Charles Peirce, Inference and Category Theory

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1 Introduction

* Charles Saunders Peirce (1839-1914), brilliantly imaginative American philosopher, with a heavy focus on the nature of enquiry. Beloved by applied category theorists for his string-diagrammatic logic calculus, the *Existential Graphs*.

* Here we look to increase the love by sketching and illustrating the thesis that the three modes of inference designated by Charles Peirce – *deduction*, *induction*, *abduction* – are represented in category-theoretic terms by *composition*, *extension* and *lift*.

* These notes are very much a work-in-progress, finally giving some attention to an idea I had in 2017. Any comments are very welcome.

* We'll start by considering these operating in contexts where all concepts have been articulated. Here we have 3 objects and 2 morphisms in a category and are looking for a third morphism to complete the triangle.

* We'll start with *Set*, but go on to consider other categories, 2-categories, maybe even double categories.

* This is the final $(3,3)$ box of the grid of Peirce's *Speculative Grammar*, the derivation of arguments. In terms of logic, this comes in the third column after the formation of types and terms of the types.

* We will also consider situations where we start out from less – perhaps a single morphism, perhaps a single object, maybe just a blank sheet – with objects, morphisms and triangles coming into focus out of the mist. This concerns the rest of Peirce's *Speculative Grammar*, the formation of concepts.

* My sincere thanks go to Matt Cuffaro and Nathan Haydon, fellow members of a Peirce reading group, for providing the space to thrash out ideas, and to Matteo Capucci for taking this thesis on.

2 Plain cases, given 2 morphisms in *Set*

* **Deduction** is straightforward. It's just the **composition** of appropriate morphisms:

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, dashed, "g \cdot f"] & & C \end{tikzcd}$$

* An **extension** (g such that $g \cdot f = h$) is like **induction**, in particular when one given arrow, f, typically mono, is taking a sample and the other arrow, h, labels the sample.

$$\begin{tikzcd} & B \arrow[dr, dashed, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"] & & C \end{tikzcd}$$

* A **lift** (f such that $g \cdot f = h$) is like **abduction**, in particular when the morphism, g, is epi, a loss of information. Casting a shadow, loss of a dimension, coarse-graining.

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[ur, dashed, "f"] \arrow[r, "h"] & C \end{tikzcd}$$

* Example: Occluded observation, where h represents the retinal image of a separate head and tail, and this is interpreted (lifted) to be seen as an animal behind a tree. For Peirce, all observation is abductive.

* One way to find a lift is to find a **section**, (k such that $g \cdot k = Id_C$). Then choose $f = k \cdot h$.

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[ur, dashed, "f"] \arrow[r, "h"] & C \arrow[u, dashed, bend right, "k"'] \end{tikzcd}$$

k is itself a lift.

$$\begin{tikzcd} & B \arrow[d, "g"] & \\ A \arrow[ur, dashed, "f"] \arrow[r, "h"] & C & C \arrow[l, "id_C"] \arrow[ul, dashed, "k"'] \end{tikzcd}$$

* Example: A section from 2D image to 3D reality.

* Similarly, given an extension problem

$$\begin{tikzcd} & B \arrow[dr, dashed, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"] & & C \end{tikzcd}$$

one way to find an extension is to find a **retraction**, (k such that $k \cdot f = Id_A$).

$$\begin{tikzcd} & B \arrow[dl, dashed, bend right, "k"'] \arrow[dr, dashed, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"] & & C \end{tikzcd}$$

Then choose $g = h \cdot k$. k is itself an extension, along the identity.

* The *nearest neighbour algorithm* does this via a retraction to the sample space.

(Note: Treatment of NN as an extension problem, M. Pugh et al. Using Enriched Category Theory to Construct the Nearest Neighbour Classification Algorithm.)

* We should worry about too small a sample, where any retraction will likely fail to represent B's variations. The sample is non-representative. (Although, this hardly matters with certain uniform kinds B and certain essential properties $B \to C$.)

* At the other extreme, A isomorphic to B would be sampling the whole set.

* There should be a dual to the nearest neighbour algorithm, the construction of a section.

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[ur, dashed, "f"] \arrow[r, "h"] & C \arrow[u, dashed, bend right, "k"'] \end{tikzcd}$$

We certainly see optimisation problems over sections, e.g, in quiver representations.

* An extreme case would have $g : B \cong C$, when we could use g^{-1}. On the other hand, we might worry about the B classification being much more fine-grained, making it implausible to find a suitable k.

* For B and C with given distance metrics, there might be some algorithm working by the choice of a function minimizing the Lipschitz constant. (Maybe one could find this directly for maps $A \to B$.)

* Parameter estimation (in an appropriate category): we parameterize distributions by some range of statistical models with parameters P. Then we cast the sample as $1 \to O^I$ and look for a lift $1 \to P$.

$$\begin{tikzcd} & P \arrow[d, "g"] \\ 1 \arrow[ur, dashed, "f"] \arrow[r, "h"] & O^I \end{tikzcd}$$

This may be done by some general process of parameter estimation: $k : O^I \to P$.

$$\begin{tikzcd} & P \arrow[d, "g"] \\ 1 \arrow[ur, dashed, "f"] \arrow[r, "h"] & O^I \arrow[u, dashed, bend right, "k"'] \end{tikzcd}$$

* Once we have a candidate extension, g, through some means, we might lift back to a retraction of the sampling f. Dually we may be able to extend back to a section.

$$\begin{tikzcd} & B \arrow[dl, dashed, bend right, "k"'] \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"] & & C \end{tikzcd}$$

* Hypothetico-deduction is about checking proposed lifts, e.g., making new deductive triangles from the codomain, B, of lift f.

$$\begin{tikzcd} D & B \arrow[l, "k"'] \arrow[d, "g"] \\ A \arrow[u, dashed, "k \cdot f"] \arrow[ur, dashed, "f"] \arrow[r, "h"] & C \end{tikzcd}$$

* Another way to test the lift, f, deductively is via a sample of its domain A:

$$\begin{tikzcd} D \arrow[r, dashed, "f \cdot k"] \arrow[d, "k"] & B \arrow[d, "g"] \\ A \arrow[ur, dashed, "f"] \arrow[r, "h"] & C \end{tikzcd}$$

* Inductive-deduction would involve forming further triangles on an inductive triangle, e.g., on the domain C of the extension, testing composites $l \cdot g$ for $l : C \to D$.

$$\begin{tikzcd} & B \arrow[r, dashed, "l \cdot g"] \arrow[dr, dashed, "g"] & D \\ A \arrow[ur, "f"] \arrow[rr, "h"] & & C \arrow[u, "l"'] \end{tikzcd}$$

* It could also involve us testing an extension, g, by measuring the accuracy of the induced classifier on some test sample, k.

$$\begin{tikzcd} & B \arrow[dr, dashed, "g"] & D \arrow[l, "k"'] \arrow[d, dashed, "g \cdot k"] \\ A \arrow[ur, "f"] \arrow[rr, "h"] & & C \end{tikzcd}$$

* These techniques are especially needed if there are many candidate lifts and extensions.

* A detective story generally has several potential lifts to account for whodunit?, but where there's a problem in supporting any of them. When all is revealed, we're relieved.

* We're beginning to point to inference over a larger network. We may start out with, say, a square, as with Spivak, Database queries and constraints via lifting problems.

$$\begin{tikzcd} D \arrow[r, "k"] \arrow[d, "l"] & B \arrow[d, "g"] \\ A \arrow[ur, dashed, "f"] \arrow[r, "h"] & C \end{tikzcd}$$

We can illustrate this by considering B as some part of reality and C as a model. Then D could be some shape of observations and A an enlargement of this. Perhaps A and D are intervals, D embedding into A. The continuation exists in the model, does this exist in reality.

* Or $l : D \to A$ might indicate the extension to a more refined path. Like a subdivided Feynman path integral.

* Simultaneous restraint for optimization that we call call lift-extension. We might say it's always been about such squares, but when the top left object is initial or the bottom right is terminal, this reduces to our earlier triangular cases.

* The capacity to lift-extend is useful for characterising orthogonality of properties of maps:

> Often it is useful to think of lifting properties as a expressing a kind of qualitative negation ("Quillen negation"): The morphisms with the left/right lifting property against those in a class P tend to be characterized by properties opposite of those in P. For example, a morphism in Sets is surjective iff it has the right lifting property against the archetypical non-surjective map $\emptyset \to \{*\}$, and injective iff it has either left or right lifting property against the archetypical non-injective map $\{x_1, x_2\} \to \{*\}$. (For more such examples see at *separation axioms in terms of lifting properties*.)

* Larger simplices are possible. We may wish to provide an account of $A \to D$, by factorisation into $A \to B \to C \to D$, given various morphisms in the tetrahedron by means of composition, extensions and lifts.

$$\begin{tikzcd} & B \arrow[d, "g"] \arrow[ddr, "j"] & \\ & C \arrow[dr, "k"] & \\ A \arrow[uur, "f"] \arrow[ur, "h"] \arrow[rr, "l"] & & D \end{tikzcd}$$

* Finally, the time-honoured way of dealing with deductive logic was via the syllogism. The form of some syllogisms may be given in terms of compositions and pullbacks.

- Some A is B
  <u>All Bs are C</u>
  Some A is C

$$\begin{tikzcd} 1 \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & A \times_D B \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ & A \arrow[r, "g"] & D \end{tikzcd}$$

Now we add the arrow $h : B \to C$

$$\begin{tikzcd} 1 \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & & \\ & A \times_D B \arrow[r] \arrow[d] & B \arrow[d, "f"] \arrow[r, "h"] & C \arrow[dl, "k"] \\ & A \arrow[r, "g"] & D & \end{tikzcd}$$

Hence there's an arrow from $A \times_D B$ to $A \times_D C$, and hence an arrow from 1 to $A \times_D C$.

* A second example:

- No B is A
  <u>All Cs are B</u>
  No C is A

$$\begin{tikzcd} & & C \arrow[d, "h"] \\ A \times_D B \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dd, "f"] \\ & 0 \arrow[dr] & \\ A \arrow[rr, "g"] & & D \end{tikzcd}$$

This requires a similar diagram with maps from pullbacks to 0.

* Perhaps then a role in inference for pullbacks.

3 A change of category, a series of thoughts

* We often don't want just any old morphism as our 'law'. If things are well-arranged, we may be able to induce based on one or two samples - like the effects of dropping atomic bombs on cities.

* If we're after a robust law in our explanations, it had better be invariant under variation. We need some Peircean *Thirdness*.

* Whether the stone drops is not dependent on whether it's rainy or sunny, or whether it's Tuesday or Wednesday.

* Robert Brandom's Kant-Sellars thesis locates necessity in invariance (Chap. 4 of my Modal HoTT book).

* One way to approach this is by working in slice categories. Some candidate explanatory law $g : B \to C$ needs to be stable, so that under variation in a rich W-context, a map $B \times W \to C \times W \to C$ factors through the projection to B and g.

* Ties in with counterfactual reasoning, pealing back the context in a dependent type theory.

* To lift $A \to C$ to $B \to C$ is to find an appropriate morphism in the slice category C/C. To extend is similarly to find a morphism in the coslice category. Note that slices often preserve logical structure better, as with toposes.

* Dependent product/sum as adjoints to context extension, $\mathbf{H} \to \mathbf{H}/W$, so Kan extensions.

* 2-category of categories: finding adjoints involves constructing Kan extensions, but may also be cast in terms of lifts:

> **In terms of Kan extensions/liftings**
>
> Given $L: C \to D$, we have that it has a right adjoint $R: D \to C$ precisely if the left Kan extension $\mathrm{Lan}_L 1_C$ of the identity along L exists and is absolute, in which case
>
> $$R \simeq \mathrm{Lan}_L 1_C.$$
>
> In this case, the universal 2-cell $1_C \to RL$ corresponds to the unit of the adjunction; the counit and the verification of the triangular identities can all be obtained through properties of Kan extensions and absoluteness.
>
> It is also possible to express this in terms of Kan liftings: L has a right adjoint R if and only if:
>
> - $R \simeq \mathrm{Rift}_L 1_D$ and this Kan lift is absolute
>
> In this case, we get the counit as given by the universal cell $LR \to 1_D$, while the rest of the data and properties can be derived from it through the absolute Kan lifting assumption.
>
> Dually, we have that for $R: D \to C$, it has a left adjoint $L: C \to D$ precisely if
>
> - $L \simeq \mathrm{Ran}_R 1_D$, and this Kan extension is absolute
>
> or, in terms of left Kan liftings:
>
> - $L \simeq \mathrm{Lift}_R 1_C$, and this Kan lifting is absolute
>
> This follows from the fact that the adjunction $L \dashv R$ induces adjunctions $- \circ R \dashv - \circ L$ and $L \circ - \dashv R \circ -$.

* But perhaps general Kan extensions are inductive:

(Dan Shiebler, *Kan Extensions in Data Science and Machine Learning*, July 2022.)

$$\begin{tikzcd} \mathbf{B} \arrow[dr, dotted, "F"] & \\ \mathbf{A} \arrow[u, "G"] \arrow[r, "K"] & \mathbf{C} \end{tikzcd}$$

> Intuitively, if we treat G as an inclusion of $\mathbf{A}$ into $\mathbf{B}$ then the Kan extensions of K along G act as extrapolations of K from $\mathbf{A}$ to all of $\mathbf{B}$. If $\mathbf{C}$ is a preorder then the left and right Kan extensions respectively behave as the least upper bound and greatest lower bounds of K.

- Section 3: Learn a classifier from a dataset of labeled examples.
- Section 4: Learn a mapping from metric spaces (X, d_X) to partitions of X.
- Section 5: Learn a mapping from datasets of labeled examples to functions.
- Section 6: Approximate a complex function with a simpler one

> In each of these applications we first define categories $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and a functor $K : \mathbf{A} \to \mathbf{C}$ such that $\mathbf{A}$ is a subcategory of $\mathbf{B}$ and $G : \mathbf{A} \hookrightarrow \mathbf{B}$ is the inclusion functor. Then, we take the left and right Kan extensions $Lan_G K, Ran_G K$ of K along G and study their behavior. Intuitively, the more restrictive that $\mathbf{B}$ is (i.e. the more morphisms in $\mathbf{B}$) or the larger that $\mathbf{A}$ is (and therefore the more information that is stored in K) the more similar $Lan_G K, Ran_G K$ will be to each other.

* Matthew Pugh, Jo Grundy, Corina Cirstea, Nick Harris, Using Kan Extensions to Motivate the Design of a Surprisingly Effective Unsupervised Linear SVM on the Occupancy Dataset and Learning Is a Kan Extension.

A classification problem, so expect extensions.

> ...maximise the distance (Δ) between the decision boundaries produced by the left and right Kan extensions.

* Ralph Hinze, Kan Extensions for Program Optimisation

* Lifts are less commonly treated in *Cat*.

* It was mentioned earlier that parameter estimation involves lifting. In some suitable category, maybe a Markov category, we might see Bayesian updating to a posterior distribution over parameters this way. Perhaps as an updating of the lift.

* There's the world of fibrations generating liftings, and other structure allowing similar, e.g., Paolo Perrone, Lifting couplings in Wasserstein spaces

* Elsewhere we're hearing of lifts. Matteo Capucci: On Quantifiers for Quantitative Reasoning

> ...argmax can be characterised as a certain right Kan lift in the bicategory *Rel*. The same universal property, written in *QPL*, yields softmax as its unique solution.

Pointing us away from *Cat*.

* *Rel*: Relations seem more like associations than causes, symptoms case. Since *Rel* can be conceived as a poset-enriched category, we have right Kan-extensions and lifts.

$$\begin{tikzcd} & B \arrow[dr, dashed, "S", "\shortmid" marking] & \\ A \arrow[ur, "R", "\shortmid" marking] \arrow[rr, "T", "\shortmid" marking] & & C \end{tikzcd}$$

- $S(b,c)$ iff $\forall a : A, R(a,b) \to T(a,c)$

$$\begin{tikzcd} & B \arrow[dr, "S", "\shortmid" marking] & \\ A \arrow[ur, dashed, "R", "\shortmid" marking] \arrow[rr, "T", "\shortmid" marking] & & C \end{tikzcd}$$

- $R(a,b)$ iff $\forall c : C, S(b,c) \to T(a,c)$

Three types: Patients, Countries, Diseases. We know which patients have visited which countries, and we know which diseases they have contracted. Then we would arrive at a couple of relations between countries and diseases:

- All the people who have travelled to country b have disease c.
- All the people who have disease c travelled to country b.

In this more symmetric setting, do we still have such a sense of induction/abduction? Perhaps:

- If you go to that country, you'll contract that disease.
- You have that disease, so you must have been to that country.

But less an extension/lift issue and more which way S is oriented.

* Also can access the relations:

- None of the people who have travelled to country b have disease c.
- None of the people who have disease c travelled to country b.

* Lifts are harder to find in *Cat*, because Cat^{op} has no Yoneda structure (Fosco Loregian, Co(End) Calculus).

> The deep reason why we do not have nice formulae for Kan liftings, is that $\mathsf{Cat}^{\mathrm{op}}$ lacks an internal concept of (co)end; in other words the internal language of $\mathsf{Cat}^{\mathrm{op}}$ is not expressive enough.
>
> On the contrary, the theory of extensions behaves much better, and in fact it is possible to give a neat characterisation of pointwise extensions:

* On the other hand, we have right Kan lifts in the bicategories *Rel*, *Span*, *Prof* (profunctors or bimodules between small categories, enriched in *Set* or in any other suitably nice V), as well as in any biclosed monoidal category. (Todd Trimble)

* Is there something to be said then about how abduction is a harder problem than induction in more function-like settings?

* Perhaps double categories, such as Sets, functions and relations, make a better setting. Learning a network with two styles of arrow. We find *extensions* (and *restrictions*) in Michael Lambert, Evan Patterson, Representing Knowledge and Querying Data using Double-Functorial Semantics, but in a different sense.

Although, taking one of the tight morphisms here as an identity map...? Or, say, in $\mathbb{R}el$, we might think of extending a relation, $A \times B \to 2$ along $\langle f, g \rangle : A \times B \to C \times D$.

* For some Peircean reasons to consider double categories over 2-categories, see this Zulip thread. (The triad – object, arrow, composition – corresponding in the final column of Peirce's semiotic table below to – type, term, argument – is realized at two levels in a double category.)

* What shapes then? Nerves are bisimplicial sets.

* Tetrahedra as 2-morphisms in the double category $\mathbb{S}lice(A)$

* Classifying obstructions to lifts via cohomology:

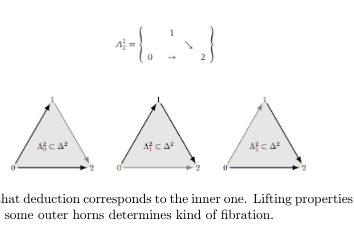
> **General**
>
> Given a fiber sequence $F \to A \xrightarrow{\mathbf{c}} B$ of classifying spaces/moduli stacks, hence $[\mathbf{c}]$ a universal characteristic class, and given an "A-structure" in the form of a morphism (cocycle) $f : X \to A$, then a lift $\hat{f}$ through $F \to A$ to an "F-structure" exists precisely if the induced B-structure $\mathbf{c}(f) : X \to B$ is trivializable in B-cohomology. One says that $[\mathbf{c}(f)]$ it is the *obstruction* to lifting the A-structure to an F-structure.
>
> $$\begin{tikzcd} & F \arrow[r] \arrow[d, "i"] & * \arrow[d, "pt_B"] \\ X \arrow[ur, "\hat{f}"] \arrow[r, "f"] & A \arrow[r, "\mathbf{c}"] & B \end{tikzcd}$$

* Classifying obstructions to extensions via cohomology:

> The formal dual of the lift obstruction problem discussed above is the following extension problem:
>
> we start with a universal characteristic map
>
> $$\mathbf{B}G \xrightarrow{\mathbf{c}} \mathbf{B}^n A$$
>
> representing a class $[\mathbf{c}] \in H^n(\mathbf{B}G, A)$ in the A-cohomology of $\mathbf{B}G$. Then given a morphism $\phi : \mathbf{B}G \to \mathbf{B}H$ we may ask for the obstruction to extending $\mathbf{c}$ along it.
>
> Now the statement is: if ϕ is a homotopy cofiber, then there is a good obstruction theory to answer this question. Namely in that situation we are looking at a diagram of the form
>
> $$\begin{tikzcd} \mathbf{B}Q \arrow[r, "f"] \arrow[d] & \mathbf{B}G \arrow[r, "\mathbf{c}"] \arrow[d, "\phi"] & \mathbf{B}^n A \\ * \arrow[r] & \mathbf{B}H \arrow[ur, "\hat{\mathbf{c}}"'] & \end{tikzcd}$$
>
> where the left square is an homotopy pushout. By its universal property, the extension $\hat{\mathbf{c}}$ of $\mathbf{c}$ exists as indicated precisely if the class
>
> $$[f^* \mathbf{c}] \in H^n(\mathbf{B}Q, A)$$
>
> is trivial.

* Inference as horn-filling:

> *Example 3.1.* The inner horn of the 2-simplex
>
> $$\Delta^2 = \left\{ \begin{smallmatrix} & 1 & \\ \nearrow & \Downarrow & \searrow \\ 0 & \rightarrow & 2 \end{smallmatrix} \right\}$$
>
> with boundary
>
> $$\partial\Delta^2 = \left\{ \begin{smallmatrix} & 1 & \\ \nearrow & & \searrow \\ 0 & \rightarrow & 2 \end{smallmatrix} \right\}$$
>
> looks like
>
> $$\Lambda^2_1 = \left\{ \begin{smallmatrix} & 1 & \\ \nearrow & & \searrow \\ 0 & & 2 \end{smallmatrix} \right\}.$$
>
> The two outer horns look like
>
> $$\Lambda^2_0 = \left\{ \begin{smallmatrix} & 1 & \\ \nearrow & & \\ 0 & \rightarrow & 2 \end{smallmatrix} \right\}$$
>
> and
>
> $$\Lambda^2_2 = \left\{ \begin{smallmatrix} & 1 & \\ & & \searrow \\ 0 & \rightarrow & 2 \end{smallmatrix} \right\}$$
>
> respectively.



Interesting to see that deduction corresponds to the inner one. Lifting properties against all/all but some outer horns determines kind of fibration.

4 Cases with less information, very sketchy

(Perhaps the most interesting part of discussions with Matt and Nathan.)

* What of abduction in broader terms?

> **On Explainable AI and Abductive Inference**
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>
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>
> **Abstract:** Modern explainable AI (XAI) methods remain far from providing human-like answers to 'why' questions, let alone those that satisfactorily agree with human-level understanding. Instead, the results that such methods provide boil down to sets of causal attributions. Currently, the choice of accepted attributions rests largely, if not solely, on the explainee's understanding of the quality of explanations. The paper argues that such decisions may be transferred from a human to an XAI agent, provided that its machine-learning (ML) algorithms perform genuinely abductive inferences. The paper outlines the key predicament in the current inductive paradigm of ML and the associated XAI techniques, and sketches the desiderata for a truly participatory, second-generation XAI, which is endowed with abduction.

("this task requires a modal system of higher-order reasoning that is able to produce clauses that encode the dictates of some universal common sense." Indications there that we should look to Peirce's *System Gamma*.)

* Here we have less information to go on. Perhaps we only have a morphism $f : A \to C$.

* This is a case of abduction where there is only a map $A \to C$, but as yet no B. Newton spoke of his 'Deduction of Universal Gravitation', but this of course isn't just coming from data.

* Consider the other direction first. Here we have an arrow $A \to B$, and would like a better grip of the situation, so we think of a C with maps from A and from B? Finding such a projection $B \to C$ can in a way be seen as inductive, especially forming the $A \to C$ map first.

* Models of the world we can see in terms of projection. But there's also the drive to get behind the world and construe what we see as a projection from something higher.

* A vivid case is given by 10d string theory/11d supergravity and compactification to our 4d world.

* The abductive formation of M-theory:

$$\begin{array}{ccc} M-\text{Theory}(?) & \overset{\text{lowenergylimit}}{\longrightarrow} & 11d\text{Supergravity} \\ {\scriptstyle \text{small coupling limit}}\downarrow & & \downarrow{\scriptstyle \text{KKreduction on}S^1} \\ \text{typeIIAstringtheory} & \overset{\text{lowenergyeffectiveQFT}}{\longrightarrow} & 10d\text{Supergravity} \end{array}$$

* In mathematics, we often call the domain of a map down into an object an 'extension', as in central extension. These extensions are detected by the cohomology of the object. (Even carrying in school arithmetic may be seen this way.) But these have more of a lift feel, and the terminology clashes with, say, 'field extension', $k \hookrightarrow K$.

* Similarly, "Formal deformation theory studies the obstruction theory of extensions to infinitesimal thickenings. A typical example of an infinitesimal thickening is a square-0-extension of a ring:" (deformation theory)

* Even, "falsity is a cocycle".

* Consider: factorisation into a deduction, and the other two patterns with arrows both out and then both in from the given arrow. Because an object isn't given, it requires concept/type formation, as well as morphism formation.

$$\begin{tikzcd} & B \arrow[dr, dashed, "g"] & \\ A \arrow[ur, dashed, "f"] \arrow[rr, "h"] & & C \end{tikzcd}$$

$$\begin{tikzcd} & B \arrow[dr, dashed, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, dashed, "h"] & & C \end{tikzcd}$$

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, dashed, "f"] \arrow[rr, dashed, "h"] & & C \end{tikzcd}$$

Depending on the order of discovery, we might see these in turn as: extension or lift; deduction or extension; deduction or lift.

* Analogy: Peirce sees analogy as the "most interesting, perhaps" type of reasoning, a "mixed" form of argument employing abduction, deduction, and induction, (CP 2.787, 1902) Peirce on Analogy. A "resemblance in form" (CP 7.498, 1898), has "all the strength of induction and more, besides" (CP 5.589, 1898). Cf. CP 2.513, W 2:46–47, 1867/1893 and CP 2.733–734, W 4:432–433, 1883.

* Fits with *analogical reasoning*:

- $s_i : S$ are similar to s^* in certain (known) respects, R.
- $s_i : S$ have some further feature Q.
- Therefore, s^* also has the feature Q.

$$\begin{tikzcd} S \arrow[r, "k"] \arrow[d, "l"] & R\&Q \arrow[d, "g"] \\ S \cup \{s^*\} \arrow[ur, dashed, "f"] \arrow[r, "h"] & R \end{tikzcd}$$

Same lift in a square as earlier. But there seems also to be a more radically creative sense of analogy.

* **Example**: Charles III is to George VI as Thursday is to what? Let $BM =$ British Monarchs. [Charles III, George V] forms a map $h : 2 \to BM$. We need to construct a linear order segment I, and a factorization of h through a map $f : 2 \to I$, such that the image of the first element comes two positions later than the image of the second, and $g : I \to BM$. Then we take DW as days of the week, and find a map $k : I \to DW$ such that $1 \xrightarrow{i_1} 2 \to I \to DW$ is Thursday. Composition with the other morphism $1 \xrightarrow{i_2} 2$ yields the answer.

$$\begin{tikzcd} & & DW & \\ & & I \arrow[u, "k"'] \arrow[dr, "g"] & \\ 1 \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & 2 \arrow[uur, "k \cdot f"] \arrow[ur, "f"] \arrow[rr, "h"] & & BM \end{tikzcd}$$

It seems to be about creating the right span. We use one domain as a *gauge* as to how to continue in the other domain. There should be some optimization account here as to why some span constructions are more satisfying than others.

* We can make the first variety of analogy look at least co-span-like, by considering maps from S to R and $\{s^*\}$ to R, and so also from the coproduct. Then given a map from S to $R\&Q$, can we refine the cospan via a map $\{s^*\}$ to $R\&Q$?

* We had pullbacks earlier, now we could point to (3/2)-pushouts and the conceptual blending described by Goguen: An Introduction to Algebraic Semiotics, with Application to User Interface Design. Cf. the classic example of boathouse and houseboat, where one replaces by 'boat' each of the positions in the schema of a 'person' sheltered in a 'house' in turn.

* Expansion of an observation may also require modification of the morphism. Say we have noticed that a full moon coincides with a Spring tide. We may also notice the same for a new moon. It's useful to combine these, $k : B' \to C$, and further generate a morphism, g, from all moon shapes, B, to height of high tide, C. We can see this as extending a map $k : B' \to C$ along an injection $i : B' \hookrightarrow B$

$$\begin{tikzcd} & B \arrow[dr, dashed, "g"] & \\ B' \arrow[ur, dashed, "i"] \arrow[rr, "k"] & & C \end{tikzcd}$$

* Then we have to further find an object, A, to depict relative earth-moon-sun positions and morphisms, f and h. This is to generate a common cause, in the style of

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, dashed, "f"] \arrow[rr, dashed, "h"] & & C \end{tikzcd}$$

* Limits as *projective* limits; colimits as *inductive* limits. Let's call the former *abductive* limits!

* In more general concept-formation, this is perhaps where other parts of Peirce's system come in.

* The rest of Peirce's *Speculative Grammar*. The three categories – Firstness, Secondness, Thirdness. The three components of semiotics – Representamen, object of thought, interpretant.

	Representamen	Object	Interpretant
Category of Firstness (Monadic)	**Qualisign** A monadic form of firstness	**Icon** A monadic form of Secondness	**Rheme** A monadic form of Thirdness
Category of Secondness (Dyadic)	**Sinsign** A firstness of a dyad	**Index** A dyadic form of Secondness	**Dicent/Dicisign** A dyadic form of Thirdness
Category of Thirdness (Triadic)	**Legisign** A firstness of a triad	**Symbol** A triadic form of Secondness	**Argument** A triadic form of Thirdness

Figure 8: The Elements of the Sign as Experienced through each Category.

	Representamen	'Object of Thought'	Interpretant
Category of Firstness (Monadic)	A vague, or indeterminate, quality	A vague, or indeterminate, object (of thought)	The possibility of a relational identity
Category of Secondness (Dyadic)	The actuality of a quality	The actuality of an object (of thought)	The actuality of a relational identity
Category of Thirdness (Triadic)	A quality as part of a 'system' or 'law'	An object (of thought) as part of a 'system' or law	A relational identity in a system of identities

Figure 9: Inserting 'Objects of Thought' in Peirce's Sign Classification.

(Chris Barnham, *The Natural History of the Sign: Peirce, Vygotsky and the Hegelian Model of Concept Formation*, De Gruyter: Mouton, 2022.)

I want to name that right-hand column: Type, Term, Derivation (Object, Morphism, Composition).

* What if neural nets are looking for triangles, they're morphisms looking for diagrams. NNs grokking – abducting? (h/t Younnesse Kaddar). Values in Vect to allow continuous flips into the next valley. Not triangles between nodes, but between activation patterns.

* Pulsating reason, simplicial reasoning. Might this tie to Simplicial neural networks, Topological neural networks, Cellular transformers?

* What if the primary arrow changes in time, e.g., is updated by new data, changing the inductive and abductive options?

* Psychotherapy seen in this light, production of a 'symbol'. Dream figure. By the end, you can tell your story. In the middle there is bemusement. Cf. Bion's *Grid*, "the inchoate "something" is transformed into a verbal statement."

* Shakespeare:

> And as imagination bodies forth
> The forms of things unknown, the poet's pen
> Turns them to shapes and gives to airy nothing
> A local habitation and a name.

* Confronted with a blank sheet, vague awareness of a something (an object) then of a relating (an arrow) then of a pattern of feelings. Then of these also in the world as things signed. Then as articulated in the logic.

* Relevance to meditative practices?