Modal and graded modal types

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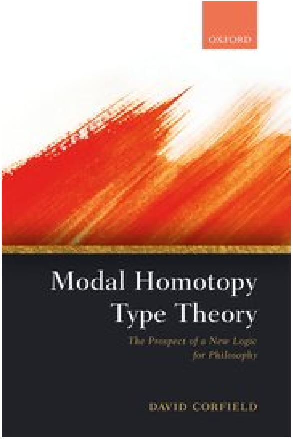
New tools for philosophy



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There's a huge amount to be done thinking philosophically about **modal types**.

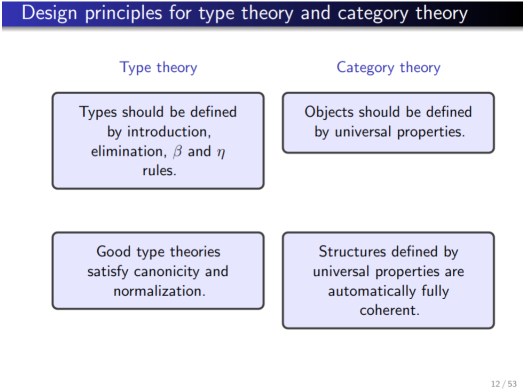
Graded modalities provide a further opportunity.

Converging lines of enquiry leading to modal dependent type theory

- Philosophy
- Logic
- Mathematics
- Programming languages
- Applied Category Theory
- Mathematical physics
- ...

Formal languages

- Type theory
- Category theory



Mike Shulman, Type 2-theories

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- Dependent sum/pair as total space
- Dependent product/function as sections
- Latter two as adjoints to context extension

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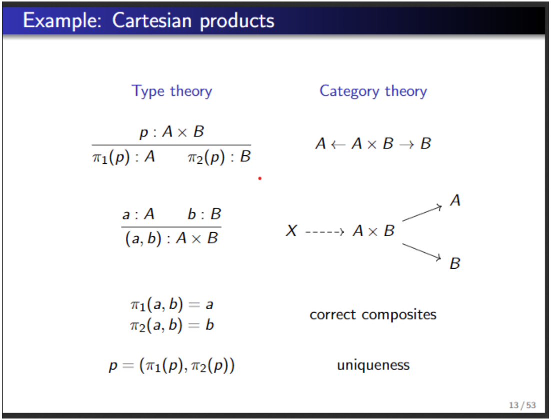
- Latter two as adjoints to context extension

Both languages are important:

> We don't get Univalent Foundations without mathematical insights expressed in category theory (pullbacks of bundles).

Product type

Even the simple case of product/conjunction is illuminated by recouching it in category-theoretic terms, e.g., elimination as the counit of the comonad generated by the duplication-product adjunction:



Formalism for modal dependent type theory

I'm inclined to think that the 'right' formalism for modal dependent type theory will emerge from the Licata-Shulman-Riley approach (sketched by Mike Shulman in his talk Type 2-theories).

If HoTT is the internal language of an $(\infty, 1)$-topos, then modal HoTT is the internal language for diagrams of $(\infty, 1)$-toposes and geometric morphisms.

Often one specifies that the (co)monads generated are idempotent.

Cutting edge

Such considerations are shaping real mathematics right now

In setting up this proof, we re-develop the theory of equivariant principal bundles from scratch by systematic use of Grothendieck's *internalization*. In particular we prove that all the intricate equivariant local triviality conditions considered in the literature are automatically *implied* by regarding G-equivariant principal bundles as principal bundles internal to the BG-slice of the ambient cohesive ∞-topos. We also show that these conditions are all equivalent. Generally we find that the characteristic subtle phenomena of equivariant classifying theory all reflect basic modal properties of singular-cohesive homotopy theory (hence of cohesive global equivariant homotopy theory).

Hisham Sati and Urs Schreiber, Equivariant Principal ∞-Bundles

Sources for modal dependent types

Even if this were to settle syntactic issues, it is still worth reflecting on insight from relevant fields as to meaning and use:

- Philosophers on, say, possibility and necessity.
- Computer scientists on effects and coeffects, etc.

Computer science as applied metaphysics

Computer science treats a small world version of philosophical ideas.

- Time - temporal flow in computation.
- Ontology - ontology of a database.

Greater or lesser amounts of structure can be included in each case.

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For a modal concept such as *possibility*, rather than some Lewisian notion of possible worlds as maximal spatiotemporally related objects, we might ask, relative to some small set of variation, does something become allowed?

Let's know explore this idea.

Brandom and the Kant-Sellars thesis about modality

...in being able to use nonmodal, empirical descriptive vocabulary, one already know how to do everything one needs to know how to do in order to deploy modal vocabulary, which according can be understood as making explicit structural features that are always already implicit in what one does in describing. (From empiricism to expressivism, 2015, p. 143)

The ability to use ordinary empirical descriptive terms such as 'green', 'rigid', and 'mass' already presupposes grasp of the kinds of properties and relations made explicit by modal vocabulary. (Between saying and doing, 2008, pp. 96-97)

The uses of nonmodal vocabulary that are made explicit by modal language include those where one describes how the state of something would be under certain kinds of **variation of its current situation**, its ranges of **counterfactual robustness**.

> *One has not grasped the concept cat unless one knows that it would still be possible for the cat to be on the mat if the lighting had been slightly different, but not if all life on earth had been extinguished by an asteroid-strike. (Between saying and doing, 2008,p. 97)*

Modal types

Almost all of the philosophical literature considers modal operators as applied only to propositions. But if we consider **propositions as (some) types**, then we might expect these operators to apply to all types.

This happens with computer science's data types, and with mathematical spaces (cohesive modalities).

Modal types

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What of the term '*possible worlds*' itself? Is this an indication of a source of possibility, variation over some type of worlds?

Then we speak of *future* events and *past* triumphs, *obligatory* payments and *known* criminals.

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(Dependent sum, dependent product and base change within a context, or interpreted in a slice category over B, if that is of any help.)

Three functions: 1

Take a mapping

$$Owner : Dog \rightarrow Person,$$

then any property of people can be transported to a property of dogs, e.g.,

$$Being\ French \mapsto Being\ owned\ by\ a\ French\ person.$$

We shouldn't expect every property of dogs will occur in this fashion.

In other words, we can't necessarily invert this mapping to send, say, 'Pug' to a property of People.

Three functions: 2

We can try...

$$\textit{Pug} \mapsto \textit{Owning some pug} \mapsto ???$$

Three functions

But then

$$Pug \mapsto Owning\ some\ pug \mapsto Owned\ by\ someone\ who\ owns\ a\ pug.$$

However, people may own more than one breed of dog.

Three functions: 3

How about

$$Pug \mapsto Owning\ only\ pugs \mapsto ???$$

Three functions

But this leads to

Pug $\mapsto$ *Owning only pugs* $\mapsto$ *Owned by someone owning only pugs*

But again, not all pugs are owned by single breed owners.

A glimpse of modality

Note the modal flavour of these properties:

- *Pug*
- *Owned by someone who owns a pug* = *possibly pug*
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Under variation over co-owned dogs, the pug property is allowed/ensured.

Modal logic

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$$w : W \vdash P(w) : \mathit{Prop}$$

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This yields *possibly* P and *necessarily* P as propositions dependent on the type *Worlds*, and as such comparable to P.

$$w : W \vdash \Diamond P(w) : Prop \qquad w : W \vdash \Box P(w) : Prop$$

Accessible worlds

More generally, we might consider an equivalence relation generated by a function: $f : W \to V$. Then

- Necessarily $P(w)$ holds at a world if P holds at all related worlds.
- Possibly $P(w)$ holds at a world if it holds at some related world.

Note that we don't need to restrict to propositions.

First glimpse of grading

Now we might consider a sequence of such V:

$$V_1 \twoheadrightarrow V_2 \twoheadrightarrow \cdots \twoheadrightarrow V_n,$$

with compatible maps $f_i : W \to V_i$. So a sequence of progressively coarser relations.

Then we have a sequence of operators, $\Box_i$ and $\Diamond_i$, so that, e.g., for $i < j$, (the truth of) $\Box_j P$ implies $\Box_i P$ and $\Diamond_i P$ implies $\Diamond_j P$.

Connections to the philosophical literature

This is not too far from what is called 'quantifier restriction' at SEP: Varieties of Modality, variation over restricted collections of possible worlds. Consider the associated equivalence classes of the actual world for the series of equivalance relations from the previous slide.

But, as in the dog and owner case, we need nothing more than variation arising from a series of functions. We don't need *worlds*.

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The outlook I’m presenting here sits well with the expressivist/ non-descriptivist approach, to which tradition the “modal normativism” of Amie Thomasson, *Norms and Necessity*, 2020, OUP, is a recent contribution.

The ideas of Brandom and Thomasson could both profitably be given a type-theoretic treatment.

In history

Since the notions of necessity and contingency assume *sets* of more or less similar events, their application is *inherently sensitive to the descriptions we use in referring to events*. To assess degree of necessity, we need to know whether the same type of event would have occurred given a certain change or intervention. Our assessment hinges, therefore, on modes of sorting and individuation, on what we consider a type, or the same type. "The war was necessary" means "a similar kind of war would have occurred in any event," hence a judgment about the degree of necessity that should be ascribed to an event will depend on how broadly or narrowly we construe the type in question. Knowing early twentieth-century European history, we may believe a war would have started sooner or later. But if the historian uses finer distinctions – a war in 1914, a war triggered by an assassination, she might lower the level of necessity she ascribes to the war, attributing increased significance to the assassination in Sarajevo. (Ben-Menahem, Y., p 124)

'Historical Necessity and Contingency', in A. Tucker (ed.) *A Companion to the Philosophy of History and Historiography*, Blackwell, 2009.

Grading

Indices of the modal operators belong to some structured set.

E.g., in the case of a sequence of n equivalence relations, for $i < j$, $\Diamond_i \Diamond_j P$ is equivalent to $\Diamond_j P$.

Obligatoriness

An idea: degrees of obligatoriness could be captured by the size of the domain of variation under which an action is required.

- So long as Y you must do X.
- It is unconditional that you do X.

Types in general

Again, the dependent types need not be propositions (Sec. 4.2.1 of my book). Think of fibres above a base space.

The injection of a fibre into the total space corresponds to saying of a type A, that As are possible As.

Coalgebras for the comonad $\Box_W$, generated by $W \to \mathbf{1}$ are the constant types, W^*A, for A a non-dependent type. Analogous to partial differential equations – how to continue to new fibres (p. 157 of my book).

Variation via spans

We can think of the modalities resulting from a map $f : W \to V$ as a passage through $W \to V \leftarrow W$, and this in turn through the associated span $W \leftarrow R \to W$, representing the equivalence relation.

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But then we might consider $R_d(t_1, t_2)$ iff $t_2 - t_1 \geq d$, allowing us to express, e.g., that something happened at least d days ago, a graded temporal operator. (There is some R_d interval ending now which began with...)

There would be an additive structure on the grading.

Intermodalities

We may generalise further to any span $A \leftarrow C \rightarrow B$. This gives rise to the 'intermodalities' of Fong, Myers, Spivak in *Behavioral Mereology: A Modal Logic for Passing Constraints* (2021):

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Here we have *intermodalities* relating properties of A to properties of B. Think of A and B as the types of behaviours of subparts of a system, where we are interested in cases where the first part being in a certain kind of state *allows* or *ensures* that the second part is in a specific kind of state.

Then we might have a variety of spans, corresponding to the degree of tightness of the constraint between the parts, or to a way to parameterise the dynamics if seen as temporal parts.

Structure type

Rather than take variation as taking place simply over a type, or a sequence of types, we might look into their inner structure.

We may stipulate additional structure on a type, such as *Time*. (Sec. 4.3 of my book.)

But let's think now about the structure provided by the *context*.

Worlds as contexts

Consider what might be the beginning of a story, or a play:

> *A man walks into a bar. He's whistling a tune. A woman sits at a table in the bar. She's nursing a drink. On hearing the tune, she jumps up, knocking over the drink. She hurls the glass at him. "Is that any way to greet your husband?", he says.*

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For Ranta (1994, *Type-theoretic grammar*), this kind of narrative should be treated as the extension of one long *context*, with its dependency structure, which begins as follows:

$x_1 : Man, x_2 : Bar, x_3 : WalksInto(x_1, x_2), x_4 : Tune, x_5 : Whistle(x_1, x_4), x_6 : Woman, x_7 : Table, x_8 : Locate(x_7, x_2), x_9 : SitsAt(x_6, x_7), x_{10} : Drink, x_{11} : Nurse(x_6, x_{10}), x_{12} : Hear(x_6, x_5), \ldots$

Contexts

In general, a context in type theory takes the form

$$\Gamma \ = \ x_0 : A_0, \ x_1 : A_1(x_0), \ x_2 : A_2(x_0, x_1), \ \ldots, x_n : A_n(x_0, \ldots, x_{n-1}),$$

where the A_i are types which may be legitimately formed. As we add an item to a context, there may be dependence on any of the previous variables.

A context need not take full advantage of this array of dependencies. For instance, in the case above, *Whistle* only depends upon x_1 and x_4 and not upon x_2 or x_3. But it certainly cannot depend on a variable ahead of it.

(Cf. Mike Shulman's *context shape* as an inverse category, Type 2-theories)

Contexts

Any such context corresponds itself to an object, the iterated dependent sum of the context. Let W represent the iterated sum, and W_i the stages of the construction of W, then the maps we considered earlier

$$W\text{-}dependent \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \mathbf{1}\text{-}dependent$$

now factor through the successive stages of the construction of the context:

$$W\text{-}dependent \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \cdots \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} W_2\text{-}dependent \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} W_1\text{-}dependent \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \mathbf{1}\text{-}dependent$$

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We could construct (graded) modal operators, etc., relative to an initial segment of the context.

Counterfactuals could work by stripping back a context until the counterfactual antecedent can hold, an idea first considered by Aarne Ranta (1991, *Constructing possible worlds*, Theoria 57 (1-2): 77-99).

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Or perhaps one might use the *context shape* (cf. first-order logic's and propositional logic's weak dependency structures) .

There is such a vast store of shared knowledge that it seems that any story can go almost anywhere its author wishes. Continuing our Western,

- An elephant escapes from the box car in which it is travels with the circus and tramples all in the saloon underfoot.
- A tornado rips through the town and takes the couple somewhere over the rainbow.

Instead, one might imagine a more controlled setting of what can occur next, where paths fan out according to circumscribed choices, as we find with computations paths in computer science or, in a more extreme form, with the collection of real numbers for the intuitionist.

To conclude

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- Variation over types; over functions; over spans (relations and more general)
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Thank you for listening!