 uses only the Yoneda lemma for the representables δ_u, and internally these are replaced by the generic point of C_0. This suggests:

Conjecture 2 (Indexed internal reflexivity)**.** *Let $\mathbb{C}$ be a thin internal category in $\mathbf{H}$. Then, within strong endofunctors of $\mathbf{H}_{/C_0}$, the response comonad $\mathsf{G}_{\mathbb{C}}$ is the universal measuring comonad of the move monad $\mathsf{F}_{\mathbb{C}}$ into itself.*

Proposition 3 then transports the indexed structure along $C_0 \to 1$. On this view the extension problem is relocated rather than removed. It reappears as the question of whether $p_* \mathsf{F}_{\mathbb{C}} p^*$ has enough global sections to separate the data seen fibrewise. The finite global frame is one sufficient condition for this. A sharper formulation would identify the exact condition under which the comparison between the transported universal measuring and the universal measuring of the transported monad is invertible.

Which category is the right test case? The tangent group category $TX \rightrightarrows X$ is *not* thin, since $s = t$ and each fibre T_xX contains many endomorphisms of x. By Proposition 4, a measuring of F into itself with the identity comonad as capacity is then the same thing as a fibrewise map $\phi_x \colon T_xX \to T_xX$. The unit and multiplication laws force

$$\phi_x(0) = 0, \qquad \phi_x(v + w) = \phi_x(v) + \phi_x(w).$$

So every fibrewise additive endomorphism of the tangent bundle, for instance $v \mapsto kv$, gives a non-transparent measuring that does not factor through G. In indexed form, then, the tangent group category is not reflexive in the sense of Proposition 5: its universal measuring comonad also contains fibrewise linear reinterpretations of tangent vectors.

The infinitesimal pair groupoid of Section 7 is thin. It is therefore the natural first case for the indexed conjecture, and its response comonad turns out to be the jet comonad.

7 Infinitesimal transformations, the Lie bracket and jets

The tangent-group comonoid captures fibrewise linear tangent structure, but it does not intrinsically contain the Lie bracket. Since $s = t$, it supplies no transport between different tangent fibres. The bracket belongs instead to the full infinitesimal pair groupoid

$$X \times_{\Im X} X \rightrightarrows X,$$

which is a genuine internal groupoid because it is a kernel pair. Being a kernel pair, it is a subobject of $X \times X$, and so it is thin: there is at most one transition between two states, namely the pair itself. Let c_X^∞ be its polynomial comonoid.

A position of

$$[c_X^\infty, y]$$

is a map $\sigma : X \to X$ with $\sigma(x)$ infinitesimally close to x. First-order vector fields determine such positions: for $\epsilon \in D$, write locally

$$\Phi_v^\epsilon(x) = x + \epsilon v(x).$$

Grading. The first-order neighbourhood is not closed under composition. Composing two first-order displacements gives $x + \epsilon v + \delta w + \epsilon\delta(\cdots)$, whose square need not vanish, although its cube does. More generally, write $X^{(k)} \subseteq X \times X$ for the k-th order neighbourhood of the diagonal. Then

$$X^{(k)} \circ X^{(l)} \subseteq X^{(k+l)},$$

so the neighbourhoods form a graded internal category, with the all-order groupoid as their union. Correspondingly, the move monad and response comonad are graded:

$$\mathsf{F}_k\mathsf{F}_l \longrightarrow \mathsf{F}_{k+l}, \qquad \mathsf{G}_{k+l} \longrightarrow \mathsf{G}_k\mathsf{G}_l,$$

and the transparent measuring respects the grading. The all-order groupoid is exactly what retains every composite.

The bracket. For an arbitrary internal category, constant inner fields turn the multiplication

$$\mu(v, W)_x = W(x)_{t(v_x)} \circ v_x$$

into ordinary composition. Hence in the infinitesimal pair groupoid

$$\mu(\sigma, \underline{\tau}) = \tau \circ \sigma.$$

In particular the position monoid contains the infinitesimal transformations Φ_v^ϵ. For these positions one has

$$(\Phi_v^\epsilon)^{-1} = \Phi_v^{-\epsilon},$$

so their group commutator is formed directly in the position monoid from constant-W Sweedler substitution and these inverses. With the rightmost map applied first, synthetic Lie theory gives

$$\boxed{\Phi_w^{-\delta}\Phi_v^{-\epsilon}\Phi_w^{\delta}\Phi_v^{\epsilon} = \Phi_{[v,w]}^{\epsilon\delta},}$$

where

$$[v, w]^i = \sum_j \left(v^j \partial_j w^i - w^j \partial_j v^i \right)$$

in local coordinates; see Kock, *Synthetic Differential Geometry*, Part I, §I.9 [4]. Related Lie-theoretic infinitesimal structure is developed in [5, §§6.6–6.8]. The bracket uses only the position monoid obtained from constant-W substitution; the full state-dependent monad multiplication contains additional structure.

Moves, responses and jets. Now apply the indexed construction to the pair groupoid, with $\eta\colon X \to \Im X$ the unit of infinitesimal shape. Its source and target are the two projections of the kernel pair of η. When η is epimorphic, as assumed in the next section, it is effective, and the Beck–Chevalley condition for the kernel-pair square gives

$$\boxed{\mathsf{F}_\infty \cong \eta^*\eta_!, \qquad \mathsf{G}_\infty \cong \eta^*\eta_* = J_X^\infty.}$$

The move monad is the infinitesimal-disk construction: a move at x is an infinitesimally nearby point together with a value there. The response comonad is the *infinite jet comonad* introduced in the next section: a response table at x is a value at every infinitesimally nearby point, that is, a jet.

Three consequences follow immediately.

(i) The transparent measuring

$$\eta^*\eta_! E \times J_X^\infty F \longrightarrow \eta^*\eta_!(E \times_X F)$$

evaluates a jet at an infinitesimally nearby point. It is the canonical measuring between infinitesimal moves and jets.

(ii) Because the pair groupoid is a groupoid, inversion exchanges s and t. The response comonad s_*t^* is therefore isomorphic to t_*s^*, the right adjoint of $s_!t^*$. So here, and unlike for a directed category, the measuring partner of the move monad is also its right adjoint:

$$\eta^*\eta_! \dashv \eta^*\eta_*.$$

(iii) By Proposition 3, the polynomial comonoid of the pair groupoid is the jet comonad with its base index summed out:

$$\boxed{c_X^\infty \;\cong\; p_!\, J_X^\infty\, p^*, \qquad c_X^\infty(Y) = \sum_{x:X} Y^{(X \times_{\Im X} X)_x}.}$$

Thus $c_X^\infty(Y)$ is the total space of the infinite jet bundle of the trivial bundle $X \times Y \to X$. Likewise $[c_X^\infty, y] \cong p_* \eta^* \eta_! p^*$.

Since the pair groupoid is thin, Proposition 5 together with the indexed conjecture of the previous section predicts

$$\{\eta^*\eta_!, \eta^*\eta_!\} \cong J_X^\infty :$$

jets would be the universal capacity for answering infinitesimal moves. This is the cleanest internal test of reflexivity available, and it involves no frame hypothesis.

The two comonoids therefore play complementary roles:

$TX \rightrightarrows X$	linear tangent structure and addition of vector fields,
$X \times_{\Im X} X \rightrightarrows X$	composition of infinitesimal motions, the Lie bracket and jets.

The first is not thin, and its indexed measurings include linear reinterpretations of tangent vectors. The second is thin, and its response comonad is the jet comonad, which is the subject of the next section.

8 Jets and differential operators

The bracket is first order in each argument. Hence there is a bundle map

$$\beta : J^1(TX) \times_X J^1(TX) \longrightarrow TX$$

with

$$\beta(j^1_x v, j^1_x w) = [v, w](x).$$

This points naturally toward the jet comonad.

Khavkine and Schreiber formulate the formal geometry of PDEs in a differentially cohesive topos [1]. For a base Σ they construct the infinite jet comonad

$$J^\infty_\Sigma : \mathbf{H}_{/\Sigma} \longrightarrow \mathbf{H}_{/\Sigma}.$$

Its co-Kleisli category encodes nonlinear differential operators, while its Eilenberg–Moore category is the category of formally integrable PDEs:

$$\boxed{\mathbf{PDE}_\Sigma \simeq \mathbf{EM}(J^\infty_\Sigma).}$$

Their notion includes open differential conditions as well as equality-defined classical diffieties.

Whenever the unit of the infinitesimal-shape modality

$$\eta_\Sigma : \Sigma \longrightarrow \Im\Sigma$$

is epimorphic, they further obtain

$$\boxed{\mathbf{PDE}_\Sigma \simeq \mathbf{H}_{/\Im\Sigma}.}$$

This hypothesis holds in the principal examples of interest; the equivalence is not specific to the Cahiers topos.

Giotopoulos, Khavkine, Sati and Schreiber subsequently prove the previously tacit preservation of the projective limits defining infinite jet bundles under passage to the Cahiers topos [2]. Thus the synthetic infinite jets used here agree with the expected reduced infinite-jet objects.

Since the bracket is first order, it determines a co-Kleisli differential operator

$$J^\infty_X(TX \times_X TX) \longrightarrow TX$$

which factors through first jets.

9 A Sweedler universe of formally integrable PDEs

The discussion so far has mostly used the residual theory y, because that is the setting of the linear double dual from the Polynomial Interlude. This should not be mistaken for a restriction to closed systems. The general loose arrow is still a runner

$$T \rightsquigarrow R,$$

and its measuring object is $M(T, R)$. Operationally, such a runner interprets a T-program statefully while being allowed to delegate part of the execution to the residual language R.

This is especially natural for differential systems. A high-level theory T might describe controller commands, while R describes lower-level plant operations such as boundary forcing, source terms, or actuator commands. The state carried by the runner may itself be a formally integrable PDE. In that case the old programming-language picture survives unchanged:

$$\boxed{\text{controller program} \longrightarrow \text{stateful differential runner} \longrightarrow \text{residual plant program.}}$$

The case $R = y$ considered above is simply the residue-free boundary case.

Fix Σ and put

$$\mathcal{C}_\Sigma := \mathbf{PDE}_\Sigma \simeq \mathbf{EM}(J^\infty_\Sigma).$$

Under the epimorphic-unit hypothesis,

$$\mathcal{C}_\Sigma \simeq \mathbf{H}_{/\Im\Sigma}$$

is a Grothendieck topos and hence locally presentable. This makes

$$\mathcal{D}_\Sigma := [\mathcal{C}_\Sigma, \mathcal{C}_\Sigma]_{\mathrm{acc}}$$

a natural accessible-endofunctor universe. The remaining structural check is whether it carries the duoidal structure needed for the Sweedler runner construction.

The cartesian structure of $\mathcal{C}_\Sigma$ has a direct systems interpretation. Products are fibre products over $\Im\Sigma$, coupling formally integrable systems over the same independent variables. A parameter object A enters as the constant family

$$\Sigma \times A$$

with trivial Cartan variation: its elements are parameters with vanishing derivatives. Thus a store of constant parameters is itself a simple PDE object. Exponentials exist because $\mathcal{C}_\Sigma$ is a topos, although their interpretation as local classical PDE constructions requires separate analysis.

If the full Sweedler hypotheses hold, a runner internal to $\mathcal{C}_\Sigma$ acts on equations together with all their differential consequences. A morphism

$$f : E \longrightarrow F$$

of PDE objects satisfies

$$\boxed{J^\infty_\Sigma(f) \circ \rho_E = \rho_F \circ f.}$$

Thus Sweedler execution commutes with infinite prolongation.

10 A first PDE test: scalar transport

Take

$$\Sigma = \mathbb{R}^2_{t,x}$$

and the trivial line bundle

$$Y = \Sigma \times \mathbb{R}_u \longrightarrow \Sigma.$$

Fix $a \in \mathbb{R}$ and consider

$$\boxed{u_t + a u_x = 0.}$$

Write

$$u_{m,n} := \frac{\partial^{m+n} u}{\partial t^m \partial x^n}.$$

Infinite prolongation gives

$$\boxed{F_{m,n} := u_{m+1,n} + a u_{m,n+1} = 0, \qquad m, n \geq 0.}$$

Formal integrability is immediate:

$$D_t F_{m,n} = F_{m+1,n}, \qquad D_x F_{m,n} = F_{m,n+1}.$$

Recursively,

$$u_{m,n} = (-a)^m u_{0,m+n}.$$

Putting

$$q_n := u_{0,n},$$

we obtain

$$D_x q_n = q_{n+1}, \qquad D_t q_n = -a q_{n+1}.$$

The last equality is a convenient formula on functions of the reduced jet coordinates q_n. On E^∞_a, the total derivative vector fields satisfy the identity

$$D_t + a D_x = \partial_t + a \partial_x.$$

Denote the resulting formally integrable PDE object by E^∞_a.

11 Vertical symmetries and superposition

Vertical translation

$$\phi_c(t, x, u) = (t, x, u + c)$$

prolongs to an Eilenberg–Moore endomorphism

$$\phi_c^\infty : E_a^\infty \longrightarrow E_a^\infty.$$

Likewise scaling

$$\psi_\lambda(t, x, u) = (t, x, \lambda u)$$

preserves the prolonged equation.

The linear equation has a much richer symmetry theory. If $s(t, x)$ is any solution, then

$$u \longmapsto u + s$$

is a vertical symmetry, since

$$(\partial_t + a\partial_x)(u + s) = (\partial_t + a\partial_x)u + (\partial_t + a\partial_x)s.$$

Thus the whole additive solution space acts by superposition, with scalar multiplication $\lambda \in \mathbb{R}$ acting on top. The case $\lambda = 0$ is still an endomorphism, while $\lambda \neq 0$ gives the invertible part. Operationally, the natural symmetry object is therefore a monoid much richer than a single unary tick.

12 Finite and infinitesimal spatial translations

Finite spatial translation

$$(t, x, u) \longmapsto (t, x + h, u)$$

moves the base point of Σ, so it is not an endomorphism in the fixed slice $\mathbf{H}_{/\Sigma}$.

Its infinitesimal symmetry nevertheless has a vertical evolutionary representative. The characteristic of x-translation is

$$Q = u_x,$$

and its prolongation acts in the reduced coordinates by

$$\boxed{q_n \longmapsto q_n + \epsilon q_{n+1}.}$$

This commutes with D_x and with the prolonged transport relation, so it is a fixed-base infinitesimal symmetry of E_a^∞.

Thus generalized infinitesimal symmetries already live naturally in $\mathbf{PDE}_\Sigma$. What requires a base-varying extension is the corresponding finite geometric transformation itself:

finite base motion may leave $\mathbf{H}_{/\Sigma}$,	its evolutionary infinitesimal symmetry need not.

13 The flat object over infinitesimal shape

Under the epimorphic-unit hypothesis, a formally integrable PDE object corresponds to an object

$$\widetilde{E} \longrightarrow \Im\Sigma$$

with

$$E \simeq \Sigma \times_{\Im\Sigma} \widetilde{E}.$$

The projection $E \to \widetilde{E}$ therefore identifies jets only along *infinitesimal* Cartan displacements. The object $\widetilde{E}$ is not, in general, a global solution space: it still lies over $\Im\Sigma$, and its fibre over a base point is the space of formal solution germs there. Smooth solutions correspond instead to sections

$$\Im\Sigma \longrightarrow \widetilde{E}.$$

This corrects a tempting but overly coarse finite-leaf picture. In the smooth category, finite Cartan translation is nonlocal on an infinite jet: the formal flow involves an infinite Taylor series, whereas the infinitesimal-shape quotient only identifies nilpotent displacements.

The classical symmetry dictionary nevertheless suggests a useful interpretation. Since $T\Im\Sigma \simeq \Im\Sigma$, vector fields on $\widetilde{E}$ are fibrewise over $\Im\Sigma$. One expects flat-compatible vertical fields on $\widetilde{E}$ to correspond to evolutionary vector fields on the diffiety that commute with the total derivatives, hence to generalized symmetries modulo Cartan-trivial ones. We retain this as a conjectural dictionary.

The scalar transport equation gives a consistency check on its action on sections. Its global solutions are

$$u(t,x) = f(x - at).$$

Accordingly,

$$\Gamma_{\Im\Sigma}(\widetilde{E}) \cong C^\infty(\mathbb{R})$$

in the evident profile parametrization. The evolutionary spatial-translation field

$$q_n \longmapsto q_n + \epsilon q_{n+1}$$

induces on sections

$$f \longmapsto f + \epsilon f'.$$

Likewise, if the solution used for superposition is $s(t,x) = g(x - at)$, then the induced action on sections is

$$f \longmapsto f + g.$$

These calculations are consequences of the proposed dictionary, not yet a proof of it.

A sharper direct test is therefore to describe the fibrewise flat structure of $\widetilde{E}$ for transport. Its fibre is represented by the formal jet coordinates $(q_0, q_1, \ldots)$, and the candidate vector fields are vertical fields compatible with the shift

$$q_n \longmapsto q_{n+1}$$

and the flat Cartan structure. If these are exactly the evolutionary fields whose characteristics are smooth functions of the jet coordinates satisfying the usual commutation equations with total derivatives, then the desired symmetry dictionary follows in this example.

If, in addition, $\widetilde{E}$ is microlinear, its tangent-group polynomial comonoid

$$c_{\widetilde{E}} = \sum_{z:\widetilde{E}} y^{T_z \widetilde{E}}$$

would have as linear-dual positions the vector fields on the flat formal-germ object. The primitive evaluation pairing would then read: a generalized infinitesimal symmetry evaluated at a formal solution germ. This is the appropriate local replacement for evaluation at a global solution profile.

14 Two ways to combine Sweedler theory with jets

There are at least two distinct constructions.

Change the ambient category. Work directly in

$$\mathcal{C}_\Sigma = \mathbf{EM}(J^\infty_\Sigma)$$

and form a Sweedler universe of accessible endofunctors or enriched polynomials on this category. Formal integrability is then built into state objects and morphisms.

Lift theories through the jet comonad. Remain in

$$[\mathbf{H}_{/\Sigma}, \mathbf{H}_{/\Sigma}]_{\mathrm{acc}}$$

and ask which monads T admit distributive laws

$$T J^\infty_\Sigma \Longrightarrow J^\infty_\Sigma T.$$

Such theories lift to the Eilenberg–Moore category.

The first asks what systems theory exists intrinsically on PDEs; the second asks which smooth operational theories preserve formal integrability.

15 Open problems

The calculations above leave a focused list of open checks.

Internal reflexivity beyond manifolds. Ordinary finite-dimensional manifolds admit a fibrewise-linear retract

$$TX \xrightarrow{j} X \times \mathbb{R}^N \xrightarrow{p} TX.$$

Characterize the weaker local-replacement property actually needed by the proof, and determine how far beyond manifolds it holds for microlinear objects.

Sweedler existence in internal Poly. Establish the measuring comonoids required for the full three-operation Sweedler calculus in the relevant internal polynomial category over the Cahiers topos.

Flat formal germs and generalized symmetries. For a general PDE object $E \leftrightarrow \widetilde{E} \to \Im\Sigma$, determine when $\widetilde{E}$ is microlinear and prove that its flat-compatible vertical vector fields identify with generalized symmetries modulo Cartan-trivial symmetries. For transport, compute this directly on the formal-jet fibre and show that the shift $q_n \mapsto q_{n+1}$ induces $f \mapsto f'$ only after passing to sections.

Jet-compatible theories. Characterize monads admitting distributive laws

$$TJ^\infty \Rightarrow J^\infty T.$$

Base-varying runners. Develop a fibred version allowing the independent-variable object Σ to change, needed for finite geometric symmetries and coordinate transformations.

Infinite-dimensional control. Return to the heat-equation example of Part XB, treat it first as a jet coalgebra, and study input/control operations by runners internal to the PDE category.

16 Conclusion

The first Sweedler dual becomes genuinely differential-geometric in a smooth topos. For a microlinear X, the tangent group bundle determines

$$c_X = \sum_{x:X} y^{T_x X},$$

and

$$\boxed{[c_X, y] \cong \sum_{v \in \mathfrak{X}(X)} y^X.}$$

Its monad multiplication is

$$\boxed{\mu(v, W)(x) = v(x) + W(x)(x).}$$

This is the exact Sweedler home of linear tangent structure and vector-field addition.

The Lie bracket belongs to the full infinitesimal pair groupoid. There, constant-W Sweedler substitution becomes composition of infinitesimal transformations, and the group commutator yields the bracket. Jets then supply formal differential dependence, while the jet comonad packages formally integrable PDEs.

In ordinary **Poly**, the vector-field linear dual is fully faithful on categorical comonoids, so

$$M([c, y], y) \cong c.$$

For manifolds in the Cahiers topos, the finite-global-frame construction gives a concrete route toward the corresponding internal reflexivity theorem; for arbitrary microlinear objects this remains conditional.

Finally, the transport equation shows that generalized PDE symmetries already live inside the fixed-base jet-coalgebra category. Under the equivalence with objects over $\Im\Sigma$, the corresponding object $\widetilde{E}$ should be viewed as a flat family of formal solution germs rather than as the global solution space; smooth solutions are its sections.

The hierarchy developed in this part is therefore

$TX \rightrightarrows X$	$\rightsquigarrow$	vector fields and addition,
$X \times_{\Im X} X \rightrightarrows X$	$\rightsquigarrow$	infinitesimal composition and Lie bracket,
J^∞_Σ	$\rightsquigarrow$	differential operators and formally integrable PDEs.

Subject to the remaining duoidal existence checks, these structures feed into Sweedler runners on differential systems.

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