Getting to Grips with Sweedler Runners for Systems

A conversational tutorial, Part IV: why duoidal?
Interchange, measuring, and changing the ambient universe

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Aim. Parts I–III fixed the ambient category and concentrated on the systems that live inside it: comodels, residual runners, tight theory maps, limits and colimits of theories, tensor products of theories, and parameter change. Part IV finally asks what made all of that possible. Why did we need a *duoidal* category? What exactly is the interchange law doing? How does it create the monad–comonad interaction laws behind runners? And what must be checked if we replace **Set** by a world of valued predicates, graded objects, cubical state, or resource-sensitive processes?

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1 The structure we have been using without looking at it

Our standing example has been

$$\mathcal{D} = [\mathbf{Set}, \mathbf{Set}]_{\mathrm{acc}},$$

the category of accessible endofunctors of **Set**.

In Parts I–III we used monads T, R in this category, their Sweedler hom

$$\mathcal{M}(T, R),$$

and coalgebras of that comonad. We then read a coalgebra as a residual runner

$$\theta_X : TX \times S \longrightarrow R(X \times S).$$

It was easy to forget that none of this is available from the category of endofunctors merely because it has composition. The Sweedler construction uses *two* monoidal structures, and it uses their compatibility in an essential way.

Let us therefore rewind.

The two monoidal structures have different jobs:

$F \diamond G$	$=$	$F \circ G$	(substitution / sequential effect structure),
$F \star G$	$=$	Day convolution	(interaction structure).

Monads are monoids for $\diamond$. The symmetric product $\star$ is the one whose closed structure lets an effectful computation interact with a coeffectful environment.

The point of a duoidal category is not just that both products happen to exist. It is that they are connected by a coherent *interchange* map. That map is what lets interaction respect substitution.

2 First monoidal structure: composition

Composition of endofunctors is the familiar monoidal structure

$$(\mathcal{D}, \diamond, \mathrm{Id}).$$

Thus

$$F \diamond G = F \circ G,$$

and the unit is the identity endofunctor.

A monoid for this product consists of

$$T \diamond T \longrightarrow T, \qquad \mathrm{Id} \longrightarrow T,$$

with the usual associativity and unit laws. This is exactly a monad. So when we say throughout this tutorial that a monad is an "effect theory", the relevant ambient product is composition. Its multiplication

$$TTX \longrightarrow TX$$

says how a computation whose continuations are themselves computations is flattened by substitution.

Dually, a comonoid for composition is a comonad, with counit $D \to \mathrm{Id}$ and comultiplication $D \to D \diamond D$. Whereas a monad's unit introduces effect-free computations and its multiplication combines nested layers of effects, a comonad's counit extracts a value from its context and its comultiplication makes that context available at successive levels. For example, the stream comonad extracts the present observation and associates to every time the entire future stream from that time onward. Thus monads organise the composition of effects, while comonads organise the repeated availability of contextual information.

This is already enough to speak about monads and comonads. It is *not* yet enough for Sweedler theory.

3 Second monoidal structure: Day convolution

Assume for the moment that the base category $\mathcal{C}$ is a locally presentable symmetric monoidal closed category, with tensor $\otimes$ and unit I, and that we restrict to accessible endofunctors. The category

$$[\mathcal{C}, \mathcal{C}]_{\mathrm{acc}}$$

then carries a symmetric monoidal closed convolution product. We write it as

$$F \star G.$$

For $\mathcal{C} = \mathbf{Set}$ with its cartesian tensor, the Day convolution may be pictured through the coend formula

$$(F \star G)(Z) \;\cong\; \int^{X,Y} \mathbf{Set}(X \times Y, Z) \times FX \times GY.$$

We do not need to manipulate this coend constantly. The more useful operational description is its universal property.

A natural transformation

$$F \star G \longrightarrow H$$

is the same sort of data as a family

$$FX \times GY \longrightarrow H(X \times Y),$$

natural in X, Y. That shape should look familiar. The interaction laws used throughout Parts I–II had exactly this character.

3.1 A generic element picture

An element of $(F \star G)(Z)$ can be represented schematically by

$$X \times Y \xrightarrow{u} Z, \qquad x \in FX, \qquad y \in GY,$$

modulo the coend identifications. One should read this as follows:

> F produces an X-shaped piece of structure, G produces a Y-shaped piece of structure, and the map $u : X \times Y \to Z$ tells us how their two interfaces combine into a Z-result.

For example, $F(X)$ might consist of lists of temperature readings and $G(Y)$ of lists of humidity readings. A function $u : X \times Y \to Z$ specifies how an individual temperature and humidity reading combine into a weather report or comfort classification. An element of the convolution $(F \star G)(Z)$ can be represented by precisely such a triple: a list of temperatures, a list of humidities, and the rule for combining their entries.

For **Set**, the convolution unit is naturally isomorphic to Id. In more general bases it is useful to keep the two monoidal units conceptually separate.

Do not confuse two tensors. Part III introduced the Kronecker/Plotkin–Power tensor $T \otimes R$ of *Lawvere theories*. That tensor says that the operations of two theories commute. The present product $F \star G$ is Day convolution in the *ambient endofunctor category*. It is the interaction tensor used to build Sweedler measuring objects. The two constructions live at different categorical levels and solve different problems.

4 Why two unrelated monoidal structures would not be enough

Suppose we had composition $\diamond$ and convolution $\star$, but no compatibility between them. Then we could certainly talk about monads under $\diamond$, and we could separately talk about convolution products. What we could not do coherently is use convolution to combine objects that themselves carry monad or comonad structure defined using composition.

The crucial map has the form

$$\boxed{(A \diamond B) \star (C \diamond D) \longrightarrow (A \star C) \diamond (B \star D).}$$

This is the interchange map. It says, very roughly:

> If two interactions each contain a sequential layer, we may first pair the inner pieces and then pair the outer pieces, obtaining a sequential composite of interactions.

The interchange map expresses the compatibility of two ways of composing interactions: sequentially, by functor composition, and in parallel, by Day convolution. On the left, we have two interacting structures, each with an outer and an inner layer: an A-structure containing B-structure, and a C-structure containing D-structure. Interchange reorganises these into a sequential composite of paired structures. At the inner level, B and D interact through Day convolution; at the outer level, A and C interact in the same way. For example, suppose each system performs two successive stages of activity, recording information at each stage. Initially, we have two separate sequences, $\ell \to k$ and $n \to m$. Interchange groups the corresponding stages into $(\ell, n) \to (k, m)$. More generally, Day convolution allows such a regrouping for arbitrary endofunctors, using the map that combines their inner interfaces to construct the interaction of the outer structures. Thus interchange is not an assertion that arbitrary effects commute. It is a canonical way of coordinating the two kinds of composition: interact first and then expose the sequential structure, or organise the interactions stage by stage. The interchange transformation relates these constructions in the indicated direction; it need not be invertible.

A duoidal category consists of two monoidal structures together with this interchange and compatible unit maps, satisfying coherence axioms. There are several equivalent conventions for which product is declared lax monoidal with respect to the other; we will keep the convention adapted to the endofunctor example above.

4.1 Seeing interchange on representatives

In the Set endofunctor case, start with a representative of

$$((A \circ B) \star (C \circ D))(Z).$$

It consists schematically of

$$u : X \times Y \to Z, \qquad a \in A(BX), \qquad c \in C(DY).$$

From u, Day convolution gives a canonical map

$$BX \times DY \longrightarrow (B \star D)(Z).$$

Now the outer elements

$$a \in A(BX), \qquad c \in C(DY)$$

can themselves be combined by Day convolution over that map, producing an element of

$$(A \star C)((B \star D)(Z)).$$

That is precisely

$$((A \star C) \circ (B \star D))(Z).$$

So the abstract interchange map has an intuitive two-stage reading:

interact the inner layer, then interact the outer layer.

4.2 Why this is the right compatibility for systems

Think of $\diamond$ as substitution. Then $A \diamond B$ means that an A-operation has B-computations sitting in its continuation positions. Likewise $C \diamond D$.

If those two composite structures are to interact, their continuation layers B, D must first interact, and then the outer layers A, C can interact on the resulting object. That is exactly the direction of the interchange map.

> Duoidal interchange is the law that lets interaction respect substitution.
>
> This is the conceptual reason that the second tensor cannot simply be bolted onto the first one after the fact.

5 Closed convolution turns interaction laws into monad maps

5.1 From interactions to transformations: the internal hom

So far we have considered two ways of combining computational structures: sequential composition and Day convolution. We now turn to a different question. Rather than combining two structures, can we describe the possible ways of transforming one into another?

The familiar analogy is currying. A function $X \times Y \to Z$ may equally be regarded as a function $X \to Z^Y$. The exponential Z^Y represents the possible ways of turning a Y-input into a Z-output. Day convolution has an analogous internal hom. We write $[D, R]_\star$ for the endofunctor characterised by the natural bijection

$$\mathcal{D}(F \star D, R) \cong \mathcal{D}(F, [D, R]_\star).$$

Thus an interaction of F with D producing R can equivalently be described as an F-structure taking values in the possible transformations from D to R.

In our setting there is a particularly useful concrete description:

$$[D, R]_\star(X) \cong \mathrm{Nat}\big(D, R(X \times -)\big).$$

An element of the internal hom at X is therefore a natural family of maps

$$\phi_Y : D(Y) \longrightarrow R(X \times Y).$$

It turns a D-structure into an R-structure, uniformly in Y, while also producing an X-result.

Take $D = \mathrm{Id}$. Then

$$[\mathrm{Id}, R]_\star(X) \cong \mathrm{Nat}(\mathrm{Id}, R(X \times -)).$$

By Yoneda for accessible endofunctors (or directly by naturality),

$$\mathrm{Nat}(\mathrm{Id}, R(X \times -)) \cong R(X).$$

Hence

$$\boxed{[\mathrm{Id}, R]_\star \cong R.}$$

This is a satisfying sanity check: if there is no nontrivial source structure to transform, the internal hom simply recovers the residual effect theory.

We can do better still with the environment comonad

$$D(Y) = E \times Y.$$

Then

$$[D, R]_\star(X) \cong R(X \times E).$$

So an element of the internal hom is an R-computation producing both an X-result and an E-value. Its multiplication uses the diagonal $E \to E \times E$ implicitly through the comonad structure, together with the multiplication of R.

Why does this form a monad? Suppose now that D is a comonad and R is a monad for composition. We would like to combine successive transformations of the above kind into a single transformation.

The two structures play complementary roles. Comultiplication

$$\delta : D \longrightarrow D \circ D$$

makes the D-structure available to successive stages, while multiplication

$$\mu : R \circ R \longrightarrow R$$

combines the residual R-effects produced by those stages.

The interchange transformation makes these operations compatible. More precisely, it induces a canonical map

$$[D, R]_\star \circ [D, R]_\star \longrightarrow [D \circ D, R \circ R]_\star.$$

Precomposing with δ and postcomposing with μ gives

$$[D, R]_\star \circ [D, R]_\star \longrightarrow [D, R]_\star.$$

Similarly, the counit of D and the unit of R give a map

$$\mathrm{Id} \longrightarrow [D, R]_\star.$$

These satisfy the monad axioms. Thus the possible transformations from a comonad D into a monad R themselves acquire a structure of sequential composition.

The significance for Sweedler theory is that implementations of computational structure can themselves be organised algebraically. We will now make this precise.

5.2 The duoidal structure: composition, convolution, and interchange

The symmetric convolution product is closed. Let

$$[D, R]_\star$$

denote its internal hom, so that

$$\mathcal{D}(F \star D, R) \cong \mathcal{D}(F, [D, R]_\star).$$

Now suppose

$$D \quad \text{is a comonad}$$

for composition and

$$R \quad \text{is a monad}$$

for composition. The duoidal interchange makes the internal-hom functor lax monoidal with respect to composition. Consequently

$$[D, R]_\star$$

inherits a monad structure. This is one of the key structural moves in the whole story.

We can therefore ask for a monad map

$$T \longrightarrow [D, R]_\star.$$

By the convolution adjunction, its underlying natural transformation corresponds to

$$T \star D \longrightarrow R.$$

In the endofunctor case this is an interaction family of the familiar shape

$$TX \times DY \longrightarrow R(X \times Y).$$

The monad/comonad laws say that this interaction respects units, substitution, counits, and duplication.

This explains a formula that appeared very early in our programme without much motivation. A residual monad–comonad interaction law is not just an arbitrary natural transformation $T \star D \to R$. Duoidality upgrades the closed convolution object $[D, R]_\star$ to a monad, so that the interaction can equivalently be required to be a *monad map*

$$T \to [D, R]_\star.$$

That is a much tighter algebraic statement.

6 The Sweedler hom as the universal interacting comonad

From one implementation to all implementations. Recall the stateful runners of Parts I and II. Given a source effect theory T, a residual effect theory R, and a state space S, a runner is described by natural maps

$$\theta_X : TX \times S \longrightarrow R(X \times S),$$

satisfying equations expressing compatibility with the unit and multiplication of T and R.

Operationally, the runner receives a request to perform a T-computation, together with the current state. It carries out that request using the effects available in R, returning a result and an updated state within an R-computation. Different state spaces and different implementation strategies give different runners. But can we understand all these implementations as instances of one universal construction? This is the purpose of the *Sweedler hom*.

From an internal hom to an interaction law. In the preceding section we saw that, given a comonad D and a monad R, the convolution internal hom

$$[D, R]_\star$$

is itself a monad. We can therefore interpret the operations of a source theory T in this new monad. Such an interpretation is a monad morphism

$$\alpha : T \longrightarrow [D, R]_\star.$$

By the internal-hom adjunction, the underlying natural transformation can equally be presented as

$$\overline{\alpha} : T \star D \longrightarrow R.$$

The requirement that α preserve the monad structure imposes the coherence conditions that make $\overline{\alpha}$ an *interaction law*.

The comonad D describes the structure available to mediate the interaction. Its comultiplication makes that structure available across successive stages, while the multiplication of R combines the residual effects produced by those stages. Thus an interaction law describes a uniform way of interpreting T-computations using R-computations in the presence of D-structure. The monad-morphism conditions ensure that interpreting a sequential computation agrees with sequentially interpreting its constituent computations.

The universal interacting comonad. So far, however, we have chosen D in advance. What happens if we allow the interacting comonad to vary? Fix monads T and R. Their *Sweedler hom*

$$\mathcal{M}(T, R)$$

is the universal comonad classifying coherent interactions of T with R.

More precisely, in the accessible-endofunctor setting under consideration, the Sweedler measuring construction gives a natural bijection

$$\boxed{\mathbf{Comon}(\mathcal{D})(D, \mathcal{M}(T, R)) \cong \mathbf{Mon}(\mathcal{D})(T, [D, R]_\star).}$$

On the right are the ways of interpreting T-effects using R-effects in the presence of a chosen comonad D. On the left are comonad morphisms

$$D \longrightarrow \mathcal{M}(T, R).$$

The bijection says that every coherent interaction with a chosen D is obtained uniquely by composing with the universal interaction associated with $\mathcal{M}(T, R)$.

It is important not to confuse this universal object with an individual implementation. The Sweedler hom is not itself a particular state machine, nor should we think of it simply as an enormous state space containing every possible runner. It is a comonad that *classifies* coherent interactions.

Recovering stateful runners. How does this abstract universal construction connect with the machines of Parts I and II? The connection is provided by coalgebras. For the accessible endofunctors considered here, the McDermott–Rivas–Uustalu correspondence identifies coalgebras of the Sweedler hom with residual stateful runners, under the hypotheses of their theorem.

Concretely, a coalgebra

$$\xi : S \longrightarrow \mathcal{M}(T, R)(S)$$

corresponds to a natural family

$$\theta_X : TX \times S \longrightarrow R(X \times S)$$

satisfying the runner equations. The maps θ_X describe what happens when a T-computation is presented to a machine in state $s \in S$: the machine implements the requested computation using R-effects, with both the result and the updated state represented inside the resulting R-computation. The unit and multiplication equations ensure that this interpretation respects the sequential structure of the source theory. For example, implementing a composite T-computation must agree with implementing its successive stages, passing the resulting state from one stage to the next.

Thus a Sweedler-hom coalgebra is a machine whose states determine how the source theory is implemented. The runner equations are not an isolated collection of constraints imposed on machines: they arise from the universal theory of interactions between monads and comonads.

Example: implementing a store. The simplest case occurs when the residual theory is the identity monad, $R = \mathrm{Id}$. The runner maps then have the form

$$\theta_X : TX \times S \longrightarrow X \times S.$$

There are no residual effects: every requested operation must be implemented entirely through the state S.

For example, let T be the usual store theory for values in a set O, with operations

$$\mathsf{get} : 1 \longrightarrow O, \qquad \mathsf{put} : O \longrightarrow 1.$$

Here we use the conventional algebraic-effects notation for operations and their result types. A runner implements these operations through maps

$$\mathsf{get}_S : S \longrightarrow O \times S, \qquad \mathsf{put}_S : O \times S \longrightarrow S.$$

Writing

$$\mathsf{get}_S(s) = (g(s), s')$$

for the observed value and successor state, the store equations force a read to leave the state unchanged, so that

$$\mathsf{get}_S(s) = (g(s), s).$$

They also require that writing a value makes that value available to subsequent reads, and that successive writes behave as the final write.

More explicitly, writing

$$p(o, s) = \mathsf{put}_S(o, s),$$

the equations include

$$g(p(o,s)) = o, \qquad p(o', p(o,s)) = p(o', s), \qquad p(g(s), s) = s.$$

These are precisely the familiar laws of a store comodel. Consequently, coalgebras for

$$\mathcal{M}(T, \mathrm{Id})$$

are the comodels of T.

The important point is that the comodel laws are inherited from the equations of the source theory. We do not first describe arbitrary state transitions and then separately ask which ones happen to obey the store laws. The Sweedler-hom coalgebras already encode lawful implementations of the specified theory.

Allowing a nontrivial residual monad R generalises this situation. A store operation might then be implemented using additional effects, such as logging, nondeterministic choice, or communication with an external service. The corresponding runner has the form

$$TX \times S \longrightarrow R(X \times S),$$

and its equations ensure that the implementation still respects the store theory.

Three ways to change a system. The construction also makes an important systems-theoretic distinction visible.

- Changing the source monad T changes what operations the system is asked to perform, and which equations those operations obey.
- Changing the residual monad R changes the effects or resources available to implement those operations.
- Changing the coalgebra
$$S \longrightarrow \mathcal{M}(T, R)(S)$$
changes how a particular stateful system implements the same source theory using the same residual repertoire.

We will encounter all three possibilities throughout the tutorial, especially when studying safety, verification, and changes of ambient category.

The constructions fit together as follows:

Composition and Day convolution
$\Downarrow$
Interchange gives $[D, R]_\star$ a monad structure
$\Downarrow$
Monad maps $T \to [D, R]_\star$ describe coherent interactions
$\Downarrow$
$\mathcal{M}(T, R)$ classifies these interactions
$\Downarrow$
$\mathcal{M}(T, R)$-coalgebras describe residual stateful runners.

The Sweedler hom is the universal classifier; a coalgebra is a particular implementation. In the special case $R = \mathrm{Id}$, these coalgebras are the comodels of T. With nontrivial R, they are residual runners.
This is the categorical machinery underlying the concrete systems studied in Parts I and II.

7 Runner composition is enriched composition

In Part II we composed runners concretely. If

$$(S, \theta) : T \rightsquigarrow R$$

and

$$(Q, \psi) : R \rightsquigarrow P,$$

then their composite uses state $S \times Q$: first run the T-computation through S, obtaining an R-computation, then run that through Q.

At the universal level, this comes from enriched composition

$$\boxed{\mathcal{M}(R, P) \star \mathcal{M}(T, R) \longrightarrow \mathcal{M}(T, P).}$$

Notice which tensor occurs here: it is the convolution tensor $\star$ on comonads. Duoidality ensures that convolution of composition-comonoids is again a composition-comonoid, so the left side is again a comonad. Passing to coalgebras turns this abstract enriched composition into execution through successive hidden states. This gives a satisfying explanation for a fact that otherwise looks like a clever ad hoc construction:

loose composition in $\mathbf{Run}(\mathcal{D})$ is Sweedler-enriched composition made concrete.

8 A checklist for changing the ambient category

Later parts will repeatedly vary the ambient category. We can now ask a disciplined question. Suppose we want a new systems universe $\mathcal{E}$. When is it sensible to hope that it supports our Sweedler constructions? One lesson will be that it is often unnecessary, and technically undesirable, to work with all accessible endofunctors, in the form of

$$\mathbf{Run}([\mathcal{E}, \mathcal{E}]_{\mathrm{acc}})$$

Indeed, a smaller category generated from a chosen enriched system of arities

$$j : \mathcal{J} \hookrightarrow \mathcal{C}$$

may already carry the composition and convolution structures needed for Sweedler theory. Part VI gives the first concrete instance, using the representable cost objects in C-Set.

For now, let us run through a checklist of requirements.

1. What are the theories?

We need a monoidal product $\diamond$ whose monoids are the structures we want to call theories. For ordinary endofunctors this is composition.

If we change the ambient setting too drastically, monoids for composition may no longer be the right notion. Petri-net process theories, for example, may live more naturally as monoids in an endoprofunctor or species category. So this question must come first.

2. What is the interaction tensor?

We need a second, preferably symmetric, monoidal product $\star$ that describes interaction and admits a useful closed structure.

For endofunctors on a monoidal base, Day convolution is the standard candidate. For endoprofunctors, Day convolution may instead encode parallel resource combination.

3. Is there a coherent interchange?

We need maps of the form

$$(A \diamond B) \star (C \diamond D) \longrightarrow (A \star C) \diamond (B \star D)$$

with the required coherence and unit data.

This is the point at which a promising pair of tensors either becomes a genuine Sweedler setting or does not.

4. Is the interaction tensor closed?

We need internal homs such as

$$[D, R]_\star$$

if we want measuring objects in the form used above.

5. Do the universal measuring comonoids exist?

Even with duoidality, the representing object

$$\mathcal{M}(T, R)$$

requires existence hypotheses. In the accessible endofunctor setting, local presentability and accessibility supply the needed adjoint-functor technology.

6. Is our chosen subcategory stable?

Often we do not want all endofunctors. We want accessible, finitary, enriched, strong, polynomial, or structure-preserving ones. We must check closure under composition, convolution, internal homs, and the constructions used to form the measuring object.

7. What are the concrete coalgebras?

Finally we ask whether a coalgebra for $\mathcal{M}(T, R)$ has a useful systems interpretation. In **Set** it is a state space with

$$TX \times S \to R(X \times S).$$

In another universe, the state object and the tensor with X may carry order, geometry, resources, predicates, or concurrency.

For every new proposed venue we should ask, in order:

theory tensor? interaction tensor? interchange? closedness?
measuring objects? stability of the chosen subcategory? concrete runner meaning?

This checklist will organise the next several parts of the tutorial.

9 Changing ambient universe should itself be functorial

The long-term proposal is not merely to construct $\mathbf{Run}(\mathcal{D})$ for many unrelated $\mathcal{D}$'s. We would like suitable duoidal functors

$$F : \mathcal{D} \longrightarrow \mathcal{E}$$

to transport the whole architecture.

At a minimum, such a functor should preserve enough of both monoidal structures and their interchange to send

$$\diamond\text{-monoids to } \diamond\text{-monoids}$$

and

$$\diamond\text{-comonoids to } \diamond\text{-comonoids},$$

and to compare the measuring objects

$$F\mathcal{M}_{\mathcal{D}}(T, R) \quad \text{and} \quad \mathcal{M}_{\mathcal{E}}(FT, FR).$$

Depending on the preservation hypotheses, this comparison may be an isomorphism or only a canonical map. This is why the likely statement is pseudofunctorial rather than naively strict.

Schematically we are aiming for something like

$$\boxed{\mathbf{Run} : \mathbf{Duoid}_{\mathrm{adm}} \longrightarrow \mathbf{DoubleCat}}$$

for an appropriately chosen 2-category of *admissible* symmetric duoidal categories.

This is not yet a theorem of our tutorial. It is one of the formal research tasks. But Part IV tells us exactly what such a theorem would have to preserve.

10 First reconnaissance: valued predicates

The first change of universe we want to explore is not concurrency or Petri nets. It is certification.

Let V be an ordered value object, eventually something like a commutative quantale. A V-valued predicate is a map

$$p : X \longrightarrow V.$$

A standard category V-**Pred** has objects (X, p) and a morphism

$$f : (X, p) \longrightarrow (Y, q)$$

when

$$p(x) \leq q(fx)$$

for every $x \in X$.

For $V = 2$, this just says that a function sends states satisfying the source predicate to states satisfying the target predicate.

There is a forgetful functor

$$U : V\text{-}\mathbf{Pred} \longrightarrow \mathbf{Set}, \qquad (X, p) \longmapsto X.$$

This is naturally a fibrational setting: over a fixed set X, the fibre consists of all V-valued predicates on X.

10.1 A candidate monoidal structure

If V is a commutative unital quantale with tensor $\otimes_V$ and unit k, then there is an evident tensor

$$(X, p) \otimes (Y, q) = \big(X \times Y,\ (x, y) \mapsto p(x) \otimes_V q(y)\big)$$

with unit $(1, k)$.

Because a quantale tensor preserves joins and has a residual, this category is a natural candidate for a symmetric monoidal closed base. A corresponding internal-hom predicate on the set Y^X is expected to measure how well a function preserves the valuation, schematically by

$$[p, q](f) = \bigwedge_{x \in X} (p(x) \multimap q(fx)),$$

where $\multimap$ is the residual of $\otimes_V$.

The exact hypotheses and variance conventions should be checked carefully when we commit to a particular V. For Part IV the point is only that V-**Pred** is not an arbitrary new category: it comes with plausible monoidal and closed structure of exactly the kind needed by the endofunctor construction.

11 Two different ways to move from Set to valued predicates

Here we should be careful. There are at least two constructions one might mean by "replace Set by V-**Pred**".

11.1 Route A: internalise everything in V-Pred

The most literal move is

$$\mathcal{D}_V = [V\text{-}\mathbf{Pred}, V\text{-}\mathbf{Pred}]_{\mathrm{acc}}.$$

If V-**Pred** satisfies the appropriate local presentability and monoidal hypotheses, then we may hope to equip this endofunctor category with

$$\text{composition} \qquad \text{and} \qquad \text{Day convolution},$$

just as before.

A monad in this category would then be a theory acting directly on certified objects. A runner state would itself be a valued predicate

$$(S, s : S \to V),$$

and the runner map would be a morphism in V-**Pred**, not just an underlying function.

This is conceptually clean, but it may change more of the theory than we want.

11.2 Route B: keep the Set theory and lift it

Often our intention is different. We already have a Set-endofunctor

$$F : \mathbf{Set} \to \mathbf{Set}$$

and want a lifted functor

$$\bar{F} : V\text{-}\mathbf{Pred} \to V\text{-}\mathbf{Pred}$$

with

$$U\bar{F} = FU.$$

Then $\bar{F}$ does not replace the underlying dynamics. It enriches the same dynamics with a rule for transporting predicates or valuations.

Likewise, a lifted monad

$$\bar{T}$$

keeps the ordinary effect theory T visible underneath while adding certified structure above it.

This is the territory of categorical TT-lifting, codensity lifting, and related fibrational constructions.

The two routes answer different questions:

$[V\text{-}\mathbf{Pred}, V\text{-}\mathbf{Pred}]_{\mathrm{acc}}$	: rebuild the whole theory inside certified objects,
liftings over $[\mathbf{Set}, \mathbf{Set}]$	: keep the old theory and add certification to it.

For our systems programme the second route may often be closer to "certify an existing machine", while the first may be cleaner for an intrinsic theory of valued systems.

12 A tiny $V = 2$ example

Take ordinary predicates. An object is

$$(X, P), \qquad P \subseteq X.$$

A morphism $f : (X, P) \to (Y, Q)$ is a function with

$$x \in P \implies f(x) \in Q.$$

Now consider the unary theory from Part I. A deterministic runner is an endofunction

$$\sigma : S \to S.$$

If S is equipped with a safety predicate $P \subseteq S$, then requiring

$$\sigma : (S, P) \to (S, P)$$

in 2-Pred says exactly

$$s \in P \implies \sigma(s) \in P.$$

So the most elementary "certified runner" is simply a transition preserving an invariant.

For a Moore machine

$$\delta : A \times S \to S, \qquad o : S \to O,$$

a lifted theory should let us say not merely that δ and o exist, but that the appropriate predicates on input, state, and output are respected.

That is the first point of contact with assume–guarantee reasoning. But notice that we have not yet derived the correct lifted monad or the correct duoidal endofunctor category. We have only seen what its simplest morphism condition ought to express.

13 What has to be proved before $\mathbf{Run}(\mathbf{Pred}_V)$ is legitimate

The notation

$$\mathbf{Run}([V\text{-}\mathbf{Pred}, V\text{-}\mathbf{Pred}]_{\mathrm{acc}})$$

is attractive, but at this stage it is a research target rather than established infrastructure. We should not write it as though all hypotheses have already been checked.

- **The base category is locally presentable.** This point can be settled quite generally. Let V be a set-sized complete lattice, and choose a regular cardinal

  $$\lambda > |V|.$$

  A V-predicate

  $$p : X \to V$$

  may equivalently be presented by its level sets

  $$X_v = \{x \in X \mid v \leq p(x)\}, \qquad v \in V.$$

  These form an antitone family of subobjects of X:

  $$v \leq w \quad \implies \quad X_w \hookrightarrow X_v,$$

and satisfy

$$X_{\bigvee_i v_i} = \bigcap_i X_{v_i}.$$

In particular,

$$X_\perp = X.$$

Conversely, an antitone family of subobjects satisfying these intersection conditions determines a unique predicate by

$$p(x) = \bigvee \{v \in V \mid x \in X_v\}.$$

Indeed, the intersection axiom ensures that

$$x \in X_v \quad \iff \quad v \leq p(x).$$

Thus V-**Pred** is equivalent to the category of models of a small λ-ary limit sketch: one has a sort X, subobject sorts X_v, the inclusions expressing antitonicity, and specified limits expressing

$$X_{\bigvee_i v_i} = \bigcap_i X_{v_i}.$$

Since categories of models of small limit sketches are locally presentable, it follows that

$$\boxed{V\text{-}\mathbf{Pred} \text{ is locally presentable.}}$$

This conclusion concerns only the underlying category and does not depend on the particular quantale tensor placed on V.

The situation is therefore better than the cautious formulation above suggested. Local presentability of the base is not an obstruction to Route A. The genuinely nontrivial questions occur one level higher: whether the appropriate category of accessible strong endofunctors carries the required duoidal interchange, and whether the corresponding measuring objects exist.

In addition, at minimum we need to establish the following.

1. **Endofunctor duoidality.** Does the accessible endofunctor category inherit composition and Day convolution with the required interchange?
2. **Choice of subcategory.** Do we want all accessible endofunctors, strong endofunctors, fibred endofunctors, or functor liftings over **Set**?
3. **Lifted theories.** Given an ordinary monad T, which liftings $\bar{T}$ are appropriate for safety, quantitative bounds, assumptions, or guarantees? TT-lifting and codensity lifting provide established techniques for constructing such lifts, but they do not by themselves prove the full Sweedler environment we want.
4. **Measuring objects.** Do the resulting categories admit universal Sweedler homs, and do their coalgebras have the intended certified-runner interpretation?
5. **Comparison with Set.** Does the forgetful or underlying-set functor induce a duoidal comparison that sends a certified runner to its ordinary runner?

This list is deliberately more cautious than saying "take $[\mathbf{Pred}_V, \mathbf{Pred}_V]_{\text{acc}}$ and proceed". Part IV is where we learn to distinguish a promising notation from a verified Sweedler setting.

14 How later examples fit the same checklist

We will not develop the later venues here, but the point of Part IV is that we now know how to approach each of them.

14.1 Ordered cost monoids and amortisation

Let C be an ordered commutative cost monoid, viewed as a thin monoidal category. The category of C-sets or copresheaves carries Day convolution, with representables satisfying the resource law

$$y(p) \otimes_{\mathrm{Day}} y(q) \cong y(p+q).$$

The question for a later part is whether an appropriate enriched/strong accessible endofunctor category

$$[C\text{-}\mathbf{Set}, C\text{-}\mathbf{Set}]_{\mathrm{acc}}$$

(or a closely related category) satisfies our duoidal checklist. If so, graded effects and amortised resources enter the same Sweedler architecture rather than being added externally.

14.2 Precubical sets and HDAs

For higher-dimensional automata, the state object is no longer merely a set of points: it contains cubes representing concurrent executions. A candidate venue is

$$[\mathbf{PreCub}, \mathbf{PreCub}]_{\mathrm{acc}}.$$

Again the first question is not "what is an HDA runner?" but "what are the two tensors and does the interchange law exist?"

14.3 Petri nets and endoprofunctors

For a palette S, a particularly suggestive category is

$$\mathcal{D}_S = [\mathbb{B}_S^{op} \times \mathbb{B}_S, \mathbf{Set}].$$

Here the two products have an immediately systems-theoretic reading:

$$\boxed{\begin{aligned} \text{profunctor composition} &= \text{ sequential process composition},\\ \text{Day convolution} &= \text{ parallel resource combination}. \end{aligned}}$$

This is therefore not merely "endofunctors on a different category". It is a genuinely different duoidal universe, and potentially one of the strongest tests of the whole Run programme.

15 What Part IV has added

The first three parts taught us how to work *inside* one Sweedler universe. Part IV has explained what such a universe must contain.

Structure	Systems reading
Composition $\diamond$	Substitution/sequential structure; its monoids are effect theories and its comonoids are coeffect environments.
Day convolution $\star$	Symmetric interaction structure; its closed hom internalises "environment interacting with residual theory".
Interchange	Lets interaction commute coherently with substitution; this is the heart of duoidality.
Closed convolution $[D, R]_\star$	Becomes a monad when D is a comonad and R a monad.
Sweedler hom $\mathcal{M}(T, R)$	Universal comonad classifying interactions of T with residual theory R.
Enriched composition	Universal source of sequential runner composition.
Change of $\mathcal{D}$	Changes what counts as state, effect, interaction, certification, resource, or process.

The single sentence to keep is

duoidality is what allows theories of substitution to have a composable theory of interaction.

16 Where we turn next

We next deploy abstract duoidal theory by taking the first candidate new universe seriously.

Part V: Certified runners in V-valued predicates.

We should begin with $V = 2$, where everything can be read as ordinary invariants and safety predicates, and only then pass to quantitative V. The questions will be concrete:

- What is a lifted unary theory?
- What is a certified Moore runner?
- What is the handler induction functor after lifting?
- How do assumptions and guarantees live on the boundary of a runner?
- Which lifting construction gives the monads we actually want?
- Can the whole lifted setting be made duoidal, rather than merely lifting individual monads?

After that, the same checklist can be applied to graded/amortised computation, higher-dimensional automata, and Petri nets.

References and orientation

For the general Sweedler construction in a duoidal category:

- I. López Franco and C. Vasilakopoulou, *Duoidal categories, measuring comonoids and enrichment*, arXiv:2005.01340. The central result is that, under suitable existence hypotheses, monoids for one product are enriched in comonoids by universal measuring objects.

For the endofunctor instance and its application to monad–comonad interactions and residual runners:

- D. McDermott, E. Rivas, and T. Uustalu, *Sweedler Theory of Monads*, FoSSaCS 2022, LNCS 13242, 428–448. Their Section 2 constructs the symmetric closed convolution on accessible endofunctors; Section 3 identifies monad–comonad interaction laws as measuring maps and develops the Sweedler hom and copower.

For the duoidal structure on endofunctors under composition and Day convolution:

- R. Garner and I. López Franco, *Commutativity*, Journal of Pure and Applied Algebra 220 (2016), 1707–1751. The endofunctor example is a standard source of the composition/Day-convolution duoidal structure used in the Sweedler theory of monads.

For V-valued predicates and functor liftings:

- S. Goncharov et al., work on fibrations of quantale-valued predicates and relations, where objects are maps $p : X \to V$ and morphisms satisfy $p \leq q \circ f$; this provides a standard quantitative-predicate base for lifting functors.
- S. Katsumata, *A Semantic Formulation of TT-lifting and Logical Predicates for Computational Metalanguage*, CSL 2005; and S. Katsumata, T. Sato, T. Uustalu, *Codensity Lifting of Monads and its Dual*, LMCS 14(4), 2018. These provide established methods for lifting monads along fibrations, but further work is needed to organise such liftings into the full duoidal Sweedler setting proposed here.

The picture to retain from Part IV is

composition: theories and substitution
⇕ interchange ⇕
convolution: interaction and measuring
⇓
Sweedler homs and runners
⇓
change the ambient duoidal world to change the kind of systems.