

Getting to Grips with Sweedler Runners for Systems

A conversational tutorial, Part X: smooth systems and differential Moore machines
Tangencies, parameterised ODEs, smooth enrichment, and the relation to Run

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Aim. Earlier parts treated a Moore machine as a state space equipped with a way of producing a next state. Continuous dynamics suggests a different reading: at a state we first choose an *infinitesimal continuation*, and trajectories arise only after these infinitesimal choices have been integrated. Synthetic differential geometry makes this literal. In a smooth topos with an infinitesimal interval D , the tangent bundle is simply

$$TX = X^D.$$

A tangent vector is therefore an infinitesimal path. This gives the differential Moore-machine picture a much more intrinsic origin, and clarifies how the tangencies of Capucci–Myers meet the Sweedler architecture.

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1 From a next state to an infinitesimal continuation

An ordinary deterministic Moore machine has data

$$S, \quad u : S \times A \longrightarrow S, \quad v : S \longrightarrow O.$$

Given a current state s and an action a , the update $u(s, a)$ chooses the next state. A control system described by an ordinary differential equation has almost the same visible interface, but its update does not jump to another point. It chooses a tangent vector based at the current point:

$$f : S \times A \longrightarrow TS, \quad \pi_S \circ f = \text{pr}_S.$$

Thus $f(s, a) \in T_s S$. If the action varies with time we obtain

$$\dot{x}(t) = f(x(t), a(t)).$$

The formal resemblance with a Moore machine is clear:

discrete Moore machine	parameterised ODE
S	S
A	A
O	O
$u : S \times A \rightarrow S$	$f : S \times A \rightarrow TS$
$v : S \rightarrow O$	$v : S \rightarrow O.$

But the right-hand column looks at first as though we have inserted a tangent functor by hand. The central question of this part is whether the tangent target can instead arise naturally from the ambient semantics.

2 Why ordinary smooth spaces are not yet enough

Finite-dimensional manifolds are too small for the categorical constructions used elsewhere in the tutorial: they are not cartesian closed, and mapping spaces need not be manifolds. A familiar enlargement is the topos of smooth spaces

$$\mathbf{SmoothSet} = \mathbf{Sh}(\mathbf{CartSp}),$$

where **CartSp** denotes Cartesian spaces with the usual smooth topology. Ordinary manifolds embed fully faithfully, while exponentials and all small limits and colimits exist [8].

This is already useful for systems theory. State spaces, parameter spaces and observation spaces may be genuinely smooth objects, and smooth maps vary in smooth families. But the passage

$$\mathbf{Man} \longrightarrow \mathbf{SmoothSet}$$

does not by itself explain why a continuous-time update should land in TS . The topos **SmoothSet** has smooth maps, but it does not contain the infinitesimal objects required for synthetic differential geometry [8].

This suggests a slightly different enlargement. Instead of stopping at smooth spaces, pass to a well-adapted model of synthetic differential geometry.

3 The Cahiers topos and the infinitesimal interval

Let $\mathcal{H}$ denote the Cahiers topos. One presentation is as the sheaf topos on formal Cartesian spaces

$$\mathbb{R}^n \times \ell W,$$

where W is a Weil algebra and ℓW is the corresponding infinitesimally thickened point [4, 9]. Thus $\mathcal{H}$ enlarges the ordinary smooth site not only by allowing generalized smooth spaces, but also by adjoining nilpotent infinitesimal directions.

The basic infinitesimal object is the dual-number interval

$$D = \{\varepsilon \in \mathbb{R} \mid \varepsilon^2 = 0\}.$$

Inside a well-adapted model of synthetic differential geometry, the tangent bundle of an object X is simply the exponential

$$TX := X^D,$$

with projection

$$X^D \longrightarrow X$$

given by evaluation at $0 \in D$ [5, 6, 9]. For an ordinary manifold embedded in $\mathcal{H}$, this synthetic tangent bundle agrees with its usual tangent bundle.

The slogan “a tangent vector is an infinitesimal path” becomes an equation:

$$TX = X^D.$$

Tangency is no longer an auxiliary structure attached to a smooth state space. It is generated by exponentiation with the infinitesimal interval already present in the ambient topos.

This changes the emphasis of the chapter. In the manifold formulation one first has a state space S and then chooses its tangent bundle. In the synthetic formulation, once the ambient smooth topos and its infinitesimal object D have been chosen, the tangent construction is canonical.

4 Differential Moore machines internally

A parameterised vector field can now be written

$$f : S \times A \longrightarrow S^D$$

over S . By exponential adjunction this is equivalent to a map

$$\tilde{f} : S \times A \times D \longrightarrow S$$

satisfying

$$\tilde{f}(s, a, 0) = s.$$

So an action a at state s does not directly choose a later state. It chooses an infinitesimal D -shaped continuation of s .

This makes the Moore analogy considerably more natural. The ordinary update

$$S \times A \rightarrow S$$

selects a successor. The differential update

$$S \times A \rightarrow S^D$$

selects an infinitesimal successor germ. Both are state-transition mechanisms; they differ in the shape of the transition that the ambient category regards as primitive.

For example, on $S = \mathbb{R}$ consider

$$\dot{x} = -x + a.$$

Classically this is represented by

$$f : \mathbb{R} \times \mathbb{R} \longrightarrow T\mathbb{R}, \quad f(x, a) = (x, -x + a).$$

Synthetically it is equivalently a map

$$\tilde{f} : \mathbb{R} \times \mathbb{R} \times D \longrightarrow \mathbb{R}, \quad \tilde{f}(x, a, \varepsilon) = x + \varepsilon(-x + a).$$

Because $\varepsilon^2 = 0$, this is already the exact first-order evolution law; there is no limiting process hidden in the notation.

5 Closing the loop remains ordinary wiring

Suppose a controller chooses an action from the observation,

$$c : O \longrightarrow A.$$

For a differential Moore machine

$$f : S \times A \rightarrow S^D, \quad v : S \rightarrow O,$$

feedback gives the vector field

$$S \longrightarrow S^D, \quad s \longmapsto f(s, c(v(s))).$$

In adjunct form this is an infinitesimal evolution

$$S \times D \longrightarrow S.$$

For $\dot{x} = -x + a$ and $c(o) = -ko$, the closed system is

$$\dot{x} = -(1 + k)x.$$

Nothing special has happened to the wiring calculus. The genuinely differential ingredient lies in the codomain S^D , not in the feedback operation.

Parallel composition is equally natural because exponentiation preserves products:

$$(S_1 \times S_2)^D \cong S_1^D \times S_2^D.$$

Hence two parameterised vector fields combine into one on the product state space. This is exactly the compatibility required for the continuous-time systems to participate in the same sort of compositional interface calculus as ordinary Moore machines.

6 Generalized Moore machines and tangencies

Capucci and Myers package the common Moore-machine pattern into the notion of a *generalized Moore machine* [1]. Instead of stipulating once and for all that an update lands back in the state space, suppose that each state comes with a specified object of admissible changes. In elementary notation we may picture this as a family

$$s \longmapsto T_s S.$$

If the action menu depends on the current observation, with actions A_o available at output o , a generalized Moore machine consists of

$$v : S \longrightarrow O$$

and an update of schematic form

$$(s \in S) \times A_{v(s)} \longrightarrow T_s S.$$

The point is not that all such systems have the same dynamics. It is that they have the same interface pattern once one abstracts the notion of change.

Several familiar examples fit the same template:

admissible change at s	resulting machine
S	ordinary deterministic Moore machine
$\mathcal{P}(S)$	nondeterministic Moore machine
$P(S)$	probabilistic Moore machine
$T_s S$	parameterised ODE on a smooth state space.

Capucci–Myers organise the choice of such a notion of change by a *tangency*, abstractly a fibration equipped with a section [1]. For the present tutorial, the essential point is simply that a tangency determines what sort of local update a generalized Moore machine is allowed to make.

The synthetic picture now explains the differential row from inside rather than outside. In the Cahiers topos the differential tangency is induced canonically by

$$T = (-)^D.$$

The fibre over a state $s : S$ consists of infinitesimal paths whose value at 0 is s . Thus the differential example does not require us to bolt an unrelated tangent doctrine onto a smooth state space; it is generated by the infinitesimal structure already present in the ambient topos.

For differential systems, the ambient semantic universe can itself determine the tangency. The extra *choice* of tangency becomes visible only when we step back to the wider theory of generalized Moore machines, where deterministic, nondeterministic, probabilistic and differential notions of change are to be treated uniformly.

This should not be overgeneralised. Not every tangency need arise as exponentiation by an infinitesimal object. The claim is narrower and more useful: the principal smooth example does.

7 Should the endofunctor category be enriched?

The passage from manifolds to a smooth topos solves one problem but raises another. The naive analogue of our original ambient category is

$$[\mathcal{H}, \mathcal{H}]_{\text{acc}}.$$

Its composition-monoids are accessible monads on the Cahiers topos. But $\mathcal{H}$ is cartesian closed and therefore enriched over itself, so Part VI suggests asking for more: if an endofunctor is to respect smoothly varying, and now infinitesimally varying, families of maps, it is natural to require it to be $\mathcal{H}$ -enriched. Equivalently, under the usual hypotheses one asks for tensorial strength with respect to the cartesian product [3].

The more faithful candidate is therefore

$$\mathcal{D}_{\text{sdg}} = \mathcal{H}\text{-}\mathbf{End}_{\text{acc}}(\mathcal{H}),$$

the accessible strong/enriched endofunctors of the smooth topos.

This requirement has an important interpretation. In **Set**, Set-enrichment is invisible because it is ordinary functoriality. In $\mathcal{H}$, enrichment asks the endofunctor to act coherently on internal mapping objects and hence on families parameterised by smooth and infinitesimal objects. It is therefore the categorical level at which the ambient differential geometry can remain visible to the operational theories we place on top of it.

There are now two related but different statements:

- the object D in $\mathcal{H}$ produces the canonical differential tangency $T = (-)^D$;
- enrichment of endofunctors over $\mathcal{H}$ asks theories and their constructions to respect the internal smooth structure of that same universe.

The first concerns the geometry of state change. The second concerns how operational constructions vary in families.

8 A candidate synthetic-smooth Run environment

If the enriched-accessible endofunctor story behaves as in Part VI, we should expect two compatible monoidal structures on $\mathcal{D}_{\text{sdg}}$:

- (1) composition of enriched endofunctors, whose monoids are strong/enriched monads;
- (2) an enriched convolution product supplying the measuring side of Sweedler theory.

This suggests

$$\mathbf{Run}(\mathcal{D}_{\text{sdg}}).$$

A loose arrow again has the familiar stateful-implementation shape

$$\theta_X : TX \times S \longrightarrow R(X \times S),$$

now interpreted internally in $\mathcal{H}$, with S a smooth state object and θ enriched-natural in X .

Such a runner would be a smooth stateful implementation of one smooth theory in another. It could already express smooth store-like systems, protocol translators, or residual probabilistic effects whenever the relevant monads are present. Crucially, however, the tangent bundle has not been manufactured by the runner construction. It is already available as S^D . The new compatibility question is therefore more precise:

if S carries $X : S \rightarrow S^D$, when does a runner preserve or transport X ?

Since $(-)^D$ comes from the closed structure of $\mathcal{H}$, this is much more intrinsic than asking an independently chosen tangent functor to cooperate with $\mathbf{Run}$.

Technical status. The synthetic differential geometry is standard; the Sweedler conclusion is not automatic. One still has to verify, perhaps first at fixed accessibility rank, that the chosen enriched endofunctors admit the convolution product, interchange law and measuring objects needed by the construction. The SDG reformulation removes an artificial-looking choice of tangent bundle; it does not settle the enriched duoidal analysis.

9 How should Run see a differential Moore machine?

Internalising the old Moore theory in $\mathcal{H}$ still gives an ordinary smooth comodel

$$u : S \times A \longrightarrow S, \quad v : S \longrightarrow O.$$

That is a smooth discrete-time machine. A differential Moore machine instead has

$$f : S \times A \longrightarrow S^D$$

over S , or equivalently

$$\tilde{f} : S \times A \times D \longrightarrow S, \quad \tilde{f}(s, a, 0) = s.$$

So one should not pretend that the old Lawvere theory of Moore machines, merely interpreted in a smooth topos, has become the theory of ODEs. The SDG reformulation tells us where the dependent target comes from, but the operational presentation still has to notice it.

There are at least two strategies.

9.1 Strategy A: make the theory infinitesimal-aware

One may seek a fibred, dependent, or generalized algebraic theory whose cooperation has precisely the type

$$S \times A \rightarrow S^D$$

over S . In this presentation the canonical tangency induced by D is built directly into the notion of theory/comodel. Ordinary single-sorted Lawvere theory is unlikely to be the final language, because the admissible update object depends on the state object itself.

9.2 Strategy B: let the ambient topos supply tangency and keep Run operational

Alternatively, **Run**($\mathcal{D}_{\text{sdg}}$) can continue to organise theories, residual effects and stateful implementations, while differential Moore machines form an additional fibred or module-like layer determined canonically by $(-)^D$. This is no longer the old picture of two wholly independent choices: the tangency is fixed by the ambient SDG universe, but Run still has to interact with the resulting infinitesimal dynamics.

At present the second strategy seems conceptually cleaner:

The smooth topos says what infinitesimal change is; Run says what operational theory is being implemented through residual state. The mathematical problem is to make those two structures interact functorially.

10 DOTS and Run: two axes of generalisation

The comparison with the double-operadic theory of systems can now be stated more sharply. Libkind–Myers organise families of open systems, their interfaces, couplings and maps; generalized Moore machines arise after choosing a tangency [2, 1]. The Sweedler proposal varies a different ingredient: given a suitable duoidal category, monoids are operational theories and measuring coalgebras encode stateful implementations between them.

In the general theory these are genuinely different axes:

DOTS / generalized Moore viewpoint	Run / Sweedler viewpoint
choose a tangency	choose an ambient duoidal category
what counts as change?	what counts as theory and implementation?
wire systems through interfaces	compose runners through residual theories.

The Cahiers example reveals an important special case: the ambient semantic category can itself generate the differential tangency. Thus the two axes remain conceptually distinct, but in smooth differential systems they are not arbitrary independent inputs.

A useful refinement of the earlier slogan is therefore

ambient geometry supplies states and, in SDG, infinitesimal change;
theory supplies operations.

DOTS organises the resulting open-system interfaces and wiring; Run organises mediation between operational theories. A satisfactory smooth systems theory should allow both descriptions to coexist.

11 What a smooth residual runner might mean

Once the ambient infinitesimal geometry and the operational level have been separated cleanly, residual effects become easier to formulate. Suppose T describes an abstract smooth command language and R a residual theory. A runner

$$T \rightsquigarrow R$$

with state S still has the ordinary Sweedler form

$$TX \times S \longrightarrow R(X \times S).$$

Nothing in this formula says that the state evolution is discrete. It says how abstract operations are implemented while hidden smooth state evolves.

For a differential system, however, that hidden evolution should also be compatible with the canonical infinitesimal dynamics on S . One might therefore seek a differential runner whose local implementation has schematic form

$$\text{abstract command} + s \longmapsto \text{residual effect} + \text{infinitesimal continuation of } s.$$

This is deliberately not yet a definition. It separates two structures that may coexist: residual effects governed by R , and infinitesimal evolution governed by D .

Examples worth testing include:

- deterministic control systems with residual sensor or actuator effects;
- stochastic differential systems, where probability and infinitesimal evolution interact;
- hybrid systems combining discrete commands with continuous flow;
- smooth protocol translators whose own residual state obeys a differential law.

These examples are precisely why “smooth runner” should not be identified with “ODE”. The runner is an implementation mechanism; the differential law is extra structure on its state evolution, albeit structure now supplied canonically by the ambient SDG world.

12 Infinitesimal semantics versus trajectory semantics

A second distinction should remain explicit. A vector field

$$X : S \longrightarrow S^D$$

is an infinitesimal rule. Its adjunct

$$\tilde{X} : S \times D \longrightarrow S$$

looks like an infinitesimal action. A trajectory appears only after solving or integrating that local rule. Under suitable existence and uniqueness hypotheses one asks for a local or global flow

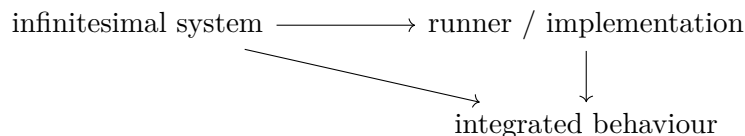
$$\Phi : \mathbb{R} \times S \longrightarrow S.$$

Thus continuous systems naturally have two semantic levels:

$$\begin{array}{ll} \text{infinitesimal evolution} & : S \times D \rightarrow S, \\ \text{integrated behaviour} & : \mathbb{R} \times S \rightarrow S. \end{array}$$

Synthetic differential geometry makes the first level intrinsic, but it does not collapse the distinction between generator and solution.

This matters for Part IX. Behavioural semantics concerns executions or observations through time; a vector field supplies only their infinitesimal generator. One should therefore expect a passage from differential runners to behaviour to contain a genuine integration or solution construction rather than merely forgetting structure. Schematically one might hope for a triangle



whose diagonal is a solution semantics.

13 Lyapunov functions as maps of infinitesimal behaviour

Lyapunov theory is an important part of smooth control theory, allowing for the certification of a system as safe or stable in some sense, as briefly discusses in Part V. The synthetic formulation is useful here. Take the closed system

$$\dot{x} = -x$$

and

$$V(x) = \frac{1}{2}x^2.$$

Classically,

$$\dot{V}(x) = dV_x(-x) = -x^2 = -2V(x).$$

In SDG the derivative of a map is obtained by applying the tangent functor

$$T = (-)^D.$$

Thus $V : S \rightarrow \mathbb{R}$ induces

$$V^D : S^D \longrightarrow \mathbb{R}^D.$$

Composing the vector field with V^D compares the infinitesimal evolution of the system with the infinitesimal evolution of the scalar observation V .

This is exactly the shape one wants for stability theory. A Lyapunov function is not merely an observable quantity; it is an observable whose infinitesimal change satisfies an order condition. Capucci–Myers formulate such conditions compositionally using assume–guarantee reasoning and generalized Moore machines [1]. From the present perspective, the derivative appearing in the certificate is not external analytic notation. It comes from functoriality of $(-)^D$.

Part V can therefore be read as supplying the ordered or valued layer, while Part X supplies the infinitesimal layer. Their meeting point should be a category in which one can express inequalities between maps of tangent objects. The simple equation

$$TV \circ X = Y \circ V$$

would say that V is a strict map of infinitesimal dynamical systems; Lyapunov theory weakens this to an inequality against a comparison dynamics Y .

14 A layered architecture

The smooth case now suggests a richer architecture than ordinary Moore machines require, but the layers are no longer a list of arbitrary choices.

Ambient geometry.

A smooth topos such as the Cahiers topos supplies smooth families, infinitesimal objects and exponentials.

Infinitesimal change.

Its infinitesimal interval D induces the canonical differential tangency $TX = X^D$.

Operational theory.

Sweedler theory acts on monads and runners in a suitable duoidal category of enriched endofunctors of the ambient topos. Tight maps of theories, coproducts and tensor products already express substantial forms of interface decomposition, parallel composition, parameter change, and wiring between open systems; loose morphisms record the corresponding residual implementations.

Interface presentation.

The same compositional structure may be organised from a more explicitly boundary-centred point of view. A DOTS-style calculus makes paired input/output interfaces and their wiring primitive, rather than treating wiring primarily through maps and combinations of theories. Thus the contrast is not between a non-compositional Sweedler semantics and a compositional DOTS layer, but between two presentations of compositional structure: theory-centred wiring in $\mathbf{Run}(\mathcal{D})$, and boundary-centred wiring in the DOTS perspective.^a

Behaviour.

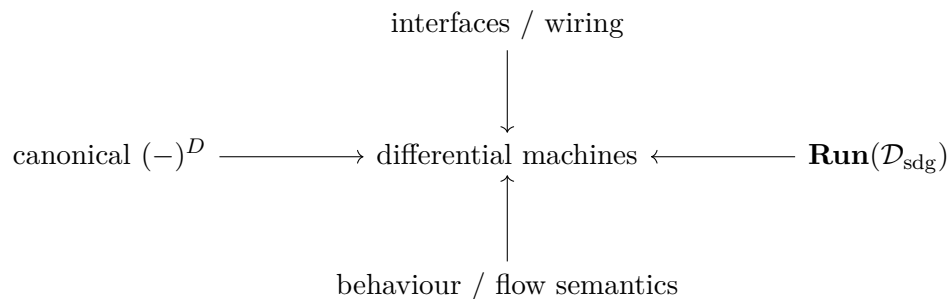
Integration passes from infinitesimal evolution to trajectories and observable histories.

Certification.

Ordered or valued structure compares the induced infinitesimal behaviour with safety, stability or Lyapunov specifications.

^aAppendix A develops a corresponding double category $\mathbf{IntRun}(\mathcal{D})$, whose objects are pairs of theories and whose loose morphisms encode prism-like interface wiring. We do not need that construction here; the point is only that the boundary-centred presentation can itself be given a Sweedler-style semantics.

One may picture the middle of the architecture as



with certification acting on the resulting dynamical maps. The important change from the earlier draft is that the left-hand input is no longer an unexplained tangent doctrine. It is induced by the ambient smooth topos.

15 Questions for the programme

The SDG reformulation resolves a presentational problem, but it also sharpens the remaining mathematics.

- Q1.** Can the enriched-accessible Sweedler construction be established cleanly for the Cahiers topos, perhaps first at fixed accessibility rank as in Part VI?
- Q2.** Can differential Moore machines $S \times A \rightarrow S^D$ be presented as comodels of a suitably fibred or dependent theory, and how does that presentation compare with treating $(-)^D$ as a canonical doctrine over $\mathbf{Run}$?
- Q3.** Which runners preserve or transport infinitesimal dynamics, and is there a useful measuring object for such differential runners?
- Q4.** Is there a universal or pseudofunctorial passage from infinitesimal runners to trajectory/flow semantics, analogous in spirit to the behaviour constructions of Part IX?
- Q5.** How should Part V's valued predicates be combined with $(-)^D$ so that Lyapunov and assume–guarantee certificates become intrinsic to the runner architecture?
- Q6.** Can residual probability be combined with the canonical tangency so as to include stochastic differential systems without rebuilding the framework from scratch?
- Q7.** More generally, which tangencies arise internally from modalities or exponentials in a semantic category, and which genuinely require additional external structure?

These questions are now better separated. SDG supplies a natural answer to “where does the tangent bundle come from?”; the unresolved work concerns how that infinitesimal geometry interacts with theories, residual effects, integration and certification.

16 Beyond smooth spaces: condensed and analytic semantics

The use of **SmoothSet** or the Cahiers topos rather than the category of finite-dimensional manifolds illustrates a general principle. The category in which a system is first encountered need not itself be the best category in which to develop its algebra of theories, runners and handlers. One may instead pass to a larger semantic universe which retains the relevant geometric structure while having the completeness, cocompleteness and monoidal properties needed by the Sweedler construction.

Functional analysis provides a particularly striking parallel. Naive categories of topological vector spaces are notoriously less well behaved categorically than ordinary module categories. One response, going back in a different form to Lawvere’s use of bornological topoi, is to replace the original category by a sheaf-theoretic environment in which analytic structure can be treated algebraically. The condensed and analytic mathematics of Clausen and Scholze pursues this strategy systematically: topological algebra is embedded into the world of condensed objects, and suitable completeness conditions lead to solid modules and, more generally, analytic rings and modules [11, 12].

From the present point of view this is suggestive for a very specific reason. If $\mathcal{C}$ is a suitable locally presentable symmetric monoidal category, then the accessible endofunctors

$$[\mathcal{C}, \mathcal{C}]_{\text{acc}}$$

are candidates for the same composition–convolution duoidal structure that underlies our basic example

$$[\mathbf{Set}, \mathbf{Set}]_{\text{acc}}.$$

Thus the passage from sets to smooth, condensed or analytic objects need not require a change in the basic architecture of the theory. Rather, it changes the semantic category $\mathcal{C}$ in which states, observations and residual state are allowed to live:

$$\mathbf{Set} \rightsquigarrow \mathbf{SmoothSet}, \mathbf{Cond}(\mathbf{Set}), \mathbf{Cond}(\mathbf{Ab}), \mathbf{Solid}_R, \dots$$

followed, where the required hypotheses hold, by

$$\mathcal{C} \rightsquigarrow [\mathcal{C}, \mathcal{C}]_{\text{acc}} \rightsquigarrow \mathbf{Run}([\mathcal{C}, \mathcal{C}]_{\text{acc}}).$$

The solid setting is especially suggestive. The category of solid abelian groups is closed under limits and colimits and carries a symmetric monoidal tensor product compatible with solidification [12]. Thus one is led to ask whether categories of solid modules provide an analytic counterpart to the smooth setting considered above:

$$\boxed{\mathbf{Run}([\mathbf{Solid}_R, \mathbf{Solid}_R]_{\text{acc}})}.$$

Here the intended systems would no longer merely have smooth state spaces and smooth updates. Their state, parameter and residual objects could carry the kinds of completed linear structure appropriate to functional analysis, while monads would describe analytic theories of operations and runners would mediate between such theories.

There is a useful methodological point here. Requiring a duoidal category does restrict the possible ambient semantics, but this restriction should not be read as forcing systems back toward discrete algebra. Much of modern categorical geometry and analysis proceeds in precisely the

opposite direction: one replaces a badly behaved concrete category by a presentable, often sheaf-theoretic, monoidal category in which the original geometry or analysis becomes amenable to algebraic methods. The requirements imposed by Sweedler theory therefore align rather well with an independent trend in modern mathematics.

One may schematically compare

$$\begin{array}{ll} \text{smooth dynamics} & : \text{manifolds} \rightsquigarrow \mathbf{SmoothSet}, \\ \text{topological algebra} & : \text{topological objects} \rightsquigarrow \text{condensed objects}, \\ \text{functional analysis} & : \text{topological modules} \rightsquigarrow \text{solid/analytic modules}. \end{array}$$

In each case the larger semantic universe is not intended to erase the original geometry. It packages it in a form in which categorical algebra behaves better.

This also clarifies the scope of the construction $\mathbf{Run}(\mathcal{D})$. The choice

$$\mathcal{D} = [\mathbf{Set}, \mathbf{Set}]_{\text{acc}}$$

is not meant to privilege discrete systems. It is the first member of a family. Varying the semantic category $\mathcal{C}$ changes what a state or residual state may be, while passing to accessible endofunctors provides the level at which theories and their interactions can be studied. The synthetic-smooth case is one instance of this principle; condensed and analytic mathematics suggest that genuinely functional-analytic systems may furnish another.

There are, however, genuine questions still to be checked. For a proposed semantic category $\mathcal{C}$, one must verify not merely that it has a useful symmetric monoidal structure but that the relevant class of endofunctors supports the convolution product and its interchange with composition, and that the resulting measuring objects exist. In the condensed and solid settings these questions should therefore be regarded as a programme rather than as established consequences of the theory developed here. What is already clear is that the categorical technology developed for modern functional analysis provides unusually promising ambient categories in which to ask them.

16.1 Convenient categories for probability

Probability theory provides another example in which the naive category of structured sets is not necessarily the best ambient category for categorical systems theory. The usual category $\mathbf{Meas}$ of measurable spaces and measurable maps supports the Giry monad, but it is not cartesian closed. This becomes a serious obstruction as soon as one wants probability distributions on spaces of functions, as occurs naturally in higher-order probabilistic programming.

Heunen, Kammar, Staton and Yang introduced the category $\mathbf{Qbs}$ of quasi-Borel spaces as a convenient replacement [13]. Rather than taking measurable subsets as primitive, a quasi-Borel space specifies an appropriate class of random elements

$$\mathbb{R} \longrightarrow X.$$

The resulting category is cartesian closed and well pointed, contains the standard Borel spaces in a suitable sense, and carries a commutative probability monad

$$P : \mathbf{Qbs} \longrightarrow \mathbf{Qbs}.$$

Thus random functions and probability distributions on function spaces become ordinary objects and morphisms of the ambient category.

This example fits the present methodology particularly closely. The category **Qbs** is a Grothendieck quasitopos and is locally presentable. Hence it is natural to consider

$$\mathcal{D}_{\text{prob}} = [\mathbf{Qbs}, \mathbf{Qbs}]_{\text{acc}}$$

as a candidate semantic universe for Sweedler runners. If the composition–convolution structure on accessible endofunctors is available in the required form, one obtains

$$\mathbf{Run}([\mathbf{Qbs}, \mathbf{Qbs}]_{\text{acc}}),$$

in which the objects need no longer be theories acting only on sets of states: they may act on state spaces supporting higher-order probabilistic structure.

This should be distinguished from merely placing a probability monad inside

$$[\mathbf{Set}, \mathbf{Set}]_{\text{acc}}.$$

For example, in earlier parts we treated probabilistic effects using monads such as **rand**. Passing instead to **Qbs** changes the ambient notion of state itself. A state space may now contain function spaces and continuous distributions internally, while the probability monad P appears as one particular theory inside the richer endofunctor category. Thus there are two independent choices:

$$\begin{aligned} &\text{the semantic category of states } \mathcal{C}, \\ &\text{the monad or theory } T \in [\mathcal{C}, \mathcal{C}]_{\text{acc}}. \end{aligned}$$

Probability can enter at either level.

This distinction is useful for systems theory. A deterministic update on a quasi-Borel state space,

$$f : X \longrightarrow X,$$

already has a measurability-compatible higher-order semantics. A probabilistic update is instead represented by a Kleisli map

$$X \longrightarrow PX.$$

More elaborate theories may combine probabilistic choice with state, observation, nondeterminism or other effects. The runner construction then asks how such theories mediate one another through residual state, rather than treating stochasticity as an isolated additional operation.

The quasi-Borel example therefore reinforces the same general lesson as the smooth and condensed examples. One need not regard the requirements of the Sweedler construction as forcing the theory into a narrow class of discrete categories. Instead one may first choose a convenient categorical home for the kind of mathematics involved and then form the appropriate category of accessible endofunctors:

$$\begin{aligned} \text{smooth dynamics} & : \mathbf{SmoothSet}, \\ \text{higher-order probability} & : \mathbf{Qbs}, \\ \text{functional analysis} & : \text{condensed or solid objects}. \end{aligned}$$

The common strategy is to retain the intended semantics while moving to an ambient category in which products, function spaces, colimits and algebraic constructions behave sufficiently well.

There is also a potentially important interaction with the probability monad itself. Since P is commutative, its probabilistic structure is already compatible with the cartesian symmetric

monoidal structure on **Qbs**. From the Sweedler point of view one may therefore ask not only for runners

$$T \rightsquigarrow P,$$

but for measuring objects such as

$$\{T, P\}$$

internal to the probabilistic semantic universe. This would provide a higher-order analogue of the probabilistic runner examples considered earlier over **Set**, with random functions and distributions on function spaces available intrinsically rather than encoded externally.

As before, one should separate what is established from what remains to be checked. Local presentability and the probabilistic monad are available in **Qbs**; the additional claim needed here is that the chosen class of accessible endofunctors carries the precise duoidal composition–convolution structure required by the Sweedler construction. The example is nevertheless especially promising because the categorical deficiencies which motivated the introduction of quasi-Borel spaces are closely related to the structural properties that our construction requires.

17 What the smooth case now tells us

The chapter began with what looked like an awkward exception to the Moore pattern:

$$S \times A \longrightarrow TS$$

rather than $S \times A \rightarrow S$. Synthetic differential geometry turns that exception into a clue. In a smooth topos with infinitesimals,

$$TS = S^D,$$

so a differential update is a choice of infinitesimal path based at the current state. For ordinary manifolds this recovers the familiar tangent bundle, while for generalized smooth objects it remains defined internally.

This makes the tangency of differential Moore machines much less programmatic. Capucci–Myers abstract from this and other examples to a general notion of admissible change; Sweedler runners address a different level, namely mediation between operational theories through residual state. The resulting division of labour is now fairly concrete:

- the smooth topos supplies the kind of states and the infinitesimal object D ;
- $(-)^D$ supplies differential tangency;
- enriched endofunctors supply the prospective world of smooth operational theories;
- runners supply stateful implementations between those theories;
- wiring supplies interconnection;
- integration supplies trajectories and behaviour;
- ordered or valued lifts supply certification.

The broader examples at the end of the chapter show why the first step matters. Smooth dynamics, higher-order probability and functional analysis each motivate replacing a poorly behaved concrete category by a better categorical home before one forms the endofunctor world in which theories and runners live. The Cahiers topos is especially revealing because the passage to the better ambient category does more than improve categorical closure: it also makes the basic differential structure internal.

The guiding picture is therefore

$$\text{smooth topos with } D \implies TX = X^D \implies \text{differential Moore machines.}$$

Sweedler theory may then be developed over that same semantic universe, rather than being asked to supply tangency from outside.

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