Interlude: Behaviour becomes visible in Poly

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Part IX described behaviour abstractly. A theory T has a behaviour category $\mathcal{B}_T$, and a runner from T to R induces a coprofunctor from $\mathcal{B}_T$ to $\mathcal{B}_R$. In the polynomial setting these constructions become unusually concrete. The reason is simple but powerful:

$$\boxed{\text{comonads in } \mathbf{Poly} \text{ are categories.}}$$

Consequently the Sweedler measuring comonad $\{T, R\}$ determines an ordinary category

$$\mathcal{B}_{T,R} := \mathrm{Cat}\{T, R\}.$$

When $R = \mathrm{Id}$ this is Garner's behaviour category; for general R it is a relative behaviour category.

The aim of this interlude is to unpack the loose direction of $\mathrm{Run}(\mathbf{Poly})$ through these relative behaviour categories. The fundamental representation is

$$\mathrm{Run}(T, R) \simeq [\mathcal{B}_{T,R}, \mathbf{Set}],$$

so objects of $\mathcal{B}_{T,R}$ are elementary relative behaviours and their representables generate arbitrary stateful runners by density. Applying behaviour to those representables produces a canonical kernel, and every runner is recovered from that kernel by a coend. Tight maps enter separately: a monad map $f : T \to R$ induces, by restriction of comodels, a cofunctor $\varphi_f : \mathcal{B}_R \to \mathcal{B}_T$. We use the convention that a tight arrow $C \to D$ in $\mathbf{CofProf}$ is a cofunctor $D \to C$, so this contravariance becomes the tight direction of the double functor Beh. The internal, enriched, graded and cubical polynomial settings below are organized around the same distinction between relative behaviour categories, their density kernels, and the cofunctorial action of tight maps.

1. Polynomial comonads are categories

Write a polynomial as

$$p = \sum_{i \in p(1)} y^{p[i]}.$$

A comonad structure for substitution on p equips its positions with the objects of a category and its directions with the arrows out of those objects; the counit supplies identities and comultiplication supplies composition. Conversely a category determines such a polynomial comonad. We write $\mathrm{Cat}(C)$ for the category corresponding to a polynomial comonad C. Morphisms of polynomial comonads correspond to *cofunctors* (retrofunctors), not ordinary functors; this variance is essential below.

Set

$$\boxed{\mathcal{B}_{T,R} := \mathrm{Cat}\{T, R\},} \qquad \boxed{\mathcal{B}_T := \mathcal{B}_{T,\mathrm{Id}}.}$$

There is a useful coalgebraic way to read the objects of this category. Under the usual accessibility hypotheses, let

$$\nu\{T, R\}$$

be the final coalgebra of the measuring comonad. Its carrier is the set of objects of $\mathcal{B}_{T,R}$. Thus an object

$$b \in \mathcal{B}_{T,R}$$

is a complete extensional behaviour of a stateful runner from T to R: it records everything that can happen when T is implemented while residual effects are handed to R, after all intensional distinctions between states with the same future behaviour have been forgotten. Every runner coalgebra

$$S \longrightarrow \{T, R\}(S)$$

therefore has a unique behaviour map

$$S \longrightarrow \mathrm{Ob}(\mathcal{B}_{T,R}).$$

When $R = \mathrm{Id}$ this is Garner's familiar final-behaviour map for T-comodels. The relative category $\mathcal{B}_{T,R}$ should be thought of as the categorical anatomy of that final relative behaviour space: its objects are complete behaviours, while its arrows describe the admissible continuations or probes available from them.

Garner's cosemantics theorem identifies comodels of an accessible theory with copresheaves on its behaviour category,

$$\boxed{T\text{-Comod} \simeq [\mathcal{B}_T, \mathbf{Set}],}$$

compatibly with the forgetful functors to $\mathbf{Set}$.[1] Thus an arbitrary $b \in \mathcal{B}_T$ is an elementary behaviour, while $\mathcal{B}_T(b, -)$ is the generic comodel with a distinguished state of behaviour b.

The coalgebras of $\{T, R\}$ are the stateful runners, and the polynomial comonad–category correspondence therefore gives

$$\boxed{\mathrm{Run}(T, R) \simeq [\mathcal{B}_{T,R}, \mathbf{Set}].}$$

This is the relative form of Garner's representation theorem: a runner is a copresheaf on the category of elementary relative behaviours. One may think of an object $b \in \mathcal{B}_{T,R}$ as a local execution profile: if the runner is currently in a state of type b, then b records how source commands can be answered using the residual theory R, together with the ways that profile can evolve after a command is executed. The copresheaf tells us how many concrete states of each such profile the runner possesses.

2. Relative behaviours and representables

For an arbitrary object $b \in \mathcal{B}_{T,R}$, Yoneda gives the representable

$$\mathcal{B}_{T,R}(b, -) : \mathcal{B}_{T,R} \to \mathbf{Set}.$$

Under $\mathrm{Run}(T, R) \simeq [\mathcal{B}_{T,R}, \mathbf{Set}]$, let $\boldsymbol{y}_b$ denote the corresponding generic runner. If M corresponds to X_M, co-Yoneda gives

$$X_M \cong \int^{b \in \mathcal{B}_{T,R}} X_M(b) \times \mathcal{B}_{T,R}(b, -),$$

hence

$$\boxed{M \simeq \int^{b} X_M(b)\, \boldsymbol{y}_b.}$$

[1] R. Garner, *The costructure–cosemantics adjunction for comodels for computational effects*, arXiv:2011.14520. The same paper sends maps of the relevant presheaf comonads to cofunctors on behaviour categories.

The density decomposition ranges over all relative behaviours. A state $x \in X_M(b)$ has relative behaviour b, and the representables $\mathcal{B}_{T,R}(b,-)$ are the elementary pieces from which arbitrary runners are assembled.

3. Cofunctors and the tight direction in CofProf

Part IX used the double category **CofProf**. We use the convention

$$\boxed{\text{a tight arrow } C \to D \text{ in } \mathbf{CofProf} \text{ is a cofunctor } D \to C.}$$

Thus cofunctors are read backwards in the tight direction. This is the natural convention for behaviour: restriction of comodels is contravariant on behaviour categories, while reversing cofunctors makes Beh covariant as a double functor.

A loose arrow

$$P : C \rightsquigarrow D$$

is a coprofunctor

$$\boxed{P : D^{op} \times C \longrightarrow \mathbf{Set}.}$$

Thus

$$\mathbf{CofProf}(C, 1) \simeq [C, \mathbf{Set}],$$

so Garner's equivalence

$$T\text{-Comod} \simeq [\mathcal{B}_T, \mathbf{Set}]$$

identifies comodels with loose arrows $\mathcal{B}_T \rightsquigarrow 1$.

If

$$P : C \rightsquigarrow D, \qquad Q : D \rightsquigarrow E,$$

their composite is

$$\boxed{(Q \odot P)(e,c) = \int^{d \in D} Q(e,d) \times P(d,c).}$$

Now let $\varphi : C \to D$ be a cofunctor. Under the convention just fixed, it represents a tight arrow

$$D \longrightarrow C$$

in **CofProf**. Its defining lifting operation induces a functor over underlying sets

$$\varphi_! : [C, \mathbf{Set}] \longrightarrow [D, \mathbf{Set}],$$

with

$$\boxed{(\varphi_! Y)(d) = \coprod_{\varphi c = d} Y(c),}$$

and with arrows of D acting by the chosen lifts along φ. We denote the associated loose arrow by

$$\varphi^\natural : D \rightsquigarrow C.$$

Its underlying coprofunctor is

$$\boxed{\varphi^\natural(c,d) := \coprod_{c' \in \varphi^{-1}(d)} C(c,c'),}$$

which is contravariant in $c \in C$ and covariant in $d \in D$, exactly as required for a loose arrow $D \rightsquigarrow C$. Equivalently,

$$\varphi^\natural(c,d) = (\varphi_! C(c,-))(d).$$

Thus $\varphi^\natural$ is the companion of the tight arrow $D \to C$ represented by the cofunctor $\varphi : C \to D$. For an ordinary functor the analogous formula is the familiar representable profunctor; here the cofunctor formula and the reversed tight orientation are the relevant ones.

Garner's functoriality is exactly of this form. A monad map $f : T \to R$ induces restriction

$$f^* : R\text{-Comod} \longrightarrow T\text{-Comod}$$

over **Set**, and hence a cofunctor

$$\boxed{\varphi_f : \mathcal{B}_R \longrightarrow \mathcal{B}_T.}$$

With our convention, the cofunctor $\varphi_f : \mathcal{B}_R \to \mathcal{B}_T$ is read as the tight arrow

$$\mathrm{Beh}(f) : \mathcal{B}_T \longrightarrow \mathcal{B}_R.$$

Its companion is the loose arrow $\varphi_f^\natural : \mathcal{B}_T \rightsquigarrow \mathcal{B}_R$. Thus Beh preserves the direction of tight maps and their companions once cofunctors are read backwards.

4. Runners act on comodels

A runner

$$M : T \rightsquigarrow R$$

takes an R-comodel and runs it as a T-comodel. We therefore have a change-of-comodel functor

$$M^* : R\text{-Comod} \longrightarrow T\text{-Comod}.$$

Using Garner's equivalences,

$$R\text{-Comod} \simeq [\mathcal{B}_R, \mathbf{Set}], \qquad T\text{-Comod} \simeq [\mathcal{B}_T, \mathbf{Set}],$$

we may regard this as

$$M^* : [\mathcal{B}_R, \mathbf{Set}] \longrightarrow [\mathcal{B}_T, \mathbf{Set}].$$

The behaviour construction of Part IX represents this cocontinuous functor by a coprofunctor

$$\boxed{\mathrm{Beh}(M) : \mathcal{B}_T \rightsquigarrow \mathcal{B}_R.}$$

Its values can be read directly from representables:

$$\boxed{\mathrm{Beh}(M)(r,t) = (M^* \mathcal{B}_R(r,-))(t).}$$

The interpretation is useful. The representable

$$\mathcal{B}_R(r,-)$$

is the generic R-comodel with a state of behaviour r. Running it through M produces a T-comodel. The set

$$\mathrm{Beh}(M)(r,t)$$

is the set of states of T-behaviour t appearing in this transformed generic R-behaviour. Thus $\mathrm{Beh}(M)$ tells us, behaviour by behaviour, how the residual behaviour of R can support behaviour of T.

5. The behaviour kernel

The density decomposition gives a canonical kernel indexed by all relative behaviours. For $b \in \mathcal{B}_{T,R}$, define

$$\boxed{K_{T,R}(b; r, t) := (\boldsymbol{y}_b^* \mathcal{B}_R(r, -))(t).}$$

For fixed b this is a coprofunctor $\mathcal{B}_T \rightsquigarrow \mathcal{B}_R$, and Yoneda variance gives

$$\boxed{K_{T,R} : \mathcal{B}_{T,R}^{op} \to \mathrm{CofProf}(\mathcal{B}_T, \mathcal{B}_R).}$$

Let M correspond to X_M. Since $M \simeq \int^b X_M(b)\boldsymbol{y}_b$, it remains only to justify passage through this colimit. On carriers, a runner N with state set S_N acts on an R-comodel with carrier X by $S_N \times X$. Colimits in $[\mathcal{B}_{T,R}, \mathbf{Set}]$ are computed pointwise, hence on state sets, and $(-) \times X$ preserves colimits in **Set**. Therefore

$$\boxed{\mathrm{Beh}(M)(r, t) \cong \int^{b \in \mathcal{B}_{T,R}} X_M(b) \times K_{T,R}(b; r, t).}$$

Equivalently, $\mathrm{Beh}(M) = X_M \star K_{T,R}$.

Tight maps act on behaviour categories through Garner's cofunctoriality. If $f : T \to R$ is a monad map, restriction of comodels induces

$$\varphi_f : \mathcal{B}_R \longrightarrow \mathcal{B}_T,$$

and, with the orientation convention of §3, the corresponding tight arrow in **CofProf** is $\mathcal{B}_T \to \mathcal{B}_R$. Its companion loose arrow is

$$\boxed{\varphi_f^\natural(r, t) = \coprod_{r' \in \varphi_f^{-1}(t)} \mathcal{B}_R(r, r').}$$

Thus the representable kernel describes arbitrary loose runners, while tight maps act functorially on the absolute behaviour categories by cofunctors.

6. Garner's theorem as the boundary case

Take $R = \mathrm{Id}$. Then $\mathcal{B}_{T,R} = \mathcal{B}_T$ and $\mathcal{B}_R \simeq 1$. For every $b \in \mathcal{B}_T$, the representable runner is the generic comodel represented by b, and

$$\boxed{K_{T,\mathrm{Id}}(b; *, t) = \mathcal{B}_T(b, t).}$$

Thus the kernel is Yoneda and, for $X : \mathcal{B}_T \to \mathbf{Set}$,

$$X(t) \cong \int^{b \in \mathcal{B}_T} X(b) \times \mathcal{B}_T(b, t).$$

This is co-Yoneda. Hence Garner's representation theorem is the $R = \mathrm{Id}$ boundary case of the representable kernel construction.

7. The linear example: relative behaviours and tight maps

Let A, B be monoids and take $T = Ay$, $R = By$. The absolute behaviour categories are $\mathbf{B}A$ and $\mathbf{B}B$, but the relative category is generally not $[\mathbf{B}A, \mathbf{B}B]$. Let E_A be the right-translation category of A: its objects are $x \in A$, and for each $a \in A$ there is an arrow

$$x \xrightarrow{a} xa,$$

with composition induced by multiplication. A functor $E_A \to \mathbf{B}B$ is equivalently a B-valued cocycle on translation arrows. Left translation by A acts on $\mathrm{Fun}(E_A, \mathbf{B}B)$, and

$$\boxed{\mathcal{B}_{Ay,By} \simeq \mathrm{El}_A(\mathrm{Fun}(E_A, \mathbf{B}B)),}$$

the category of elements for the left-translation action of A.

Concretely, a runner with state set S is a right A-action together with

$$\beta : S \times A \to B$$

satisfying

$$\beta(s, 1) = 1, \qquad \beta(s, a_1 a_2) = \beta(s, a_1)\, \beta(s \cdot a_1, a_2).$$

For B a group, objects of the relative category may be written as pointed functions $h : A \to B$, $h(1) = 1$, with arrows

$$h \xrightarrow{a} h_a, \qquad h_a(x) = h(a)^{-1} h(ax).$$

Thus arrows are labelled by source commands $a \in A$, not by intertwiners in B. The relative behaviour category therefore records how source commands translate the available B-valued response data.

A tight map $f : Ay \to By$ is simply a monoid homomorphism $f : A \to B$. Its companion runner has one state and cocycle

$$\beta(*, a) = f(a).$$

The induced cofunctor

$$\varphi_f : \mathbf{B}B \longrightarrow \mathbf{B}A$$

has the unique object map and lifts the command $a \in A$ by the residual arrow $f(a) \in B$. Hence its companion loose arrow has carrier B with the right A-action through f:

$$\boxed{\varphi_f^{\natural}(*, *) \cong B.}$$

Thus the linear example displays both structures explicitly: $\mathcal{B}_{Ay,By}$ is the translation/cocycle category governing arbitrary runners, while monoid homomorphisms $A \to B$ act on the absolute behaviour categories through the corresponding cofunctors.

8. Two coends

There are now two different coends in the behaviour story, and it is useful not to confuse them. The first builds the behavioural image of a single runner:

$$\boxed{\mathrm{Beh}(M) = \int^{b \in \mathcal{B}_{T,R}} X_M(b) \times K_{T,R}(b).}$$

Here we integrate over *relative behaviours*.

The second composes two behavioural coprofunctors. If

$$M : T \rightsquigarrow R, \qquad N : R \rightsquigarrow S,$$

then

$$\boxed{(\mathrm{Beh}(N) \odot \mathrm{Beh}(M))(s,t) = \int^{r \in \mathcal{B}_R} \mathrm{Beh}(N)(s,r) \times \mathrm{Beh}(M)(r,t).}$$

Here we integrate over *intermediate R-behaviours*.

The two constructions perform different jobs:

$$\int^{b \in \mathcal{B}_{T,R}} \quad \text{assembles one runner from elementary relative behaviours,}$$

$$\int^{r \in \mathcal{B}_R} \quad \text{composes two already assembled behavioural transformations.}$$

This helps explain the double functor

$$\mathrm{Beh} : \mathrm{Run}(\mathbf{Poly}) \longrightarrow \mathbf{CofProf}.$$

Runner composition eliminates an intermediate residual theory; behavioural composition eliminates the corresponding intermediate behaviour by a coend. In the linear example this is just the balanced tensor product of bisets.

Thus the double functor is not an accidental comparison between two unrelated structures. In **Poly**, it is already latent in the measuring comonoids.

9. Free theories and cofree behaviour

Let F^* be the free polynomial monad on a polynomial F. We use the Dirichlet internal hom $[-,-]$, right adjoint to the symmetric Dirichlet tensor $\otimes$. Symmetry gives

$$\mathbf{Poly}(F, [C,R]) \cong \mathbf{Poly}(C \otimes F, R) \cong \mathbf{Poly}(C, [F,R]),$$

so the extra notation $\langle F, R\rangle$ is unnecessary. The universal properties of free monads, measuring comonoids and cofree comonads give

$$\boxed{\{F^*, R\} \cong \mathfrak{c}_{[F,R]}.}$$

Indeed,

$$\begin{aligned} \mathrm{Comon}(C, \{F^*, R\}) &\cong \mathrm{Mon}(F^*, [C,R]) \\ &\cong \mathbf{Poly}(F, [C,R]) \\ &\cong \mathbf{Poly}(C, [F,R]) \\ &\cong \mathrm{Comon}(C, \mathfrak{c}_{[F,R]}). \end{aligned}$$

For $F = Ay$, $[Ay, R] \cong R^{\times A}$, so

$$\boxed{\{A^* y, R\} \cong \mathfrak{c}_{R^{\times A}}.}$$

For $F = y$, this specializes to $\{\mathbb{N}y, R\} \cong \mathfrak{c}_R$. With $R = y$, one recovers streams. Thus running a freely generated theory produces cofree behaviour.

10. Polynomial branching: List_+

We can also obtain branching behaviour while remaining entirely inside **Poly**.

Consider the nonempty-list monad

$$\mathrm{List}_+(X) = \coprod_{n\geq 1} X^n.$$

This is polynomial. Its multiplication concatenates nonempty lists. Taking it as residual gives

$$\boxed{\{\mathbb{N}y, \mathrm{List}_+\} \cong \mathfrak{c}_{\mathrm{List}_+}.}$$

The corresponding behaviours are leafless finitely branching planar trees: every node has a finite, ordered, nonempty family of children. In particular there are no terminal nodes, so every branch continues indefinitely. Thus changing the residual changes the geometry of behaviour:

$$\begin{array}{rcl} R = y & \rightsquigarrow & \text{one continuation: streams,} \\ R = \mathrm{List}_+ & \rightsquigarrow & \text{finitely many ordered continuations: branching trees.} \end{array}$$

The qualification "ordered" matters. List_+ remembers both order and multiplicity:

$$[x, y] \neq [y, x], \qquad [x, x] \neq [x].$$

It therefore models an intensional form of branching rather than ordinary relational nondeterminism.

If we instead use

$$\mathrm{List}(X) = \coprod_{n\geq 0} X^n,$$

leaves are also permitted. The empty list can represent a state with no continuation, providing a simple polynomial representation of termination or deadlock. There is a canonical quotient

$$\mathrm{List} \longrightarrow \mathcal{P}_{\mathrm{fin}}$$

which forgets order and multiplicity. The finite powerset monad is no longer polynomial, but it lies naturally in the larger accessible-endofunctor setting. Thus one can regard ordinary relational branching as a commutative-idempotent quotient of the more intensional branching visible in **Poly**.

A useful slogan is therefore

Poly remembers the shape of branching;	**Rel** remembers only the resulting alternatives.

For a set A of labels, take

$$T = A^*y,$$

the free monad generated by Ay. With residual List_+, the one-step coalgebraic form is

$$S \longrightarrow \mathrm{List}_+(S)^A.$$

Thus each state and action determine a finite ordered nonempty family of successor states.

These are polynomial analogues of finitely branching labelled transition systems. Passing from List_+ to the finite-powerset quotient forgets order and repetition and gives the familiar nondeterministic form

$$S \longrightarrow \mathcal{P}_{\mathrm{fin}}(S)^A.$$

11. The linear double dual and categorical vector fields

Let c be a category, regarded as a polynomial comonad, and let $[-,-]$ denote the Dirichlet internal hom. For any residual monad R, the identity on $[c,R]$ transposes to a canonical comonoid map

$$\boxed{c \longrightarrow \{[c,R],R\}.}$$

For a general residual R there is no reason for this comparison map to be invertible. The case $R = y$, however, is special.

Write

$$c = \sum_{x \in \mathrm{Ob}\, c} y^{c[x]}, \qquad \Gamma(c) = \prod_{x \in \mathrm{Ob}\, c} c[x].$$

An element $v \in \Gamma(c)$ chooses one outgoing arrow $v_x \in c[x]$ at every object x; it is therefore natural to regard v as a categorical vector field. The important point is that $[c,y]$ does *not* forget the objectwise directions. Rather,

$$\boxed{[c,y] \cong \sum_{v \in \Gamma(c)} y^{\mathrm{Ob}\, c} = \Gamma(c)\, y^{\mathrm{Ob}\, c}.}$$

Thus the *positions* of the linear dual are vector fields, while the directions over each vector field are the objects of c. The monad structure records how these object-indexed choices substitute and compose.

This distinction is essential for understanding the second dual. An object of

$$\{[c,y],y\}$$

initially amounts to choosing, for every vector field v, one of the directions over v, hence an object

$$x_v \in \mathrm{Ob}\, c.$$

Among the vector fields is the identity field

$$\mathbf{1}_c = (1_x)_{x \in \mathrm{Ob}\, c}.$$

Evaluation at this field therefore extracts an object

$$x_{\mathbf{1}_c} \in \mathrm{Ob}\, c.$$

The measuring laws force all the other choices x_v to be determined by this one: they are the evaluations of the corresponding vector fields at the chosen object. Likewise, the directions in the measuring comonoid recover the arrows emanating from that object, and the comonoid laws recover identities and composition. In this sense there is no further independent data in an object of the second dual.

Consequently the canonical comparison is an isomorphism:

$$\boxed{c \cong \{[c,y],y\}.}$$

Thus every polynomial category is the Sweedler dual of its linear dual. The point is not that the bare *set* of vector fields determines the category, but that the full polynomial monad $[c,y]$ retains both the vector fields and their objectwise directions. The second Sweedler dual reconstructs the category from these compatible evaluation data.

The discrete two-object category is a useful sanity check. There is only one vector field, the identity field, but there are still two directions over it. Hence

$$[c, y] \cong y^2.$$

For a one-object category $c = \mathbf{B}A$, the direction set has only one element. In that case

$$[c, y] \cong Ay, \qquad \mathbf{B}A \cong \{Ay, y\}.$$

This suggests a more precise slogan:

a category is reconstructed from its vector fields together with their objectwise evaluation data.

Varying the residual monad R gives the more general comparison maps

$$c \longrightarrow \{[c, R], R\}.$$

The case $R = y$ is reflexive; for other residual theories the second dual can add genuinely new states, so these maps should instead be read as residual-dependent completions or observability constructions.

12. Beyond Set: $\mathbf{Poly}_{\mathcal{E}}$

Shapiro and Spivak's *Structures on Categories of Polynomials* constructs $\mathbf{Poly}_{\mathcal{E}}$ for a finite-limit category $\mathcal{E}$, with substitution and Dirichlet tensor, and proves that substitution-comonoids are precisely internal categories in $\mathcal{E}$ whose source morphism is exponentiable; coalgebras correspond to internal copresheaves.[2] Thus

comonads in $\mathbf{Poly}_{\mathcal{E}}$ are suitable internal categories in $\mathcal{E}$.

If the required Sweedler measuring construction exists, one should expect $\{T, R\}_{\mathcal{E}}$ to determine an internal relative behaviour category $\mathcal{B}^{\mathcal{E}}_{T,R}$, with internal copresheaves providing the corresponding loose runners. Tight maps should act contravariantly on these internal behaviour categories by the appropriate internal analogue of Garner's cofunctors.

Schematically one would like

$$\boxed{\{T, R\}_{\mathcal{E}} \longrightarrow \mathcal{B}^{\mathcal{E}}_{T,R} \longrightarrow K^{\mathcal{E}}_{T,R} \longrightarrow \mathrm{CofProf}_{\mathcal{E}}.}$$

The final arrow is a programme: one must specify internal distributors, prove the relevant cocontinuity, formulate the internal representable kernel, and identify the internal cofunctorial action induced by tight maps.

13. Internal versus locally graded categories

This point is especially relevant to the $\mathcal{C}$-Set examples considered elsewhere in the tutorial. Let $\mathcal{C}$ be a small monoidal category and write

$$\mathcal{E} = \widehat{\mathcal{C}} = [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}].$$

[2] B. T. Shapiro and D. I. Spivak, *Structures on Categories of Polynomials*, arXiv:2305.00167.

We equip $\mathcal{E}$ with Day convolution $\star$. Locally $\mathcal{C}$-graded categories in this sense go back to Wood; Levy gives a convenient modern account.[3] If $\mathcal{C}$ is not symmetric, we use the right internal-hom convention

$$(-) \star A \dashv [A, -]_{\mathrm{Day}},$$

so that

$$[A, [B, X]] \cong [A \star B, X].$$

There are two distinct ways in which categorical structure can arise from $\mathcal{E}$, and it is important not to conflate them.

First, one may form ordinary polynomials internal to the presheaf topos:

$$\mathbf{Poly}_{\mathcal{E}}.$$

Its substitution-comonoids are suitable *internal categories in* $\mathcal{E}$. Their objects and arrows are themselves presheaves, with source, target, identity and composition maps internal to the cartesian category $\mathcal{E}$.

Second, using this Day convolution, a category enriched in

$$(\mathcal{E}, \star, yI)$$

has an ordinary set A of objects and, for each $a, b \in A$, a hom-presheaf

$$H_{ab} \in \mathcal{E}.$$

Equivalently, for every grade $c \in \mathcal{C}$ there is a set

$$H_{ab}(c)$$

of c-graded arrows $a \to b$, functorial contravariantly in c. Composition is a morphism of presheaves

$$H_{ab} \star H_{bc} \longrightarrow H_{ac},$$

and the identities are maps

$$yI \longrightarrow H_{aa}.$$

This is precisely a *locally $\mathcal{C}$-graded category*. Thus locally graded categories are not merely analogous to enriched categories: they are exactly categories enriched in the Day-convolution presheaf category. The interpretation is simple: the hom from a to b is no longer a set of executions but a whole presheaf of executions classified by resource, cost, dimension, or whatever the objects of $\mathcal{C}$ represent. Composition must then combine those grades as well as compose the underlying executions.

The same statement has a useful matrix form. Let

$$\mathbf{Mat}(\mathcal{E})$$

be the bicategory whose objects are sets and whose matrices $H : A \nrightarrow B$ have entries $H_{ab} \in \mathcal{E}$, interpreted as arrows $a \to b$, with composition

$$(H \odot K)_{ac} = \coprod_{b} H_{ab} \star K_{bc}.$$

[3] R. Wood, *Indicial methods for relative categories* (1976) and *V-indexed categories* (1978); P. B. Levy, *Locally graded categories* (2019).

Then

$$\boxed{\text{locally } \mathcal{C}\text{-graded categories } = \text{ monads in } \mathbf{Mat}(\widehat{\mathcal{C}}_{\mathrm{Day}}).}$$

The monad multiplication is exactly graded composition. This is the precise graded analogue, on the matrix side, of the familiar fact that ordinary categories are monads in $\mathbf{Mat}(\mathbf{Set})$.

There is moreover a polynomial-looking contravariant realization which explains why the same structure should appear as a comonoid. Fix the object set A, and for an $\mathcal{E}$-valued graph $H = (H_{ab})$ define an endofunctor

$$P_H : \mathcal{E}^A \longrightarrow \mathcal{E}^A$$

by

$$\boxed{(P_H X)_a = \prod_{b \in A} [H_{ab}, X_b]_{\mathrm{Day}}.}$$

Here $[-,-]_{\mathrm{Day}}$ is the internal hom for Day convolution. The key calculation is

$$\begin{aligned}(P_H^2 X)_a &= \prod_b \left[H_{ab}, \prod_c [H_{bc}, X_c] \right] \\ &\cong \prod_{b,c} [H_{ab}, [H_{bc}, X_c]] \\ &\cong \prod_c \left[\coprod_b H_{ab} \star H_{bc}, X_c \right].\end{aligned}$$

Consequently

$$\boxed{P_H^2 \cong P_{H \odot H}.}$$

The variance is now exactly the one familiar from ordinary polynomial containers. A graded composition map

$$H \odot H \longrightarrow H$$

induces, contravariantly in the first argument of the internal hom,

$$P_H \longrightarrow P_{H \odot H} \cong P_H^2,$$

while the identity matrix $I_A \to H$ induces

$$P_H \longrightarrow P_{I_A} \cong \mathrm{Id}.$$

Associativity and the unit laws become the comonad axioms. Hence every locally graded category canonically determines a comonad of this Day-enriched polynomial form:

$$\boxed{\text{locally } \mathcal{C}\text{-graded category} \quad \longmapsto \quad \text{Day-enriched polynomial comonad.}}$$

This is already an exact statement at the matrix/enriched-representable level. What remains to be formalized is the intrinsic analogue of ordinary **Poly** in which these endofunctors and their cartesian or lens-like transformations form the ambient polynomial category.

The representable case makes the grading transparent. For $c \in \mathcal{C}$, let $y(c)$ be the representable presheaf. Since Yoneda is strong monoidal for Day convolution,

$$y(c) \star y(d) \cong y(c \otimes d).$$

The corresponding basic monomial is

$$R_c(X) = [y(c), X]_{\mathrm{Day}},$$

and enriched Yoneda gives

$$\boxed{R_c(X)(d) \cong X(d \otimes c).}$$

Thus R_c is literally a shift of grade by c, and

$$R_c R_d \cong R_{c \otimes d}.$$

More generally, if

$$E = \coprod_{i=1}^{n} y(c_i),$$

then

$$[E, X]_{\mathrm{Day}}(d) \cong \prod_i X(d \otimes c_i),$$

while

$$E \star E' \cong \coprod_{i,j} y(c_i \otimes d_j).$$

This is exactly the closure calculation expected for the J_0 and J_+ arity levels discussed elsewhere: representable grades compose by tensoring, and finite coproducts of representables remain of the same form.

There is, however, an important boundary to the present result. If one passes from the pure monomials $[E, -]$ to coefficient-bearing enriched-polynomial generators

$$F_{E,K}(X) = [E, X]_{\mathrm{Day}} \star K,$$

then

$$F_{E,K} F_{E',K'}(X) = [E, [E', X] \star K'] \star K.$$

This reduces to a single generator

$$[E \star E', X] \star K' \star K$$

only when the canonical strength map

$$[E, Y] \star K' \longrightarrow [E, Y \star K']$$

is invertible, or when the chosen enlarged polynomial class is nevertheless closed under the displayed composite. Dualizable arities are sufficient, but far too restrictive for cost categories such as $(\mathbb{N}, +, 0)$. The more promising route is therefore the arities programme: present coefficients by the chosen arities and use eleuthericity and accessibility to extend substitution from the arity level. Thus the locally graded comonoid calculation is settled for the pure-direction fragment, while closure of the full coefficient-bearing enriched polynomial universe remains one of the substantive checks.

The distinction between internal and locally graded categories becomes particularly vivid for precubical sets. Let $\square$ be the cube category and

$$\mathbf{pCub} = [\square^{\mathrm{op}}, \mathbf{Set}]$$

with its standard Day tensor. Ordinary internal polynomials over **pCub** lead to categories internal to precubical sets: cubically varying objects and arrows. The locally graded construction instead gives categories whose individual homs are precubical sets,

$$H_{xy} \in \mathbf{pCub}.$$

Thus $H_{xy}([n])$ is the set of n-cube-graded ways of passing from x to y, with all cubical face maps built into the hom-object. These may be thought of as *locally cubically graded categories*: categories whose spaces of processes carry explicit concurrency geometry. An ordinary transition category merely says which executions go from x to y; here the hom-object also records whether two executions are faces of a genuine square, whether three belong to a filled cube, and so on. Independence is therefore local structure of the hom itself, rather than information reconstructed afterwards from interleavings.

A tiny test already distinguishes true concurrency from mere interleaving. Write

$$\square^n = y[n]$$

for the representable n-cube, and let

$$\partial\square^2 \hookrightarrow \square^2$$

be the boundary of the square. Take $H_{xy} = \square^2$ when two actions a, b are genuinely independent, and $H'_{xy} = \partial\square^2$ when only the two boundary interleavings are present. If a third independent action c is represented by $\square^1$, Day convolution gives

$$\square^2 \star \square^1 \cong \square^3,$$

whereas

$$\partial\square^2 \star \square^1 \hookrightarrow \square^3$$

is the cubical shell consisting of the four lateral square faces, with no 3-cube filler and with the two end faces witnessing a-b concurrency absent. For the standard precubical geometric tensor this map is indeed monic: degreewise

$$(X \star Y)_n \cong \coprod_{p+q=n} X_p \times Y_q,$$

so tensoring with a fixed precubical set preserves monomorphisms. Thus the distinction between a filled concurrency square and its interleaving boundary survives further enriched-polynomial composition. In the contravariant realization this inclusion is detected by

$$[\square^3, X] \longrightarrow [\partial\square^2 \star \square^1, X].$$

So the enriched polynomial construction does not collapse true concurrency to its interleavings.

This example also exposes a useful subtlety. The Day tensor models addition of an *independent* cubical direction. Thus $\square^2 \star \square^1 = \square^3$ means that c is concurrent with the two directions already present. It should not be read as strict temporal sequencing "first the a, b-block, then c". Such causal gluing would instead require a different operation, for example a suitable pushout identifying the terminal vertex of one execution block with the initial vertex of the next. For concurrency this suggests that two compositions may ultimately need to coexist:

independence/parallel combination	:	$X \star Y$,
causal or sequential gluing	:	$X ; Y$.

This is precisely the sort of distinction for which a duoidal refinement is natural. It also links the present interlude back to Part VII: the cubical enrichment remembers fillers, conflict and missing higher-dimensional cells locally in each hom, while a second composition could encode the causal assembly of execution blocks.

The two polynomial programmes should therefore be kept distinct but compared systematically:

$\mathbf{Poly}_{\mathcal{C}\text{-Set}}$	$\rightsquigarrow$	internal behaviour categories,
Day-enriched polynomial functors over $\mathcal{C}$-Set	$\rightsquigarrow$	locally graded behaviour categories.

The matrix theorem and the construction $H \mapsto P_H$ make the second line precise for the pure-direction fragment. The full ambient enriched-polynomial category is still to be constructed, especially its coefficient-bearing substitution calculus and the extension from the chosen arities. This is the natural setting for an enriched analogue of the slogan

comonads in **Poly** are categories.

14. What Poly teaches us

The polynomial picture makes the loose behaviour theory concrete, and it also changes the intuition. Relative observability itself is not peculiar to **Poly**: in any Sweedler setting the final $\{T, R\}$-coalgebra classifies complete behaviours of T as seen through the residual theory R. What is special here is that the relative behaviour object is a polynomial comonad and hence carries an ordinary category structure. Observability therefore has more than a set of final values: it has a local geometry of available continuations and probes.

Instead of thinking of a runner primarily as a state set carrying operations, we can think of it as a collection of elementary execution profiles together with the ways those profiles evolve and compose. A measuring comonoid determines a relative behaviour category

$$\{T, R\} \rightsquigarrow \mathcal{B}_{T,R},$$

and runners are its copresheaves:

$$\boxed{\mathrm{Run}(T, R) \simeq [\mathcal{B}_{T,R}, \mathbf{Set}].}$$

This refines the final-coalgebra picture in two ways. First, positions of the final $\{T, R\}$-coalgebra come equipped with directions: from a relative behaviour one can ask not only *which* complete behaviour is present, but which experiments, continuations or residual responses are available from it and how they compose. Second, a concrete runner is not represented merely by its map into the final carrier. The copresheaf records how its states are distributed over relative behaviours and how those realizations vary along behaviour morphisms. In this sense **Poly** separates extensional behaviour from its realizations while retaining the categorical structure of the available observations.

The representables are the elementary density generators. Writing $\boldsymbol{y}_b$ for the runner corresponding to $\mathcal{B}_{T,R}(b, -)$, define

$$K_{T,R}(b) = \mathrm{Beh}(\boldsymbol{y}_b).$$

Then every runner M, represented by X_M, satisfies

$$\boxed{\mathrm{Beh}(M) \cong \int^{b \in \mathcal{B}_{T,R}} X_M(b) \times K_{T,R}(b).}$$

Thus the loose behavioural image is assembled by a coend over elementary relative behaviours.

Tight maps have a complementary description. A monad map $f : T \to R$ induces restriction of comodels and hence a cofunctor

$$\varphi_f : \mathcal{B}_R \longrightarrow \mathcal{B}_T.$$

With the convention that a tight arrow $C \to D$ in **CofProf** is a cofunctor $D \to C$, this is the tight arrow

$$\mathrm{Beh}(f) : \mathcal{B}_T \longrightarrow \mathcal{B}_R.$$

Its companion loose arrow is

$$\boxed{\varphi_f^{\natural}(r,t) = \coprod_{r' \in \varphi_f^{-1}(t)} \mathcal{B}_R(r,r').}$$

This gives the tight and loose parts of the behaviour double functor without identifying tight maps with objects of the relative behaviour category.

For $R = \mathrm{Id}$, the representable kernel reduces to Garner and Yoneda. For linear monads, the relative category is the translation/cocycle category of §7, while a monoid homomorphism $A \to B$ induces the corresponding cofunctor $\mathbf{B}B \to \mathbf{B}A$. For free source theories the measuring comonad is cofree on $[F, R]$. With List_+ residual it produces leafless ordered branching trees.

The later enriched and internal polynomial constructions can therefore be judged against three questions:

1. what replaces the relative behaviour category;
2. what plays the rôle of the representable density kernel;
3. how tight maps act on behaviour categories and their associated loose arrows.

The locally graded construction of §13 gives a strong answer to the first question on the enriched side and an explicit enriched-polynomial realization of its composition. The remaining task is to develop the corresponding enriched density kernel and tight cofunctoriality inside the full enriched polynomial universe.