

Getting to Grips with Sweedler Runners for Systems

A conversational tutorial, Part VI: enriched runners for resources
Strong monads on cost-indexed copresheaves and the shape of amortised execution

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Aim. Part VI asks what happens when the states, theories and runners themselves live in the resource-sensitive world used for amortised analysis. Let C be an ordered commutative monoid of costs and let

$$\mathcal{E} = C\text{-}\mathbf{Set} = [C, \mathbf{Set}]$$

carry Day convolution. We begin concretely: potential is represented by tensoring with a representable, and an amortised data structure can literally store that potential in its state. The functional queue is our running picture: enqueue deposits potential with an element; a later expensive dequeue consumes that stored potential. Only then do we ask what categorical machinery is needed above this picture. The answer points to

$$(C\text{-}\mathbf{Set})\text{-}[C\text{-}\mathbf{Set}, C\text{-}\mathbf{Set}]_{\text{acc}},$$

the strong/enriched accessible endofunctors. We then use the Cayley map $X \mapsto X \otimes -$ to pass from resource objects to resource-sensitive endofunctors. This avoids pretending that $C\text{-}\mathbf{Set}$ itself already carries the particular duoidal structure needed for the runner construction. Only after that conceptual bridge is in place do we construct the second tensor, interchange and the Sweedler machinery. The technical existence argument is deliberately moved to an appendix. In the main text we use only the working conclusion: this is a highly plausible and well-supported ambient category for resource-sensitive runners.

Contents

1	Amortisation in the resource-sensitive base	2
1.1	Why amortisation is a good test of the architecture	3
1.2	Cost and potential already live in the base category	3
1.3	A running picture: the functional queue	4
1.3.1	Enqueue deposits potential	4
1.3.2	Dequeue consumes what an earlier operation stored	5
1.4	Relation to coalgebraic amortised analysis	6
2	From resource objects to resource-sensitive theories	6
2.1	The first correction: ordinary endofunctors are too weak	6
2.2	Strong monads as resource-sensitive theories	7
2.3	The candidate ambient category	8
2.4	From graded objects to graded endofunctors: the Cayley bridge	8

3	The duoidal structure and Sweedler existence	9
3.1	The second tensor: enriched Day convolution of endofunctors	9
3.2	The interchange calculation	10
3.3	Existence status: measuring objects versus runner classification	11
3.4	What is established, and what remains open	11
4	What resource-sensitive runners look like	12
4.1	What would an enriched runner look like?	12
4.1.1	Runner composition	12
4.2	First example: the unary theory again	13
4.3	Reading resource states: representables are rigid	13
4.3.1	A comodel-side diagnostic: observations use a tensor too	14
4.4	Do not conflate two appearances of grading	14
4.5	Where the amortisation adjunction meets Run	15
5	Changing and comparing resource semantics	15
5.1	Change of grading: variance and a projection formula	15
5.2	Threshold embeddings: Day tensor versus cartesian product	16
6	Synthesis and outlook	17
6.1	Returning to the queue: what has been gained	17
6.2	Probabilistic amortisation: a conjectural extension	18
6.3	Checklist for other resource-sensitive bases	19
6.4	Where Part VI leaves us	20
A	Technical appendix: why enriched Sweedler theory should work	21
A.1	Fix an accessibility rank	21
A.2	A smaller arity-generated Sweedler universe	23
	References and further reading	25

1 Amortisation in the resource-sensitive base

Amortised analysis is a way of proving bounds on a whole run of a stateful computation when the cost of individual operations can fluctuate wildly. An operation that is cheap now may prepare the state so that a later operation becomes expensive; conversely, an expensive operation may be paid for by work or “credit” accumulated earlier. Rather than bounding every step separately by its worst case, one assigns each state a non-negative *potential* Φ and defines the amortised cost of a transition $s \rightarrow s'$ by

$$\widehat{c}(s, s') = c(s, s') + \Phi(s') - \Phi(s).$$

Summing along a run makes the potential terms telescope:

$$\sum_i c_i = \sum_i \widehat{c}_i + \Phi(s_0) - \Phi(s_n).$$

Thus, when the initial potential is fixed—most often $\Phi(s_0) = 0$ —and potential is non-negative, a uniform bound on amortised costs yields a bound on the total actual work. The familiar language

of “banking credit” is therefore not merely a metaphor: amortisation is intrinsically about resource carried by state across a sequence of interactions.

We begin below the level of theories. The aim is first to see what the cost-indexed base category already knows about potential, and then to let the functional queue show why persistent runner state is exactly the right place for amortised resource to live.

1.1 Why amortisation is a good test of the architecture

The point of changing the ambient category is not to decorate the same old systems with a new label. It should change what counts as an object, a theory, a handler, and a runner in a way that reflects the phenomenon being modelled.

Amortised analysis is an unusually good test because the semantic category is already doing real work. In *Categorical Models of Amortized Cost: An Adjoint Relationship between Cost and Potential*, cost and potential are organised by an adjoint pair of graded functors; the cost functor carries compatible graded monad and graded comonad structure; and one of the concrete models is built from copresheaves on a monoidal category of costs. Products and function spaces are interpreted using Day convolution and its right adjoint. Thus resource combination is not external bookkeeping: it is already part of the monoidal geometry of the semantic category.

Let $C = (C, +, 0, \leq)$ be an ordered commutative monoid, regarded as a thin symmetric monoidal category. Following the copresheaf convention, write

$$\mathcal{E} = C\text{-}\mathbf{Set} = [C, \mathbf{Set}].$$

If one prefers presheaves, one can reverse C ; nothing essential in the discussion depends on this variance choice.

The Day tensor on $\mathcal{E}$ will be written $\otimes$. For representables,

$$y(p) \otimes y(q) \cong y(p + q).$$

This is the basic resource law: placing p and q units of potential side by side gives $p + q$.

The central question of Part VI is therefore not merely “can we put monads on C -sets?” It is: *can we build the whole Sweedler architecture so that its theories and runners respect the Day resource tensor intrinsically?*

1.2 Cost and potential already live in the base category

Before discussing monads, it is worth seeing what the base category is buying us. For a cost $p \in C$, let $y(p)$ be the corresponding representable. Tensoring with it gives a resource shift

$$P_p A := y(p) \otimes A.$$

In the amortisation semantics this is the shape of a potential modality: $P_p A$ is an A -value carrying p units of potential. Closedness gives the right adjoint

$$M_p A := [y(p), A]_{\mathcal{E}},$$

so that

$$P_p = - \otimes y(p) \dashv [y(p), -] = M_p.$$

The symbols P_p and M_p are only schematic here; the point is the geometry. Potential is represented by an actual object of $C\text{-Set}$ and cost is controlled by the tensor–hom adjunction. The equation

$$y(p) \otimes y(q) \cong y(p + q)$$

then says that stored resources add in exactly the way the cost monoid says they should.

A useful mental shift is this: potential is not a number written next to an ordinary state. It is part of the *type of the state object*. Amortised analysis can therefore become a story about ordinary morphisms in a resource-sensitive category.

1.3 A running picture: the functional queue

Consider the familiar functional FIFO queue implemented by two lists. Elements are enqueued onto a first list l_1 . Dequeue normally pops the second list l_2 ; if l_2 is empty, the first list is reversed into the second and then popped. A single dequeue can therefore be expensive, although the amortised cost is constant.

The resource-sensitive presentation explains the usual potential argument very directly, but one tensor choice must be made explicit. Here lists are *Day lists*:

$$L_{\otimes}(A) := \coprod_{n \geq 0} A^{\otimes n},$$

not cartesian lists $\coprod_n A^{\times n}$. This matters because Day tensor adds potential, whereas cartesian product only combines the information available at a common level. Ignoring length indices for the moment, take the implementation state to have the schematic form

$$S = L_{\otimes}(P_2\tau) \otimes L_{\otimes}(\tau).$$

An element waiting in the first list does not merely have type τ . It has type

$$P_2\tau = y(2) \otimes \tau.$$

Thus every such element literally carries the potential that will later pay for moving it to the second list.

1.3.1 Enqueue deposits potential

The queue analysis charges three units to enqueue. One unit pays for the immediate list construction; the remaining two are stored with the newly enqueued element. Suppressing the length indices and currying, we can picture the operation as a morphism of the form

$$P_3 1 \otimes \tau \otimes S \longrightarrow M_0 S.$$

The important point is not the exact surface syntax of this map. It is what happens to the resource: the operation turns an ordinary τ -element into a $P_2\tau$ -element in the first list. Part of today's resource has become tomorrow's stored potential.

1.3.2 Dequeue consumes what an earlier operation stored

Suppose now that the second list is empty. The queue reverses the first list into it. Every element in the first list arrives carrying its two units of potential, and that potential pays for the list construction involved in the transfer. Schematically,

$$L_{\otimes}(P_2\tau) \rightsquigarrow L_{\otimes}(\tau)$$

while the latent potential is discharged as the work is performed. After the accounting, the queue operation can again have zero unpaid residual cost, hence the appearance of M_0 in the type-level account.

The key phenomenon is temporal:

$$S_0 \xrightarrow{\text{enq}} S_1 \xrightarrow{\text{enq}} S_2 \xrightarrow{\text{deq}} S_3.$$

The resource deposited in the first two transitions is still present in the state S_2 when the third transition needs it.

The gist of amortisation in Run language. A system operation may deposit potential in the state of a runner; a later operation can consume it because both operations are realised by the same persistent stateful implementation. What is cheap is not each transition in isolation, but the behaviour of the implementation across a run.

This suggests two Run-style presentations, but they have sharply different force. Let T_Q be an abstract queue theory. A map

$$T_Q \rightsquigarrow \text{Id}$$

with state S can express a *resource-respecting state transition*: potential may be discarded, but the morphisms of the positive copresheaf model do not create it from nothing. That is a conservation or upper-bound statement. It does *not* say that stored potential suffices to pay the implementation's actual work. In particular, nothing in a residual-Id runner distinguishes an expensive reversal from a zero-cost state transition.

For amortisation proper the residual theory must expose primitive cost:

$$T_Q \rightsquigarrow T_{\text{cost}}.$$

Now the realisation must express reversal and list construction through costed residual operations, while the abstract interface charge can be compared with that actual cost. The distinction is exactly the one needed for the usual potential inequality

$$\sum_i c_i \leq \sum_i \hat{c}_i + \Phi(s_0) - \Phi(s_n).$$

To obtain the familiar bound from zero initial potential one must moreover choose an initial state

$$s_0 : I = y(0) \longrightarrow S,$$

lying at resource level zero. Amortised soundness is therefore a statement about the reachable runs of a *pointed* realisation, not about every abstract state in isolation.

The full queue types also track list lengths and conditions such as non-emptiness. Those dependent indices point toward a later indexed/fibred refinement of Run. They are not needed for the present lesson: stored potential already shows why changing the base from Set to C -Set matters, while the residual cost theory records what that potential is required to pay for.

1.4 Relation to coalgebraic amortised analysis

There is important prior work very close to this viewpoint. Grodin and Harper formulate amortised analysis coalgebraically in a category of cost algebras: a morphism of coalgebras is a first-class generalisation of a potential function, and imprecise upper bounds are treated by colax morphisms of coalgebras. They later use coalgebras for an endoprofunctor so that algebraic and coalgebraic structure can coexist while potential combines through a monoidal product.

This is not the same construction as a Sweedler runner, but it gives an important comparison point. In their account the potential function is itself a morphism between coalgebraic realisations. In the present account the potential can be internalised into the resource-sensitive state object, while the runner mediates between an abstract theory and a residual cost theory. A satisfactory worked queue example should therefore compare

$$T_Q \rightsquigarrow T_{\text{cost}}$$

with Grodin–Harper’s coalgebraic potential map, and in particular ask whether their colax coalgebra morphisms appear as a distinguished lax or colax form of comodel morphism in the four-corner Sweedler picture. This is a comparison to establish, not an identification we shall assume.

2 From resource objects to resource-sensitive theories

The queue motivates the semantic base, but Sweedler runners need theories acting on that base. The next step is therefore to identify the correct endofunctors, monads and the relation between the small resource world and the larger endofunctor world.

2.1 The first correction: ordinary endofunctors are too weak

In the Set-based tutorial we could write

$$\mathcal{D} = [\mathbf{Set}, \mathbf{Set}]_{\text{acc}}$$

and barely think about extra structure on an endofunctor. Cartesian Set hides a great deal. Over a genuinely resource-sensitive monoidal base, ordinary endofunctors are too weak for two related reasons.

The operational reason is strength. For

$$F : \mathcal{E} \rightarrow \mathcal{E}$$

to be compatible with the resource tensor, we ask for a tensorial strength

$$\text{st}_{X,Y} : X \otimes F(Y) \longrightarrow F(X \otimes Y),$$

natural in X, Y , satisfying the usual unit and associativity axioms. If $F(Y)$ represents an F -computation involving Y , strength says how extra ambient resource is threaded through that computation coherently.

There is also a more structural reason, and it is the decisive one. If one forms the ordinary Set-enriched Day convolution of endofunctors of $\mathcal{E}$, its unit has the form

$$J_{\text{ord}}(X) = \mathcal{E}(I, X) \cdot I.$$

For $\mathcal{E} = C\text{-}\mathbf{Set}$ with positive C and $I = y(0)$, Yoneda gives $\mathcal{E}(I, X) \cong X(0)$, while positivity makes $y(0)$ terminal. Hence

$$J_{\text{ord}}(X) \cong X(0) \cdot y(0),$$

the constant copresheaf generated by the level-zero part of X . It forgets all higher resource levels. Thus the ordinary convolution does not have Id as unit and the endofunctor duoidal structure fails the normality property used in the familiar $\mathbf{Set}$ -based Sweedler story.

Enrichment repairs exactly this defect. Because $\mathcal{E}$ is symmetric monoidal closed, tensorial strength is equivalent, under the standard tensored hypotheses, to enrichment over $\mathcal{E}$. The self-enrichment has internal hom objects

$$[X, Y]_{\mathcal{E}} \in \mathcal{E},$$

and a strong endofunctor is equivalently an $\mathcal{E}$ -enriched functor

$$F : \mathcal{E} \rightarrow \mathcal{E}.$$

For the enriched Day convolution the unit becomes

$$J(X) = [I, X]_{\mathcal{E}} \cong X,$$

so normality is restored.

Enrichment is therefore essential, not merely decorative. Strength gives the operational reading—resource can be threaded through a computation—while self-enrichment gives the structural payoff: the Day-convolution unit is again the identity endofunctor rather than a collapse to resource level zero.

2.2 Strong monads as resource-sensitive theories

Composition gives a monoidal product on enriched endofunctors:

$$F \diamond G := F \circ G.$$

The composite of strong functors is strong. Schematically,

$$X \otimes F(GY) \longrightarrow F(X \otimes GY) \longrightarrow F(G(X \otimes Y)).$$

The identity functor is strong as well.

A monoid for composition is therefore an enriched monad, equivalently a strong monad

$$T : \mathcal{E} \rightarrow \mathcal{E}$$

with

$$\eta : \text{Id} \rightarrow T, \quad \mu : T \circ T \rightarrow T$$

strong natural transformations.

In Part VI our “theories” are not arbitrary monads on $C\text{-}\mathbf{Set}$. They are accessible *strong* monads. The theory itself must know how its effects sit inside a resource context.

This is already a conceptual improvement over merely attaching a cost grade to a $\mathbf{Set}$ -based theory. The base objects are resource-indexed, the theory respects resource combination, and later the runner itself will be resource-sensitive.

2.3 The candidate ambient category

We therefore propose

$$\mathcal{D}_C = \mathcal{E}\text{-}[\mathcal{E}, \mathcal{E}]_{\text{acc}}, \quad \mathcal{E} = C\text{-}\mathbf{Set}.$$

Here the prefix $\mathcal{E}$ - indicates enriched functors and enriched natural transformations. At the ordinary level one may think of this as the category of accessible strong endofunctors and strong natural transformations.

There is an immediate size caution. The category of *all* accessible endofunctors need not itself be accessible; this is already true over $\mathbf{Set}$. For a given pair of accessible theories one may therefore choose a regular cardinal κ for which they are κ -accessible and work first with

$$\mathcal{D}_{C,\kappa} = \mathcal{E}\text{-}[\mathcal{E}, \mathcal{E}]_{\kappa}.$$

For small C , the base category itself is locally finitely presentable and its finitely presentable objects are closed under Day tensor: representables satisfy $y(p) \otimes y(q) \cong y(p + q)$, and finite colimits of representables remain finite under a colimit-preserving tensor. Thus there is no need to enlarge κ merely to repair tensor-closure of the base; larger ranks enter only to accommodate theories that are not finitary.

We will treat the fixed-rank world as the technically controlled core and the all-accessible world as the natural union one ultimately wants.

2.4 From graded objects to graded endofunctors: the Cayley bridge

Put

$$\mathcal{E} = C\text{-}\mathbf{Set}, \quad \mathcal{D}_C = \mathcal{E}\text{-}[\mathcal{E}, \mathcal{E}]_{\text{acc}}.$$

It is tempting to speak at once of both $\mathbf{Run}(\mathcal{E})$ and $\mathbf{Run}(\mathcal{D}_C)$, but that would suppress a choice: $\mathbf{Run}(\mathcal{E})$ is not defined until a duoidal structure on $\mathcal{E}$ has been specified. We do not need to make that choice here. What we do need is a canonical passage from resource objects into the enriched endofunctor world.

For each $X \in \mathcal{E}$, tensoring with X gives an enriched endofunctor

$$L_X : \mathcal{E} \longrightarrow \mathcal{E}, \quad L_X(A) = X \otimes A,$$

and hence a Cayley functor

$$L : \mathcal{E} \longrightarrow \mathcal{D}_C.$$

Associativity and the unit law already give

$$L_{X \otimes Y} \cong L_X \circ L_Y, \quad L_I \cong \text{Id}_{\mathcal{E}}.$$

There is a second compatibility which is easy to obscure if one reasons only from the comparison $\otimes \rightarrow \times$. Using enriched co-Yoneda twice,

$$\begin{aligned} (L_X \star L_Y)(Z) &\cong X \otimes Y \otimes \int^{A,B} [A \otimes B, Z]_{\mathcal{E}} \otimes A \otimes B \\ &\cong X \otimes Y \otimes Z. \end{aligned}$$

Thus

$$L_X \star L_Y \cong L_{X \otimes Y} \cong L_X \circ L_Y.$$

On the image of L , composition and enriched convolution coincide. In this precise sense the Cayley functor is strong from the “trivial” two-tensor structure $(\mathcal{E}, \otimes, \otimes)$ into the two-tensor endofunctor world.

Consequently, a monoid T for Day convolution in $C\text{-Set}$ determines a monad

$$L_T = T \otimes -$$

on $C\text{-Set}$. If

$$T \otimes T \longrightarrow T$$

is its multiplication, then at A the induced monad multiplication is

$$T \otimes (T \otimes A) \cong (T \otimes T) \otimes A \longrightarrow T \otimes A.$$

In particular, the potential shift $P_p A = y(p) \otimes A$ is simply $L_{y(p)}$.

For a positive cost monoid there is also a natural comparison

$$X \otimes Y \longrightarrow X \times Y.$$

It has a different role. It yields a lax monoidal comparison from the cartesian structure on $\mathcal{E}$ to composition after applying L : schematically,

$$L_X \circ L_Y \cong L_{X \otimes Y} \longrightarrow L_{X \times Y},$$

with unit $\text{Id} = L_I \rightarrow L_1$. This is weaker than, and should not be confused with, the strong $\star/\circ$ compatibility above.

The Cayley bridge does not require us to posit a runner double category directly on $C\text{-Set}$. It says instead that the elementary resource calculus embeds into the ambient category where resource-sensitive theories live, and on that image the two endofunctor tensors agree.

3 The duoidal structure and Sweedler existence

Once the ambient category has been identified, the remaining structural question is the familiar one from earlier parts: what is the second tensor, how does it interact with composition, and how much of the Sweedler existence theorem may we safely use?

3.1 The second tensor: enriched Day convolution of endofunctors

Part IV taught us that one monoidal structure is not enough. Composition makes monads, but Sweedler theory needs a second product describing interaction.

Choose a regular κ for which the particular endofunctors under discussion are κ -accessible. For small C , the base $\mathcal{E} = [C, \text{Set}]$ is already locally finitely presentable and its finitely presentable objects are closed under $\otimes$; a larger rank is needed only when the endofunctors themselves demand it.

For enriched κ -accessible endofunctors F, G , define their enriched convolution by the small enriched coend

$$(F \star G)(X) = \int^{A, B \in \mathcal{E}_\kappa} [A \otimes B, X]_{\mathcal{E}} \otimes F(A) \otimes G(B).$$

This is the enriched Day extension of

$$(A, B) \mapsto F(A) \otimes G(B)$$

along the tensor

$$\mathcal{E}_\kappa \otimes \mathcal{E}_\kappa \rightarrow \mathcal{E}_\kappa.$$

Because the indexing category is small, the coend is legitimate. Because tensor and the relevant internal homs preserve the required colimits, the result remains κ -accessible.

The unit is

$$J(X) = [I, X]_{\mathcal{E}} \cong X,$$

so, as in the Set case, the convolution unit is isomorphic to the identity endofunctor.

Do not confuse this $\star$ with the Plotkin–Power/Kronecker tensor $T \otimes R$ of Lawvere theories from Part III. Here $\star$ is a tensor on the *ambient endofunctor category*; the Kronecker tensor is a construction on already formed theories.

3.2 The interchange calculation

Now comes the central point. We want a natural interchange map

$$(F \circ G) \star (H \circ K) \longrightarrow (F \star H) \circ (G \star K).$$

Day convolution has canonical structure maps

$$F(A) \otimes H(B) \longrightarrow (F \star H)(A \otimes B).$$

Apply this at GA, KB :

$$F(GA) \otimes H(KB) \longrightarrow (F \star H)(GA \otimes KB).$$

The convolution structure for G, K gives

$$GA \otimes KB \longrightarrow (G \star K)(A \otimes B).$$

Since $F \star H$ is enriched, applying it gives

$$(F \star H)(GA \otimes KB) \longrightarrow (F \star H)((G \star K)(A \otimes B)).$$

Composing,

$$F(GA) \otimes H(KB) \longrightarrow ((F \star H) \circ (G \star K))(A \otimes B).$$

By the universal property of enriched Day convolution this induces

$$(F \circ G) \star (H \circ K) \longrightarrow (F \star H) \circ (G \star K).$$

The coherence diagrams reduce to the associativity, symmetry and unit coherence of the tensor on $\mathcal{E}$.

Thus the old slogan from Part IV survives unchanged:

interchange lets interaction respect substitution.

3.3 Existence status: measuring objects versus runner classification

The enriched Day construction gives a duoidal setting of exactly the form used by López Franco and Vasilakopoulou. Their general existence theorem applies when the ambient category is locally presentable, the monoid tensor is accessible, and the measuring tensor is biclosed. In our notation the relevant assignment is: composition is the monoid tensor whose monoids are the strong monads, while enriched Day convolution $\star$ is the biclosed measuring tensor. At a fixed accessibility rank, these are the hypotheses that must be checked.

This yields a universal measuring comonoid

$$\mathcal{M}_{C,\kappa}(T, R)$$

for strong κ -accessible monads T, R , conditional on those fixed-rank local-presentability and biclosedness checks. What it does *not* by itself yield is the desired identification

$$\mathcal{M}_{C,\kappa}(T, R)\text{-Coalg} \simeq \{\text{resource-sensitive runners } T \rightsquigarrow R\}.$$

That second statement is a runner/comonadicity theorem and should be kept logically separate.

One direction is nevertheless immediate and is worth recording. Given a comonad map

$$C \longrightarrow \mathcal{M}_{C,\kappa}(T, R)$$

and a C -coalgebra $S \rightarrow CS$, the measuring action produces

$$TX \otimes S \longrightarrow CS \otimes TX \longrightarrow (C \star T)(S \otimes X) \longrightarrow R(S \otimes X).$$

No diagonal $S \rightarrow S \otimes S$ is required. Thus measuring coalgebras certainly generate runners. The converse—recovering the coalgebra structure universally from an arbitrary enriched runner—is precisely the comonadicity argument still to be supplied.

Working convention for Part VI. We use

$$\mathcal{D}_C = (C\text{-Set})\text{-}[C\text{-Set}, C\text{-Set}]_{\text{acc}}$$

as the intended ambient category of accessible strong endofunctors. Fixed-rank measuring objects are supported by the general duoidal theory once its hypotheses are checked. The classification of enriched runners as coalgebras for those measuring objects, and its unrestricted all-accessible extension, remain conjectural until the direct comonadicity proof is completed.

3.4 What is established, and what remains open

For the rest of the tutorial we shall use the following distinction.

1. **Established background:** C -copresheaves carry Day convolution; strong functors are self-enriched functors; strong monads are enriched monads; enriched Day convolution has unit Id ; and the composition/convolution interchange is obtained by the usual Day universal property.
2. **Fixed-rank measuring objects:** under the López Franco–Vasilakopoulou hypotheses—local presentability, accessibility of the monoid tensor and biclosedness of the measuring tensor—the universal measuring comonoids exist.

3. **Runner theorem still to prove:** one must show that the resulting measuring comonoid has as coalgebras exactly the enriched runners

$$TX \otimes S \longrightarrow R(X \otimes S).$$

The direction from measuring coalgebra to runner is explicit above; the converse is the missing comonadicity argument.

4. **Unrestricted extension still to prove:** the union over accessibility ranks introduces the familiar size issue and should be handled pairwise rather than by assuming that all accessible endofunctors form a locally presentable category.

Appendix A records the technical picture in this more cautious form.

4 What resource-sensitive runners look like

We can now return from the ambient machinery to systems. The general runner formula changes only one symbol, but replacing cartesian product by Day tensor changes the interpretation of hidden state substantially.

4.1 What would an enriched runner look like?

Suppose T, R are strong monads on $\mathcal{E}$ and $S \in \mathcal{E}$ is a resource-indexed state object. The enriched analogue of our familiar runner should have components

$$\theta_X : T(X) \otimes S \longrightarrow R(X \otimes S),$$

now natural in X in the $\mathcal{E}$ -enriched sense and compatible with the monad structures.

Compare with the Set formula

$$TX \times S \rightarrow R(X \times S).$$

Nothing has changed syntactically except $\times$ becoming the Day resource tensor $\otimes$, but the meaning is richer. The state S is itself a C -indexed object, and combining X with S means combining their resource profiles.

If $S = y(p)$ is representable, then

$$X \otimes y(p)$$

means, roughly, “ X with p additional units of resource available.” A runner with state $y(p)$ therefore has a literal resource-state interpretation.

4.1.1 Runner composition

Suppose

$$T \rightsquigarrow R \quad \text{with state } S,$$

and

$$R \rightsquigarrow P \quad \text{with state } Q.$$

Their composite should carry state

$$S \otimes Q$$

rather than a cartesian product. Schematically,

$$T(X) \otimes S \otimes Q \longrightarrow R(X \otimes S) \otimes Q \longrightarrow P(X \otimes S \otimes Q).$$

This is a particularly attractive feature of the enriched setting: hidden states combine according to the same resource tensor that governs the underlying cost semantics.

4.2 First example: the unary theory again

It is useful to revisit the simplest theory from Part I inside the enriched world. Let T be the free strong monad on one unary operation. A $\{T, \text{Id}\}$ -runner should again amount to an endomorphism

$$f : S \rightarrow S,$$

but now S is an object of $C\text{-}\mathbf{Set}$ and f is a C -set morphism.

So the old “set with an endofunction” becomes

resource-indexed state object with a resource-compatible endomorphism.

For a general filtration-like S , compatibility means that transition respects all the structure maps between resource levels. A computation cannot arbitrarily scramble the resource indexing.

This small example is worth keeping in mind. Enrichment does not replace the machines from Parts I–II by alien objects. It lifts their state spaces and structure maps into the resource-sensitive base.

It also shows a limitation. A $\{T, \text{Id}\}$ -runner need not by itself express *consumption* of resource: it is simply a morphism of the chosen resource-indexed state object. Explicit cost-changing behaviour may instead be carried by the residual theory R , by nontrivial grading structure in T , or by the way the state object itself represents resource shifts. The enriched setting tells us where such information can live; it does not decree one encoding in advance.

4.3 Reading resource states: representables are rigid

Take

$$C = (\mathbb{N}, +, 0, \leq)$$

and let $y(p)$ denote the representable corresponding to p . Then

$$y(p) \otimes y(q) \cong y(p + q).$$

Strength provides

$$y(p) \otimes T(X) \longrightarrow T(y(p) \otimes X),$$

which says that a T -computation can be placed coherently in a context carrying p additional units of resource.

There is, however, an important warning. In the thin category $\mathbb{N}$, Yoneda gives

$$\text{End}(y(p)) \cong \{\text{id}\}.$$

So a runner whose entire state is a single representable $y(p)$ cannot model gradual resource consumption by changing that state: the representable is too rigid. Consumption first appears in states assembled from several resource levels. For example, with

$$S = \coprod_{p \in \mathbb{N}} y(p),$$

an endomorphism chooses for each level p some level $f(p) \leq p$. On the finite truncation $0, \dots, n$, there are

$$\prod_{p=0}^n (p+1) = (n+1)!$$

such choices. The non-creation law is now visible as an honest state transition.

This also sharpens the queue example. Potential should not be pictured as a single representable hidden state that is mysteriously “spent.” It is distributed through a composite state such as a Day list of potential-carrying elements, and actual expenditure is witnessed by the residual cost theory.

4.3.1 A comodel-side diagnostic: observations use a tensor too

The same distinction matters for observation. If an observation is to return an O -value while leaving the state available, its comodel shape is

$$S \longrightarrow O \otimes S.$$

Such a map is not automatic. In particular, for $p > 0$, the representable $y(p)$ is not a comonoid for Day tensor: a diagonal would require

$$y(p) \longrightarrow y(p) \otimes y(p) \cong y(2p),$$

which by Yoneda would amount to $2p \leq p$. Thus the Fox-style “which tensor supports copying/observation?” diagnostic reappears on the comodel side. Whenever Part VI speaks of observations, it must say whether they duplicate state cartesianly or interact with it through the Day resource tensor.

4.4 Do not conflate two appearances of grading

There are now two constructions involving the same monoidal category C , and they should initially be kept distinct.

A C -graded monad on a category $\mathcal{X}$ is a lax monoidal functor

$$C \longrightarrow (\text{End}(\mathcal{X}), \circ),$$

that is, a family M_c with structure maps

$$M_p M_q X \longrightarrow M_{p+q} X.$$

By contrast, our enriched proposal uses an ordinary strong monad

$$T : C\text{-}\mathbf{Set} \longrightarrow C\text{-}\mathbf{Set}.$$

Thus

$\begin{array}{ll} \text{graded route} & : C \rightarrow \text{End}(\mathcal{X}), \\ \text{coplesheaf route} & : T : C\text{-}\mathbf{Set} \rightarrow C\text{-}\mathbf{Set}. \end{array}$
--

These are related but not definitionally identical. The first keeps the grades external as an index on operations. The second internalises resource variation into the objects themselves.

For amortisation, this distinction is potentially fruitful rather than inconvenient. The original semantics already has graded cost/potential functors; the copresheaf model simultaneously packages resource levels into a Day-convolution category. Part VI should therefore ask for the precise comparison theorem rather than silently identify the two descriptions.

4.5 Where the amortisation adjunction meets Run

The queue example already tells us where cost and potential enter operationally: potential is carried by the state object and later discharged by a transition. There is also a semantic echo of the same idea on handlers.

In the Set-based story, a runner with state S acts on a residual handler with an underlying exponential such as A^S . In the present closed monoidal category, exponentials are replaced by the internal hom

$$[S, A]_{\mathcal{E}}.$$

For the elementary resource state $y(p)$, this is exactly the Day tensor–hom adjunction

$$- \otimes y(p) \dashv [y(p), -].$$

Thus the closed structure supplies

$$\boxed{\text{tensor by a resource object}} \quad \dashv \quad \boxed{\text{internal hom}}.$$

This adjunction exists for every object of any closed monoidal category, so its mere appearance on both sides is not yet evidence that the amortisation adjunction has been recovered from Run. The substantive comparison must use the graded monad/comonad structure of the cost semantics and the residual-cost runner. The question is therefore sharper: under what additional structure does the Binder–Corfield–Orchard–Rajani cost/potential adjunction arise from, or act on, the Sweedler four-corner picture?

For the functional queue, the slogan would be: the state side remembers the P_2 ’s deposited with waiting elements, while the handler side provides the universal way to ask what those stored resources buy when the computation is finally evaluated.

5 Changing and comparing resource semantics

The point of the construction is not tied to one cost monoid. Two comparisons are especially revealing: changing the resource base itself, and recovering the predicate world of Part V as a special filtered case.

5.1 Change of grading: variance and a projection formula

Suppose

$$f : C \rightarrow C'$$

is a strong symmetric monoidal functor between cost/resource categories. Restriction and left Kan extension give an adjunction

$$f_! : [C, \mathbf{Set}] \rightleftarrows [C', \mathbf{Set}] : f^*.$$

For a monad T on $C'\text{-}\mathbf{Set}$, conjugation along this adjunction produces a monad

$$f^* T f_!$$

on $C\text{-}\mathbf{Set}$. Thus the most immediate endofunctor-level transport runs contravariantly in the base map:

$$\mathcal{D}_{C'} \longrightarrow \mathcal{D}_C, \quad T \longmapsto f^* T f_!.$$

The opposite direction requires different data and should not be asserted automatically.

Transporting runners needs more than transport of monads. To move a state through the Kan extension one needs a Frobenius or projection-formula comparison of the form

$$f_!(X \otimes f^*S) \longrightarrow f_!X \otimes S,$$

preferably an isomorphism in the examples of interest. Only under such a condition can the tensor-state expression in

$$T(X) \otimes S \longrightarrow R(X \otimes S)$$

be carried coherently across change of base.

This makes the examples case-sensitive. Forgetting a cost grading can satisfy the required projection formula, while a rescaling map need not. Likewise a map from a multidimensional resource monoid to total cost must be checked rather than listed as automatic. The correct slogan is therefore:

change the notion of resource, then transport theories by adjunction and transport runners only when the required projection formula is available.

5.2 Threshold embeddings: Day tensor versus cartesian product

The predicate comparison becomes more informative when the value monoid is non-idempotent. Let V be an ordered commutative monoid with finite meets and let

$$p : X \rightarrow V.$$

Its threshold object is

$$\Phi_p(v) = \{x \in X \mid v \leq p(x)\},$$

with the variance chosen to match the order convention. Predicates correspond to the principal-filtration objects among general V -sets.

Two products on these threshold objects encode two different logical/resource combinations. Day convolution gives

$$\boxed{\Phi_p \otimes_{\text{Day}} \Phi_q \cong \Phi_{p \otimes_V q}},$$

where the grades combine by the monoidal product of V . Cartesian product gives instead

$$\boxed{\Phi_p \times \Phi_q \cong \Phi_{p \wedge q}}.$$

Thus the choice between $\otimes_V$ and $\wedge$ in the valued-predicate calculations of Part V is not cosmetic: it is the choice between Day tensor and cartesian product in the ambient V -set category. A mixed $(\wedge, \otimes_V)$ formula is naturally read through the interacting pair $(\times, \otimes_{\text{Day}})$.

This also explains why $V = 2$ is a poor diagnostic. With conjunction as the monoidal product,

$$\otimes_V = \wedge,$$

and for presheaves on the cartesian thin category 2 , Day convolution agrees with pointwise cartesian product. The two structures collapse exactly in the Boolean case, so any accidental switch between them becomes invisible.

A better revealing example is a non-idempotent value monoid, such as a finite Łukasiewicz chain or $(\mathbb{N}, +, 0, \leq)$. There

$$p \otimes_V q \quad \text{and} \quad p \wedge q$$

are genuinely different, so the two tensors can no longer be conflated.

There is a small variance point worth remembering. In the copresheaf convention used for positive costs, the monoidal unit 0 is bottom; after reversing to presheaves, the corresponding condition is integrality, with the unit top. In both conventions the practical statement is the same: the Day unit is terminal.

Part V can therefore be reread as studying principal filtrations inside a larger V -set universe, but Part VI adds a warning: that universe has both a cartesian product and a Day resource tensor, and they encode different connectives except in degenerate idempotent cases such as 2.

6 Synthesis and outlook

We finish by returning to the queue with the full architecture now visible, extracting a reusable checklist, and recording exactly what Part VI has established for the rest of the tutorial.

6.1 Returning to the queue: what has been gained

It is worth returning once more to the queue before leaving the resource setting. The ordinary potential method says that an expensive operation is paid for by credits deposited during earlier cheap operations. In the present categorical picture, that statement becomes structural:

1. the credits are objects such as $P_2\tau$, not external annotations;
2. the queue representation is a state object such as $L_{\otimes}(P_2\tau) \otimes L_{\otimes}\tau$;
3. enqueue and dequeue are morphisms respecting the resource tensor;
4. the same *pointed* runner state persists from one reachable operation to the next;
5. a residual cost theory records the primitive work actually performed, so stored potential can be compared with—rather than merely coexisting with—that work.

This suggests a useful research slogan:

amortisation is naturally a property of a stateful realisation across a run.

It is not that the individual dequeue suddenly becomes cheap. Rather, the runner carries forward enough resource information for a composite history of operations to satisfy the desired bound.

A future fully worked amortisation example should define an abstract queue theory T_Q , use a primitive cost theory as residual theory, choose a zero-potential initial state, and construct the corresponding C -Set-valued runner. The residual-Id version remains useful as a conservation example, but not as a proof of sufficiency. The length-indexed types from refined amortised type systems would then be recovered in a fibred refinement rather than forced into this first example.

6.2 Probabilistic amortisation: a conjectural extension

There is a natural probabilistic continuation of the preceding picture. Rajani, Barthe and Garg’s $p\lambda$ -amor develops a graded modal type theory for expected costs of higher-order probabilistic programs: potential, cost and probability are all visible at the type level, and the soundness argument relates potential to probabilistic coupling. We do not try to reproduce that type theory here. Instead we ask what the corresponding residual theory should look like from the runner viewpoint.

The first point is that probability should not replace the C -grading. The grading still records potential—the resource available in the state—while probability belongs to the residual computation performed by a runner. Thus the deterministic picture

$$T \rightsquigarrow T_{\text{cost}}$$

should have a probabilistic analogue

$$T \rightsquigarrow R_{\text{prob}, \text{cost}},$$

where $R_{\text{prob}, \text{cost}}$ is a monad for composition in the enriched duoidal category. The Day tensor continues to be the tensor used to combine and measure resources.

A first candidate is a resource-enriched version of Spivak’s polynomial probability monad **rand**. We use **rand** throughout; this is Spivak’s current name for the monad formerly called **lott**. At the most naive level one might try

$$(\text{rand}_C X)(c) = \text{rand}(X(c)),$$

and ask whether this levelwise construction is accessible and carries the enrichment or tensorial strength required to define a monad

$$\text{rand}_C \in \text{Mnd}(\mathcal{D}_C).$$

This should be regarded as a test problem, not as an established theorem. In particular, the enriched naturality and strength have to be checked rather than inferred from the ordinary Set-based monad.

Probability by itself is not yet probabilistic amortisation. A runner must also expose the actual work carried out on each random branch. The residual object should therefore combine **rand**_C with the primitive cost theory. Schematically one expects something of the form

$$R_{\text{prob}, \text{cost}} \simeq \text{rand}_C T_{\text{cost}} \quad \text{or} \quad T_{\text{cost}} \text{rand}_C,$$

but neither composite is automatically a monad. The missing datum should be a distributive law expressing how probabilistic branching interacts with accumulated cost. Determining the correct law is one of the concrete open problems suggested by this part.

The desired systems meaning is nevertheless clear. A probabilistic resource-sensitive runner should have the familiar shape

$$T(X) \otimes S \longrightarrow R_{\text{prob}, \text{cost}}(X \otimes S).$$

From a current state it realises one abstract operation by a random residual computation, producing both a next state and branchwise actual cost. If Φ denotes the potential represented by the resource-sensitive state, the deterministic potential inequality is replaced by the expected inequality

$$\mathbb{E}[c + \Phi(S')] \leq \Phi(s) + \hat{c},$$

or equivalently

$$\hat{c} \geq \mathbb{E}[c] + \mathbb{E}[\Phi(S')] - \Phi(s).$$

Iterating and using the tower property of conditional expectation gives the stochastic analogue of telescoping:

$$\mathbb{E}\left[\sum_{i < n} c_i\right] \leq \sum_{i < n} \hat{c}_i + \Phi(s_0) - \mathbb{E}[\Phi(S_n)].$$

For a zero-potential initial point and non-negative potential this yields the expected-cost bound. Thus the pointing issue from the deterministic queue does not disappear under probability: probabilistic amortisation is still a statement about the reachable runs of a pointed realisation, now equipped with their induced probability law.

This also reconnects Part VI with the use of **rand** in Part II. There we treated random branching itself as a residual theory. Here we are asking whether that residual theory admits a resource-enriched lift and whether it can be combined with an explicit cost theory. The expectation operation has the same broad form as the entropy algebra considered there: a valuation of branches is averaged against their probabilities. Cost, entropy, loss and related quantitative observables may therefore be different algebras over closely related probabilistic residual structure.

Conjectural probabilistic extension. Seek an enriched lift

$$\mathbf{rand}_C \in \mathbf{Mnd}(\mathcal{D}_C)$$

and a distributive law combining $\mathbf{rand}_C$ with T_{cost} . A resulting probability-with-cost residual theory should make coalgebras/runners express expected-potential inequalities of the form

$$\mathbb{E}[c + \Phi(S')] \leq \Phi(s) + \hat{c}.$$

This is the probabilistic analogue of the deterministic programme of Part VI, but at present it should be read as a research proposal rather than a theorem.

6.3 Checklist for other resource-sensitive bases

When we encounter another quantitative base, we should now ask:

1. Is the resource category C small and symmetric monoidal?
2. Is $C\text{-Set}$ locally presentable and symmetric monoidal closed under Day convolution?
3. Should theories be strong/enriched monads rather than ordinary monads?
4. Which accessibility rank is required by the particular theories? (For small C , tensor-closure of finitely presentable base objects is already available.)
5. Does enriched convolution preserve that accessibility rank?
6. Does the composition/convolution interchange exist, and is the required measuring tensor biclosed?
7. Do the López Franco–Vasilakopoulou hypotheses hold at fixed rank?

8. Has the runner-coalgebra comonadicity theorem actually been proved, rather than inferred from measuring-object existence?
9. What is the systems interpretation of a resource-indexed state object S , and is a pointed initial state required?
10. Which tensor governs observations and copying on the comodel side?
11. How does change of resource base $C \rightarrow C'$ transport theories, and does the needed projection formula hold for runners?

This checklist is the resource-enriched counterpart of the ambient-category checklist from Part IV.

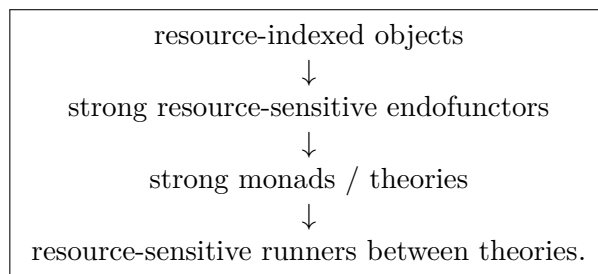
6.4 Where Part VI leaves us

The main conceptual gain is now clear. In the original Set-based theory, hidden runner state was combined cartesianly. In the resource setting, the same architecture becomes

$$T(X) \otimes S \longrightarrow R(X \otimes S),$$

where $\otimes$ is the Day tensor encoding resource combination.

Thus the hierarchy is



The amortisation semantics gives this hierarchy a concrete reason to exist. Potential is not merely a number attached to an ordinary state; it is represented in a monoidal category where resource can be combined, shifted and internalised. Strength makes theories respect that geometry. Enriched Sweedler theory then asks for implementations that respect it too.

The working conclusion of Part VI. The enriched endofunctor category

$$(C\text{-}\mathbf{Set})\text{-}[C\text{-}\mathbf{Set}, C\text{-}\mathbf{Set}]_{\text{acc}}$$

is the right candidate ambient category because enrichment both expresses tensorial strength and restores the normal Day unit. At fixed rank the general duoidal theory supports universal measuring objects under its stated local-presentability, accessibility and biclosedness hypotheses. What remains to be proved is the specifically runner-theoretic statement that their coalgebras are exactly maps $TX \otimes S \rightarrow R(X \otimes S)$, together with the unrestricted all-accessible extension.

The next parts can now change the geometry of state in a different direction. Precubical sets will replace linear resource grading by higher-dimensional execution geometry; Petri-net

endofunctors will replace pointwise state by resource/process structure. In each case the same question returns: what are the two tensors, what does interchange mean, and what counts as a theory and a runner in that ambient world?

A Technical appendix: why enriched Sweedler theory should work

This appendix records the argument suppressed in the main tutorial, and makes a start on a new one using a different technique.

A.1 Fix an accessibility rank

The purpose of this first part of the technical appendix is not to settle every size issue, but to make precise why the working ambient category is more than wishful notation.

Let $\mathcal{E} = C\text{-}\mathbf{Set}$. For small C , the underlying copresheaf category is locally finitely presentable, and its finitely presentable objects are finite colimits of representables. Since Day tensor preserves colimits in each variable and

$$y(p) \otimes y(q) \cong y(p + q),$$

finitely presentable objects are closed under $\otimes$. For a particular pair of accessible endofunctors choose a regular κ large enough that they are κ -accessible; no enlargement is needed merely for tensor-closure of the base.

Define

$$\mathcal{D}_{C,\kappa} = \mathcal{E}\text{-}[\mathcal{E}, \mathcal{E}]_{\kappa}.$$

A κ -accessible enriched endofunctor is controlled by its restriction to the small enriched category $\mathcal{E}_{\kappa}$, which supplies the required local-presentability behaviour.

Composition

Composition preserves enrichedness and κ -filtered colimits, so

$$\circ : \mathcal{D}_{C,\kappa} \times \mathcal{D}_{C,\kappa} \longrightarrow \mathcal{D}_{C,\kappa}$$

is the monoidal product whose monoids are strong κ -accessible monads. The accessibility of the bifunctor is controlled by the standard diagonal/cofinality argument for filtered colimits.

Enriched Day convolution

For $F, G \in \mathcal{D}_{C,\kappa}$, define

$$(F \star G)(X) = \int^{A, B \in \mathcal{E}_{\kappa}} [A \otimes B, X]_{\mathcal{E}} \otimes F(A) \otimes G(B).$$

Because the indexing category is small and $A \otimes B$ remains κ -presentable, the enriched hom preserves the relevant filtered colimits. Tensor and coends are colimit constructions, so $F \star G$ remains κ -accessible. Standard enriched Day theory gives the symmetric monoidal closed, hence biclosed, structure.

Interchange

The Day structure maps give

$$F(GA) \otimes H(KB) \longrightarrow (F \star H)(GA \otimes KB),$$

while the Day structure for G, K gives

$$GA \otimes KB \longrightarrow (G \star K)(A \otimes B).$$

Applying the enriched functor $F \star H$ and using the universal property of Day convolution yields

$$(F \circ G) \star (H \circ K) \longrightarrow (F \star H) \circ (G \star K).$$

The coherence reduces to the coherence of the tensor of $\mathcal{E}$.

Measuring comonoids at fixed rank

The theorem of López Franco and Vasilakopoulou applies to a duoidal category $(\mathcal{V}, \diamond, \star)$ when $\mathcal{V}$ is locally presentable, the monoid tensor $\star$ is accessible, and the measuring tensor $\diamond$ is biclosed. In the present application the monoid tensor is composition, while the measuring tensor is enriched Day convolution $\star$. Thus a fixed-rank existence claim for

$$\mathcal{M}_{C,\kappa}(T, R)$$

requires those properties of $\mathcal{D}_{C,\kappa}$ to be checked explicitly; it is not a consequence of interchange alone.

Assuming these hypotheses, the universal measuring object exists. But the runner interpretation is an additional theorem. A measuring comonad C acting on T toward R , together with a coalgebra $S \rightarrow CS$, yields a runner by

$$TX \otimes S \longrightarrow CS \otimes TX \longrightarrow (C \star T)(S \otimes X) \longrightarrow R(S \otimes X).$$

The converse direction is the missing comonadicity statement.

Why we still flag the unrestricted case

The union over all accessibility ranks is subtler. Even in the original Set case, the category of all accessible endofunctors need not itself be accessible. Thus one should not infer the unrestricted theorem by simply dropping κ from the previous paragraph.

The natural route is instead pairwise: define the category of enriched residual runners

$$\theta_X : T(X) \otimes S \rightarrow R(X \otimes S)$$

and adapt the ordinary local-presentability/comonadicity argument. If that succeeds, it will both identify runner states with coalgebras for the appropriate measuring object and remove the artificial dependence on one global accessibility rank. This is the remaining theorem suggested by the programme.

For the main tutorial none of this changes the systems interpretation. The enriched runner formula itself is perfectly meaningful. What remains conditional is the universal classification of all such runners by a measuring comonad, first at fixed rank and then without a global rank.

A.2 A smaller arity-generated Sweedler universe

The preceding construction used the large candidate

$$\mathcal{E}\text{-}[\mathcal{E}, \mathcal{E}]_{\text{acc}}, \quad \mathcal{E} = C\text{-}\mathbf{Set},$$

of accessible strong endofunctors. There is, however, a smaller ambient category which is better controlled and already contains the resource-sensitive theories needed here.

The general idea is to choose a small collection of objects of $\mathcal{E}$ which will serve as the allowed *arities* of operations. We write

$$j : \mathcal{J} \hookrightarrow \mathcal{E}$$

for the corresponding small full $\mathcal{E}$ -enriched subcategory. Thus $\mathcal{J}$ is not a second ambient category: it records the particular objects of $\mathcal{E}$ through which the operations of our theories are allowed to probe their arguments. This is the enriched “system of arities” viewpoint.

For the resource category there is an especially natural choice. Let C denote the small commutative monoidal cost category and write

$$y_c = C(c, -) \in \mathcal{E}$$

for the representable cost object at c . We take $\mathcal{J}$ to be the full $\mathcal{E}$ -enriched subcategory of $\mathcal{E}$ whose objects are precisely these representables:

$$\text{ob } \mathcal{J} = \{y_c \mid c \in C\}.$$

Thus the elementary costs themselves are the admissible arities.

The first important calculation is

$$y_c \otimes_D y_d \cong y_{c+d},$$

so the chosen arities are closed under the Day tensor. Moreover, since $\mathcal{E}$ is closed under Day convolution,

$$[y_c, X]_{\mathcal{E}}(p) \cong \mathcal{E}(y_p \otimes_D y_c, X) \cong \mathcal{E}(y_{p+c}, X) \cong X(p+c).$$

Thus homming out of the elementary cost arity y_c is simply the cost-shift

$$S_c X(p) = X(p+c).$$

Since colimits in $C\text{-}\mathbf{Set}$ are computed pointwise, every S_c preserves all colimits.

This places the representable cost objects in the setting of enriched systems of arities. In the terminology of Lucyshyn-Wright, the relevant condition is *eleuthericity* ([9]): left Kan extensions along

$$j : \mathcal{J} \hookrightarrow \mathcal{E}$$

must exist and be preserved by the enriched homs $[y_c, -]_{\mathcal{E}}$. In the present case the latter are the colimit-preserving shifts S_c , so the cost representables provide an especially well-behaved system of enriched arities.

We may therefore work with the $\mathcal{J}$ -ary strong endofunctors

$$\mathcal{D}_C := \text{End}_{\mathcal{J}}^{\mathcal{E}}(\mathcal{E}) \simeq [\mathcal{J}, \mathcal{E}]_{\mathcal{E}}.$$

Here “ $\mathcal{J}$ -ary” means that the endofunctor is determined by its values on the chosen cost arities and is reconstructed from them by enriched left Kan extension. If

$$f : \mathcal{J} \rightarrow \mathcal{E},$$

its associated endofunctor is

$$\widehat{f}(X) \cong \int^{c \in \mathcal{J}} [y_c, X]_{\mathcal{E}} \otimes_D f(c) \cong \int^c S_c X \otimes_D f(c).$$

Thus a $\mathcal{J}$ -ary resource-sensitive operation is assembled from elementary cost shifts weighted by resource-graded coefficients.

The two monoidal products

Composition of $\mathcal{J}$ -ary endofunctors remains $\mathcal{J}$ -ary. Transported across

$$\mathcal{D}_C \simeq [\mathcal{J}, \mathcal{E}]_{\mathcal{E}},$$

it becomes a substitution product, which we denote by $\star$. Explicitly,

$$(f \star g)(c) = \widehat{f}(g(c)) \cong \int^a [y_a, g(c)]_{\mathcal{E}} \otimes_D f(a) \cong \int^a S_a(g(c)) \otimes_D f(a).$$

Because evaluation, cost-shift, Day tensoring and the coend all preserve the relevant colimits, this substitution product is accessible; in fact, in this example it is separately cocontinuous.

The second product is enriched Day convolution on the small monoidal category of cost arities $\mathcal{J}$:

$$(f \diamond g)(c) = \int^{a, b \in \mathcal{J}} \mathcal{J}(y_a \otimes_D y_b, y_c) \otimes_D f(a) \otimes_D g(b).$$

Using

$$y_a \otimes_D y_b \cong y_{a+b},$$

this is the convolution induced directly by addition of costs. On the elementary generators

$$H_{c,K}(X) = [y_c, X]_{\mathcal{E}} \otimes_D K$$

one obtains

$$H_{c,K} \diamond H_{d,L} \cong H_{c+d, K \otimes_D L}.$$

Thus arity costs add while coefficient resources combine by Day tensor.

Because $\mathcal{J}$ is small and $\mathcal{E}$ is closed, enriched Day convolution on

$$[\mathcal{J}, \mathcal{E}]_{\mathcal{E}}$$

is biclosed. Its right internal hom has the standard form

$$[g, h]_{\diamond}(a) = \int_b [g(b), h(a \otimes_D b)]_{\mathcal{E}},$$

with the corresponding left internal hom defined similarly. The canonical compatibility between substitution and convolution gives the duoidal interchange

$$(f \star g) \diamond (h \star k) \longrightarrow (f \diamond h) \star (g \diamond k).$$

There is also an important improvement over the ordinary endofunctor construction. Since the Day unit of $\mathcal{E}$ is y_0 ,

$$[y_0, X]_{\mathcal{E}} \cong X,$$

so the convolution unit corresponds to the identity endofunctor. Hence the resulting duoidal structure is normal.

Measuring comonoids

We have therefore reduced the resource-sensitive construction to the small enriched functor category

$$\boxed{\mathcal{D}_C \simeq [\mathcal{J}, \mathcal{E}]_{\mathcal{E}}, \quad \text{ob } \mathcal{J} = \{y_c \mid c \in C\}.}$$

Under the standard enriched local-presentability hypotheses on the small functor category, the conditions of López Franco–Vasilakopoulou ([10]) apply: the underlying category is locally presentable, $\diamond$ is biclosed, and the substitution product $\star$ is accessible. It follows that the category of $\star$ -monoids is tensored and cotensored enriched in the category of $\star$ -comonoids. In particular, for $\mathcal{J}$ -ary strong monads T and R there is a universal measuring comonoid

$$\mathcal{M}(T, R),$$

together with the corresponding Sweedler copower.

This gives a firmer foundation for the resource-sensitive runners of this part than the unrestricted category of all accessible strong endofunctors. We need not claim that

$$\mathcal{E}\text{-}[\mathcal{E}, \mathcal{E}]_{\text{acc}}$$

itself satisfies a general enriched Sweedler existence theorem. The smaller arity-generated universe already does the required work.

A principle to carry forward

The construction suggests a broader strategy. Given an enriched ambient category $\mathcal{C}$, first choose a small full enriched subcategory

$$j : \mathcal{J} \hookrightarrow \mathcal{C}$$

whose objects are to serve as the admissible arities of operations. One then asks whether these arities are closed under the ambient tensor and whether the homs

$$\mathcal{C}(J, -), \quad J \in \mathcal{J},$$

preserve the Kan-extension colimits generated by j . If so, the category of $\mathcal{J}$ -ary enriched endofunctors may be represented by the small enriched functor category

$$[\mathcal{J}, \mathcal{C}]_{\mathcal{V}},$$

with substitution as one monoidal product and Day convolution as the other.

The cost representables y_c give the first concrete example of this principle. Here the arities have a direct systems-theoretic meaning: they are the elementary resource probes. Later ambient settings should be revisited from the same arity-generated point of view.

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