Getting to Grips with Sweedler Runners

Part XIII. Sweedler Runners in Domains

Abstract

A runner can report a partial observation, fail to supply a next state, or hand control to a continuation. Domains let us give these possibilities different types while keeping track of information order. After constructing the accessible domain-enriched setting, we use a bilimit to obtain cofree runners for free signatures with full continuation residuals and prove their measuring universal property. For stabilising ticks we construct the lifting-residual runner. We then exhibit an equalizer on which the continuation functor loses injectivity.

Why use domains?

Imagine a device whose only command is *tick*. The device may give a reading that is still unknown and nevertheless be able to tick again. Or the tick itself may fail to supply a next state. A handler offers a third possibility. Before returning an answer, it can ask what the rest of the computation would do with different responses. We keep the command fixed and vary the residual so that these distinctions are visible in the behaviour of one example.

For an ordinary tick, the stream of readings is a universal description of behaviour. Every observed tick runner has a unique behaviour-preserving map into the stream runner. Can we find a corresponding universal runner when a tick can use the full continuation monad? Over sets, even one freely repeatable tick with a trivial observation gives a size obstruction. The universal carrier would have to be in bijection with its own continuation space, which is strictly larger for any answer set with at least two elements. Domains change what counts as a continuation by requiring it to respect increasing information. This gives us a way to assemble behaviours from successive approximations.

Section 1 starts with streams and separates partial readings from failed transitions. Sections 2 and 3 recover the usual Sweedler construction for accessible endofunctors and explain why the full continuation monad lies outside it. A continuation can respond continuously to an increasing supply of information while ranging over arbitrarily many possible inputs. Section 4 uses precisely that continuity to construct the cofree runner for free commands. Sections 5 and 6 then ask what happens when we require the commands to obey an equation. With lifting, a successful transition supplies a successor whose behaviour we can restrict. With continuations, we must also account for how restricting the state space changes the available tests.

1 Ticks, streams, and partial information

Work in $\mathcal{V} = \omega\mathbf{CPO}$, the category of posets with suprema of increasing ω-chains and maps preserving those suprema. We allow the empty domain 0. A domain is *pointed* when it has a least element. Write 1 for the singleton and $[X, Y]$ for the domain of continuous maps ordered pointwise. Products and these function spaces make $\mathcal{V}$ cartesian closed [1, §4.1].

An identity-residual runner for the free tick monad $\mathsf{Ny}X = \mathbb{N}_{\mathrm{disc}} \times X$ has a continuous transition $f : S \to S$. Given an observation $o : S \to Y$, its behaviour is

$$s \longmapsto \big(o(s), o(f(s)), o(f^2(s)), \ldots\big).$$

The head and tail make the pointwise ordered stream domain $Y^{\mathbb{N}}$ the final coalgebra for $Y \times -$. The coordinates of any behaviour map are forced by the coalgebra law and each is continuous. Refining a state can now refine the observations along its run.

Let $LX = X_\perp$ denote lifting. Write $\perp_L$ for the new bottom and $\eta^L(x)$ for an old point included in LX. A lifting-residual tick is $f : S \to LS$. For $S = 1$ and $Y = \{\perp_Y < \top\}$, the ticks $f(*) = \eta^L(*)$ and $f(*) = \perp_L$ can both have observation $o(*) = \perp_Y$. The first can keep ticking indefinitely. The second supplies no successor. Bottom in the observation domain expresses partial information about a state, while the new bottom in the lifting residual marks failure to deliver a next state.

2 The domain-enriched duoidal category

Local continuity means that the action of a functor on each hom-domain is monotone and preserves increasing ω-suprema. This is enrichment over $\mathcal{V}$. A rank-λ functor preserves λ-filtered colimits of objects. The category $\omega\mathbf{CPO}$ is locally $\aleph_1$-presentable, with the $\aleph_1$-presentable objects closed under finite products [1, Thm. 5.33 and Ex. 6-9]. Fix $\lambda = \aleph_1$, or a regular rank with these properties, and let $J : \mathcal{A} \hookrightarrow \mathcal{V}$ be a small full enriched category of representatives of its presentable arities.

The two products of Part IV now act on locally continuous rank-λ endofunctors. Composition has unit Id. Enriched Day convolution along the cartesian product has the arity formula

$$(F \star G)(Z) = \int^{A,B \in \mathcal{A}} [A \times B, Z] \times FA \times GB.$$

For $A \in \mathcal{A}$, the internal hom $[A, -]$ preserves λ-filtered colimits. To see this, test the comparison against a presentable B and use $B \times A$ presentable. The arities detect isomorphisms. Thus enriched left Kan extensions from $\mathcal{A}$ preserve λ-filtered colimits, and a functor of this rank is recovered by extending its restriction, since each object is such a colimit of arities. On the arities the formula is ordinary enriched Day convolution on the small monoidal category $\mathcal{A}$. Its unit extends to $[1, Z] \cong Z$.

The interchange has direction

$$(F \circ G) \star (H \circ K) \longrightarrow (F \star H) \circ (G \star K).$$

Its component on a convolution generator uses $GX \times KY \to (G \star K)(X \times Y)$. First use the convolution map for F, H into $(F \star H)(GX \times KY)$, then apply $F \star H$ to that canonical map. The argument of [2, §§2–3] transfers to these enriched functors. Both sides of each coherence diagram agree on convolution generators by naturality and the product coherences in $\mathcal{V}$, so they agree by the enriched coend universal property. A measuring is an enriched-natural family $TX \times DY \to R(X \times Y)$ satisfying the monad–comonad interaction laws. At state S, a runner corresponds to an enriched monad map from T to the state transformer

$$\mathsf{St}^R_S(X) = [S, R(X \times S)].$$

The cartesian strength and the self-enrichment determine one another here. The concrete description thus applies to any locally continuous residual monad, including those outside the rank-bounded category. For accessible T, R, McDermott, Rivas and Uustalu give the accessible existence theorem [2, Thm. 1]. See also [2, Prop. 8] for the enriched description of runners as generalised algebras.

3 Handlers and continuation residuals

For an answer domain C, write $K_C X = [[X, C], C]$. Its unit evaluates at the supplied input, and the Kleisli extension of $u : X \to K_C Y$ acts by $q \mapsto (k \mapsto q(x \mapsto u(x)(k)))$. The action on

hom-domains is continuous by continuity of composition and evaluation. Function spaces of this kind are used in the classical continuation semantics of control [5].

3.1 The handler as a runner

For a locally continuous monad T, a T-algebra $a : TC \to C$ gives the enriched monad map

$$\widehat{a}_X(t)(k) = a(Tk(t)).$$

The right side is continuous in t, k by local continuity of T and continuous evaluation. The algebra laws imply the monad-map laws. Conversely, evaluating a monad map $\alpha : T \to K_C$ at id_C gives $a(t) = \alpha_C(t)(\mathrm{id}_C)$. Naturality at $k : X \to C$ recovers the displayed formula, and the monad-map laws evaluated at id_C give the algebra laws. A return clause $r : A \to C$ evaluates the residual by $t \mapsto \widehat{a}_A(t)(r)$. Thus the operation clauses give a singleton-state runner. The return clause supplies its final continuation. This is the identity-comonad instance of the continuation-based runners in [2, §4, Continuation-Based Runners]. More generally, currying gives $\mathsf{St}_S^{K_C} \cong K_{[S,C]}$ as monads. Both units evaluate at the input, and both binds pass the updated state to the next computation. The formulas use only the answer domain and its function spaces, so C may be unpointed.

3.2 The rank-bounded continuation monad

For locally continuous H, put $Q_\lambda H = \mathrm{Lan}_J(HJ)$, using enriched left Kan extension. The arity argument shows that $Q_\lambda H$ has rank λ. If T has that rank, the Kan extension adjunction and full faithfulness on arities give

$$\mathrm{Nat}(T, Q_\lambda H) \cong \mathrm{Nat}(TJ, HJ) \cong \mathrm{Nat}(T, H).$$

These are isomorphisms of hom-domains. The induced correspondence is postcomposition with the counit $Q_\lambda H \to H$. The inclusion of rank-λ endofunctors contains the unit and is strong monoidal for composition. Concretely $Q_\lambda H \circ Q_\lambda G \to Q_\lambda(H \circ G)$ is the unique lift of the horizontal composite of the two counits. Uniqueness gives associativity and the unit coherence for the lax monoidal structure. Hence $K_{C,\lambda} = Q_\lambda K_C$ inherits a monad structure and its counit is a monad map. Every handler map $T \to K_C$ of source rank λ lifts uniquely to a monad map $T \to K_{C,\lambda}$. The lifted map respects multiplication and unit by applying the same uniqueness to $T \circ T$ and Id.

3.3 Local continuity without accessibility

Take $C = \Sigma = \{\bot < \top\}$ and X discrete. The functional

$$q : [X, \Sigma] \to \Sigma, \qquad q(k) = \top \iff (\exists x \in X)\ k(x) = \top$$

is continuous. A witness at the supremum of an increasing chain already appears at a stage. Yet q depends on every coordinate. For any uncountable regular λ, take $|X| \geq \lambda$ and express X as the λ-filtered union of its subsets A of size $< \lambda$. Every functional coming from $K_\Sigma A$ tests only the restriction to A. Hence q comes from no stage. Underlying sets compute these λ-filtered colimits in $\omega\mathbf{CPO}$, since the data of any countable chain in the colimit can be realised at a common stage [1, Lem. 5.4 and Cor. 5.5]. The comparison map for the colimit of the $K_\Sigma A$ is therefore not surjective. For any nontrivial pointed C, one can similarly test whether some input coordinate is nonbottom and return a fixed nonbottom answer on success. Since preservation at a smaller regular rank would imply preservation at a larger one, these full continuation monads are inaccessible.

4 Cofree runners for a signature

Take a set of operations indexed by i, with parameter domain P_i and response domain B_i. A signature runner is a domain S with continuous implementations

$$c_i : P_i \times S \to R(B_i \times S).$$

A runner map $f : S \to S'$ commutes with the implementations using $R(B_i \times f)$ on returned responses and states. This definition works for arbitrary arities. When the signature functor $\coprod_i P_i \times [B_i, -]$ has an enriched free monad T, enriched Yoneda identifies the implementations with enriched natural maps from the signature functor to St_S^R, hence with enriched monad maps $T \to \mathsf{St}_S^R$. Equality on generators also gives the same runner maps. This passage from an operation to its value at the generic response is the enriched Yoneda argument underlying generic effects [3]. The following construction uses the signature runner description directly.

For an observation domain Y, define

$$H(S) = \prod_i [P_i, R(B_i \times S)], \qquad F_Y(S) = Y \times H(S).$$

An F_Y-coalgebra is exactly a signature runner together with a continuous observation $o : S \to Y$. Starting with Y records what we can observe before issuing a command. Each application of F_Y adds one layer of command behaviour, with the behaviour of successors described to the previous depth. The first projection forgets that extra layer. Its section fills the new layer with the least informative implementation. Here depth counts successive commands. Even at a single layer a continuation can use the full space of tests available at that depth.

Theorem 4.1 (Cofree signature runners)**.** *Let R be a locally continuous endofunctor of $\omega\mathbf{CPO}$, and let Y be an observation domain for which $H(Y)$ has a least element. This holds if each $R(B_i \times Y)$ has a least element. Then the limit of*

$$W_0 = Y \xleftarrow{p_0 = \pi_Y} W_1 = F_Y Y \xleftarrow{p_1 = F_Y p_0} W_2 = F_Y^2 Y \xleftarrow{p_2} \cdots$$

is the final F_Y-coalgebra and hence the cofree signature runner on Y.

Proof. Write $F = F_Y$. Use the least element of $H(Y)$ to define the first section. The map $e_0(y) = (y, \bot_{H(Y)})$ satisfies $p_0 e_0 = \mathrm{id}$ and $e_0 p_0 \leq \mathrm{id}$. The functor F is locally continuous, by local continuity of R, products and postcomposition. Applying F repeatedly gives embedding–projection pairs (e_n, p_n).

Let $W = \lim_n W_n$, with projections q_n. Each q_n has a section $j_n : W_n \to W$ obtained by using projections at earlier stages and embeddings at later stages. Then $q_n j_n = \mathrm{id}$, and $j_n q_n$ increases to id_W, since it is eventually the identity on any fixed coordinate. This is the Smyth–Plotkin bilimit criterion [4, Defs. 5 and 7, Thms. 2–3]. Their definition of an O-category only requires increasing ω-suprema in the hom-posets and continuity of composition, so bottom in hom-posets is unnecessary. Local continuity sends these identities to the corresponding identities for Fq_n, Fj_n. A compatible cone into the FW_n therefore gives a map to FW by composing each leg with Fj_n and taking the increasing supremum. The identities give uniqueness, so Fq_n are limit projections. Dropping stage zero yields an isomorphism $\delta : W \to FW$ with $Fq_n \circ \delta = q_{n+1}$, and $\pi_Y \circ \delta = q_0$.

For $\gamma : S \to FS$, put $h_0 = \pi_Y \circ \gamma$ and $h_{n+1} = Fh_n \circ \gamma$. The identity $\pi_Y \circ Fg = \pi_Y$ gives $p_0 h_1 = h_0$, and induction gives $p_n h_{n+1} = h_n$. Let $h : S \to W$ be the induced map. Composing with every Fq_n shows $\delta h = Fh \circ \gamma$. For any coalgebra map $g : S \to W$, the law first forces $q_0 g = h_0$, then forces $q_{n+1} g = F(q_n g) \circ \gamma = h_{n+1}$ by induction. Thus $g = h$. □

Starting at Y uses the projection onto the observation. Starting at 1 would require a point of Y, and an embedding–projection pair from 1 would require a least point in $F_Y(1)$. The seed at Y can be used for unpointed Y and also for $Y = 0$, when the cofree runner is empty. The proof uses pointedness only at the seed. When cofree runners exist for all Y, the recursion for h_n shows that cofree extension is continuous on hom-domains in each observation map. It follows that the right adjoint and its induced comonad are locally continuous. The forgetful functor from H-coalgebras reflects isomorphisms and creates equalizers of pairs with split equalizer in the base. Thus it is comonadic by Beck's theorem.

The theorem covers lifting and full K_C with pointed C. It also applies to identity-residual ticks when Y is pointed. For arbitrary Y the direct stream construction in Section 1 supplies the cofree identity-residual tick runner. The change of ambient category is decisive already for ticks with $Y = 1$. Over **Set**, for $|C| \geq 2$, a final coalgebra for $Z \mapsto C^{C^Z}$ would have $Z \cong C^{C^Z}$ by Lambek's lemma, contrary to Cantor's theorem.

4.1 The measuring universal property

Proposition 4.2 (Representation of enriched measurings)**.** *Suppose the signature has a locally continuous enriched free monad T, the residual R is a locally continuous monad, and cofree signature runners exist as above for every observation domain. Write G for the comonad of the cofree-runner adjunction. For every locally continuous comonad D on ω**CPO**, enriched-natural measurings $TX \times DY \to R(X \times Y)$ correspond naturally to comonad maps $D \to G$.*

Proof. The ordinary version of the first step is [2, Prop. 5]. We give the argument for enriched families directly. Write ε^D, Δ^D for the counit and comultiplication of D. For a D-coalgebra (S, s), a measuring ψ defines the runner

$$\rho^s_X = \psi_{X,S} \circ (1_{TX} \times s).$$

Naturality in the state object gives the runner-map law for D-coalgebra maps. The measuring unit gives the runner unit because $\varepsilon^D_S s = \mathrm{id}$. For multiplication, substitute s in the measuring law, use $\Delta^D_S s = D(s)s$, and use naturality of $\psi_{TX,-}$ at s. The resulting right-hand side is $\mu^R R(\rho^s_X)\rho^s_{TX}$, the runner multiplication law.

Conversely, for a functor from D-coalgebras to runners over $\mathcal{V}$, use the induced runner on each cofree D-coalgebra to define

$$\psi_{X,Y} = R(1_X \times \varepsilon^D_Y) \circ \rho^{\Delta^D_Y}_X.$$

Naturality in Y follows from the runner-map law at Dg for all $g : Y \to Y'$. Enriched naturality in X, Y holds since the hom-domains of $\mathcal{V}$ are concrete and the equations between the induced continuous maps hold at every point. The runner unit gives $\psi_{X,Y}(\eta^T_X \times 1) = \eta^R_{X\times Y}(1_X \times \varepsilon^D_Y)$. The runner-map law at the coalgebra map $s : (S, s) \to (DS, \Delta^D_S)$, followed by the counit, recovers $\rho^s_X = \psi_{X,S}(1 \times s)$. In particular $\rho^{\Delta^D_Y}_{TX} = \psi_{TX,DY}(1 \times \Delta^D_Y)$. Postcompose the runner multiplication law at (DY, Δ^D_Y) with $R(1_X \times \varepsilon^D_Y)$. Naturality of μ^R gives

$$\psi_{X,Y}(\mu^T_X \times 1) = \mu^R_{X\times Y}\, R(\psi_{X,Y})\, \psi_{TX,DY}(1 \times \Delta^D_Y),$$

which is the measuring multiplication law. Starting with ψ, reconstruction on a cofree coalgebra recovers it by naturality at ε^D_Y and $D\varepsilon^D_Y \Delta^D_Y = \mathrm{id}$. The constructions are therefore inverse.

Finally comonadicity identifies runners over $\mathcal{V}$ with G-coalgebras over $\mathcal{V}$. If a lifting gives a G-structure $g_s : S \to GS$ on each D-coalgebra, the corresponding comonad map has

component $G\varepsilon^D_Y \circ g_{\Delta^D_Y} : DY \to GY$. Naturality at the cofree coalgebras gives its naturality in Y. The G-counit law gives compatibility with counits. To see the comultiplication law, denote this map by α. Naturality of the lifting at s gives $g_s = \alpha_S s$, so the G-coalgebra law on (DY, Δ^D_Y), postcomposed with $GG\varepsilon^D_Y$, gives $\Delta^G_Y \alpha_Y = G\alpha_Y \, \alpha_{DY} \Delta^D_Y$. Conversely any comonad map gives the lifting by $s \mapsto \alpha_S s$. Since the hom-order on runner maps is inherited from $\mathcal{V}$, all these liftings over $\mathcal{V}$ are enriched. □

The cofree-runner theorem supplies the all-observations hypothesis for lifting and for full K_C with pointed C. For identity-residual ticks on unpointed Y, the stream construction in Section 1 supplies the cofree runner separately. Taking $D = G$ and the identity comonad map gives the universal measuring. This proves the Sweedler-hom property formulated with enriched-natural families, including in the case of full K_C outside the accessible Day-convolution setting.

Example 4.3 (Ticks and abort)**.** Give abort parameter domain P and empty response domain 0. Since $K_C 0 \cong C$, its implementation $P \times S \to C$ can supply an answer without resuming the program. Equivalently the implementation is an observation $S \to [P, C]$. The cofree tick-and-abort runner on Y is the cofree tick runner on $Y \times [P, C]$. For $P = 1$ the extra factor is C. A general P still allows absorption into the observation domain, but now retains a continuous family of answers at each state. With lifting, $L0$ is the singleton containing only $\perp_L$, so abort can only fail to return.

5 Stabilising ticks and the role of images

Return to the quotient $t^{n+1} = t^n$ of Part I, with $n \geq 0$. A tick $u : S \to RS$ now satisfies equality of these *Kleisli* powers, with $u^{[0]} = \eta^R_S$. For the identity residual, the cofree quotient runner on Y is Y^{n+1}. The tick shifts the tuple and repeats its last entry, and the observation is the first entry. The unique behaviour map records the first $n + 1$ observations, whose compatibility with the shift follows from the equation.

5.1 The lifting quotient

Proposition 5.1 (The quotient with lifting residual)**.** *Let W be the unrestricted cofree lifting-residual tick runner on Y, with tick $u : W \to LW$. Its subdomain*

$$E_n = \{w \in W \mid u^{[n+1]}(w) = u^{[n]}(w)\}$$

with inherited observation and restricted tick is the cofree runner for $t^{n+1} = t^n$.

Proof. The two powers are continuous, so their equalizer $i : E_n \to W$ is an ω-closed subposet with the inherited order. If $w \in E_n$ and $u(w) = \eta^L(v)$, applying the Kleisli extension of u to the equality at w gives $u^{[n+2]}(w) = u^{[n+1]}(w)$. Since the first step succeeds, this reads $u^{[n+1]}(v) = u^{[n]}(v)$. Thus every defined successor belongs to E_n. The map ui takes values in the image of $Li : LE_n \to LW$. This is an order embedding and its image is closed under increasing ω-suprema. Hence the induced tick on E_n is continuous. The equation holds there because Li is injective and i commutes with the Kleisli powers.

For any observed runner with tick $v : S \to LS$ satisfying the equation, let $h : S \to W$ be its unrestricted cofree map. Then $u^{[k]} h = Lh \, v^{[k]}$ for each k, by the runner-map law. Thus h lands in E_n and factors continuously through i. Injectivity of Li shows that the factor is itself a runner map. Uniqueness follows from uniqueness into W and injectivity of i. □

A run governed by the equation can fail before completing n successful ticks. Otherwise the state reached after n ticks has a defined self-loop, so its observation repeats indefinitely. Indeed, if the n-th Kleisli power returns a state, the $(n+1)$-st must return the very same state. For $n = 0$ every state has a defined self-loop. The proof uses both closure of the equalizer under increasing limits and closure of its successors under restriction.

5.2 The continuation obstruction

Take $A = \{a, b\}_{\mathrm{disc}}$, $Z = \{a < b\}$, and let $i : A \to Z$ be the identity on elements. It is a monomorphism. The restriction $[Z, \Sigma] \to [A, \Sigma]$ has image the chain of bit pairs $(0,0), (0,1), (1,1)$. The two functionals

$$q_1(k) = k(b), \qquad q_2(k) = k(a) \vee k(b)$$

agree on this chain and differ at $(1,0)$. Thus $K_\Sigma i$ identifies distinct points. The six monotone functionals on the square restrict to the four on the chain. In **Set**, every test on a subset extends to the whole set when the answer set is nonempty, so the double-dual functor there preserves injections.

Proposition 5.2 (Failure on an equalizer)**.** *There is an ω-closed order inclusion $i : A \hookrightarrow Z$ which is an equalizer in ω**CPO** and for which $K_\Sigma i$ is not injective.*

Proof. Let Z have a chain $c_0 < c_1 < \cdots$ with supremum z_∞, and incomparable points $b_n < z_\infty$ with $c_m \leq b_n$ precisely when $m \leq n$. These relations and reflexivity give its order. Any increasing chain either eventually stabilises or has supremum z_∞. Give $A = \{b_n \mid n \in \mathbb{N}\} \cup \{z_\infty\}$ the induced order. Its increasing chains are eventually constant, so A is ω-closed in Z. To exhibit the inclusion as an equalizer, use the target obtained by replacing the chain of c_n's with two copies of it, each with the same relations to the shared b_n's and z_∞. The two maps that choose different copies and agree on these shared points are continuous. They agree exactly on A.

The predicate $k_\infty = \chi_{\{z_\infty\}}$ is continuous on A. Every continuous Σ-valued test on Z taking $\top$ at z_∞ takes $\top$ at some c_n, hence at b_m for all $m \geq n$. The continuous functionals on $[A, \Sigma]$ given by

$$Q_1(k) = k(z_\infty), \qquad Q_2(k) = k(z_\infty) \wedge \bigvee_n k(b_n)$$

agree on all restricted tests. At k_∞ their values differ. The countable disjunction is continuous because a true value has a single witness, and the binary conjunction is continuous because the witnesses for both factors occur together at some stage of an increasing chain. Thus $K_\Sigma i$ identifies Q_1 and Q_2. □

The open sets defined by continuous Σ-valued tests on A can therefore be finer than the subspace topology induced from Z. The example rules out an argument based on preservation of all equalizers or all ω-closed order inclusions. For the specific cofree tick runner one would still have to establish the restriction of the transition and justify reflecting its equations. Existence of cofree equation-respecting runners for the full continuation residual is left open here.

6 Perspective on equations via predicate algebras

The proof for free signatures uses the order on partial implementations to construct the cofree runner. The lifting quotient additionally uses the fact that a successful transition gives an actual successor. Full continuation residuals offer a different description of the equation.

Currying and the handler correspondence identify a K_C-residual runner of T on S with a T-algebra on $[S, C]$. For ticks, $f : S \to K_C S$ corresponds to the continuous endomorphism $\tau(k)(s) = f(s)(k)$. The Kleisli power $f^{[m]}$ corresponds to τ^m by the continuation bind formula. A map $h : S \to S'$ of runners is exactly a map whose pullback on tests satisfies

$$\tau \circ [h, C] = [h, C] \circ \tau'.$$

Thus stabilisation is the ordinary algebra equation $\tau^{n+1} = \tau^n$. The reflection imposing this equation exists in the category of domains with a continuous unary operation, by the coequalizer of the two power maps there. In particular these algebras form the Eilenberg–Moore category of Ny, whose endofunctor preserves filtered colimits, on a locally presentable base [1, Thm. 5.19].

This gives a possible construction to test. Start with the unrestricted cofree runner W on Y, and quotient the unary algebra $[W, C]$ by the equation. If its universal quotient map could be realised as pullback along a continuous map $E \to W$, with target $[E, C]$, it would give the desired cofree runner provided the induced algebra factors of behaviour maps are themselves pullbacks. These are two separate conditions. The first concerns the quotient map as well as its codomain. The second concerns morphisms of the represented algebras. The observation on E would be the composite with the observation on W.

Example 6.1 (A two-state test of the quotient)**.** Take $Y = 1$, $C = \Sigma$, and $S = \{a, b\}$ discrete. Let σ swap a, b and use the pure tick $f(s)(k) = k(\sigma(s))$. On $[S, \Sigma] \cong \Sigma^2$, its algebra map τ swaps the coordinates. The reflection imposing $\tau^2 = \tau$ identifies the middle points $(1, 0)$ and $(0, 1)$. It gives the three-element chain with identity operation. Indeed, if an algebra map r intertwines τ with an idempotent τ', then $r = r\tau^2 = (\tau')^2 r = \tau' r = r\tau$. Identifying each point with its swap therefore gives the desired universal quotient.

Suppose an equation-respecting runner on S' maps to the swap runner by $h : S' \to S$. Its pullback $[h, \Sigma]$ is just such an algebra map, so it identifies the two middle tests. At every s' those tests would have to agree at $h(s')$. They take different values at both a and b, so S' is empty. The empty runner is thus terminal among idempotent runners mapping into the swap runner. Its predicate algebra $[0, \Sigma]$ has one point, while the algebra reflection retains three. The same representability obstruction holds over **Set** with a two-element answer set. The quotient of the unrestricted cofree continuation runner remains to be analysed.

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