Safeguarding via category-theoretic systems theory

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Independent researcher, currently funded by ARIA

3 February 2025

ARIA: Advanced Research and Innovation Agency

- Safeguarded AI program
- Providing forms of guarantee for the behaviour of cyber-physical systems

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- Providing forms of guarantee for the behaviour of cyber-physical systems
- At the centre of our approach is DOTS:

Cornell University

We gratefully acknowledge support fr

arXiv > math > arXiv:2505.18329

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Mathematics > Category Theory

[Submitted on 23 May 2025 (v1), last revised 28 May 2025 (this version, v2)]

Towards a double operadic theory of systems

Sophie Libkind, David Jaz Myers

We present a unified framework for categorical systems theory which packages a collection of open systems, their interactions, and their maps into a symmetric monoidal loose right module of systems over a symmetric monoidal double category of interfaces and interactions. As examples, we give detailed descriptions of (1) the module of open Petri nets over undirected wiring diagrams and (2) the module of deterministic Moore machines over lenses. We define several pseudo-functorial constructions of modules of systems in the form of doctrines of systems theories. In particular, we introduce doctrines for port-plugging systems, variable sharing systems, and generalized Moore machines, each of which generalizes existing work in categorical systems theory. Finally, we observe how diagrammatic interaction patterns are free processes in particular doctrines.

Subjects: **Category Theory (math.CT)**
Cite as: arXiv:2505.18329 **[math.CT]**
(or arXiv:2505.18329v2 **[math.CT]** for this version)
https://doi.org/10.48550/arXiv.2505.18329

Submission history

From: Sophie Libkind [view email]
[v1] Fri, 23 May 2025 19:35:57 UTC (463 KB)
[v2] Wed, 28 May 2025 19:20:26 UTC (471 KB)

Outline today

- Category theory and its applications
- Double categories
- Systems composition
- Systems behaviour
- Compositionality and safeguarding cyber-physical systems

Applications of category theory

(With approximate dates)

- Mathematics (from the 1940s)
- Logic/Foundations (from the 1960s)
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Category Theory	Physics	Topology	Logic	Computation
object X	Hilbert space X	manifold X	proposition X	data type X
morphism $f: X \to Y$	operator $f: X \to Y$	cobordism $f: X \to Y$	proof $f: X \to Y$	program $f: X \to Y$
tensor product of objects: $X \otimes Y$	Hilbert space of joint system: $X \otimes Y$	disjoint union of manifolds: $X \otimes Y$	conjunction of propositions: $X \otimes Y$	product of data types: $X \otimes Y$
tensor product of morphisms: $f \otimes g$	parallel processes: $f \otimes g$	disjoint union of cobordisms: $f \otimes g$	proofs carried out in parallel: $f \otimes g$	programs executing in parallel: $f \otimes g$
internal hom: $X \multimap Y$	Hilbert space of 'anti-X and Y': $X^* \otimes Y$	disjoint union of orientation-reversed X and Y: $X^* \otimes Y$	conditional proposition: $X \multimap Y$	function type: $X \multimap Y$

Table 4: The Rosetta Stone (larger version)

(Baez and Stay 2009, Physics, Topology, Logic and Computation: A Rosetta Stone)

This fusion continues: The Quantum Monadology

The Quantum Monadology

An article that we have written:

- Hisham Sati, Urs Schreiber:

  The Quantum Monadology

  Quantum Studies: Mathematics and Foundations
  Vol 12, No 25 (2025)
  doi:10.1007/s40509-025-00368-5

  download:

  - pdf (v2, some typos fixed)
  - arXiv:2310.15735

CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS

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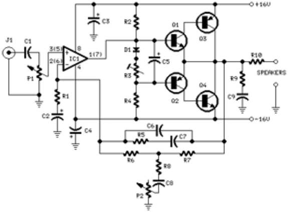
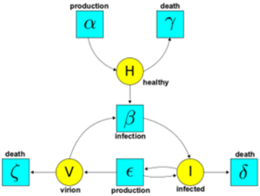
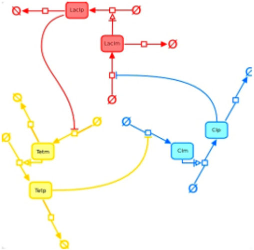
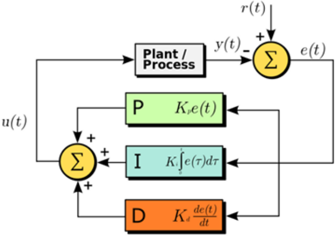
  Causality, probabilistic reasoning, statistics, learning theory, deep neural networks, dynamical systems, information theory, database theory, natural language processing, cognition, consciousness, systems biology, genomics, epidemiology, chemical reaction networks, neuroscience, complex networks, game theory, robotics, quantum computing,...

Network diagrams are everywhere...



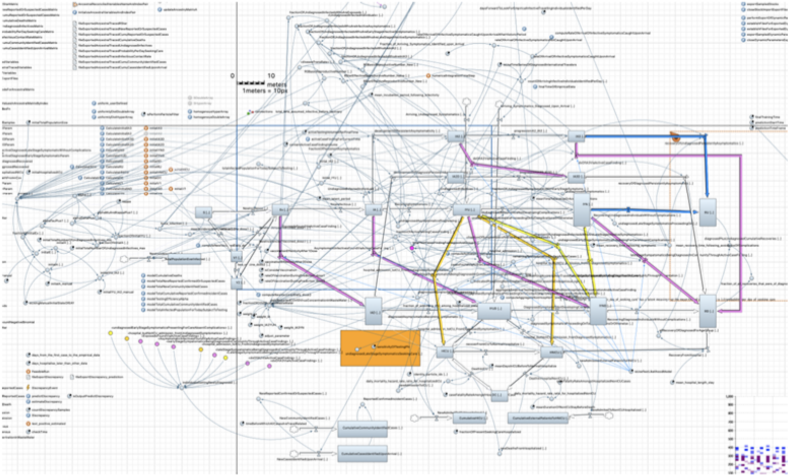






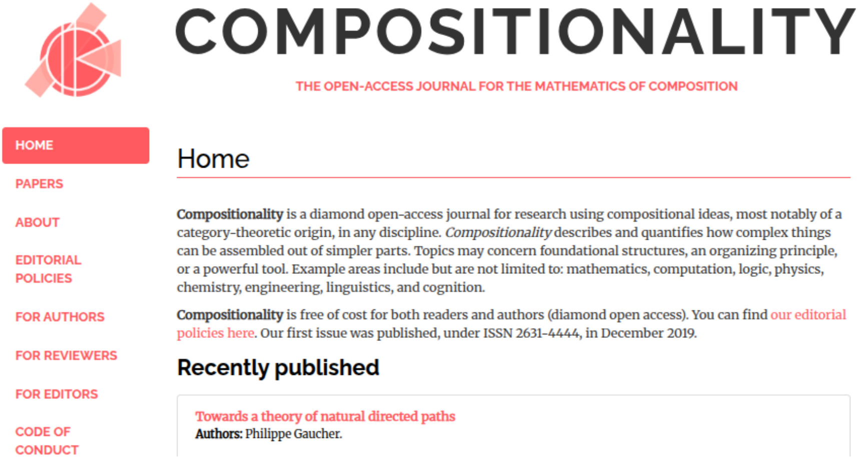
John Baez, Applied Category Theory

...and may be complex



John Baez, Applied Category Theory

A calculus for compositionality is needed


COMPOSITIONALITY

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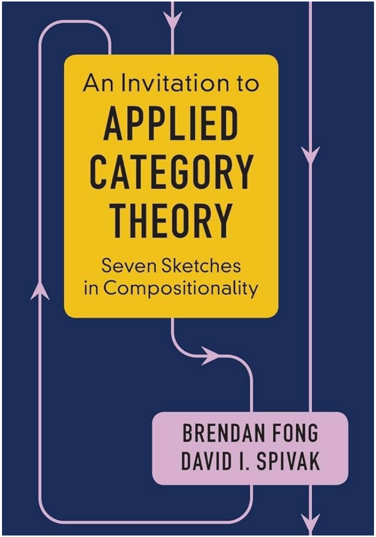
Recently published

Towards a theory of natural directed paths
Authors: Philippe Gaucher.

7 Sketches book



Contents

Double Categories

Seven Sketches needs updating.

Double categories are now everywhere in applied category theory.

They allow for the representation of two kinds of morphism:

- *Set*, the category of sets and functions
- *Rel*, the category of sets and relations

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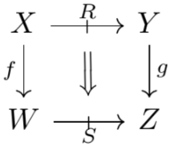
Double categories are now everywhere in applied category theory.

They allow for the representation of two kinds of morphism:

- *Set*, the category of sets and functions
- *Rel*, the category of sets and relations
- $\mathbb{R}el$, the double category, of sets, functions, relations, and cells:



whenever the implication holds, $R(x, y) \implies S(f(x), g(y))$.

- Similarly, $\mathbb{P}rof$, the double category of categories, functors, profunctors, and natural transformations.

Why Double Categories?, Evan Patterson

CatColab: "Software for making models of the world together"

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Our aim is a system in which anyone, from citizen to scientist, can contribute their piece of understanding of the world in a language in which they're comfortable, whether that's unadorned natural language, a diagram or flowchart, or a complicated system of partial differential equations. CatColab aims to enable such an unprecedentedly wide range of collaborators to combine their world models in as close to automatic a way as is consistent with finding truth. Our aim is to let everyone participate fully and transparently in the process of understanding the world.

CatColab

Rooted in double categories

> *In 2024, the authors and other collaborators at Topos Institute began work on CatColab, a web-based application for formal, compositional scientific modeling based directly on Paré's lax double functors into $\mathbb{S}$pan. CatColab interprets a small double category (perhaps with extra structure such as products) as a double (Lawvere) theory, and lax (structure-preserving) functors from such a theory into $\mathbb{S}$pan as models of the theory...*

(Carlson, Patterson, Instances of models of double-categorical theories)

Types of systems treated by categorical systems theory

- Moore machines, Mealy machines, and generalized versions (e.g., nondeterministic, stochastic,...)
- Petri nets, all flavours (colored, hierarchical, generalized stochastic Petri nets with delays/jumps)
- Stock and flow diagrams, system structure diagrams, causal loop diagrams,...

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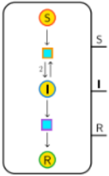
In a double category of systems,

- the *loose* direction concerns composition
- the *tight* direction concerns behaviour

David Jaz Myers, Categorical Systems Theory, DOTS lectures

Composition of systems

Petri net representing a population of susceptible (S) people undergoing infection (I) and recovering (R).



Composition of systems

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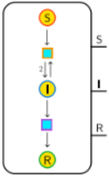
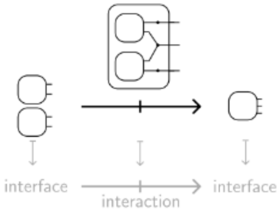


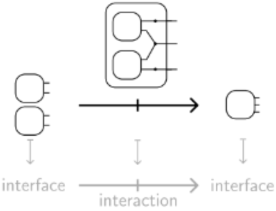
Exposed variables allow for composition via identification, as dictated by wiring diagrams.

Consider how we may see this net to have been composed.

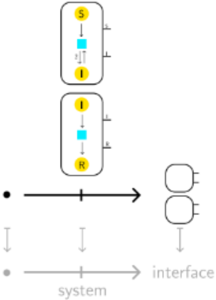
An interaction:



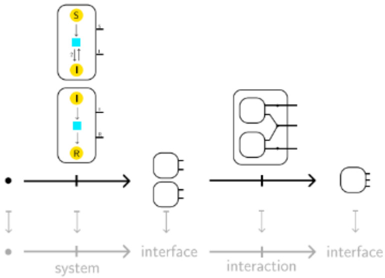
An interaction:



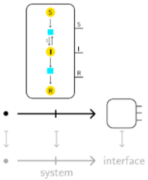
This interaction can now be set to work in wiring together two systems of the right kind:



Here now we see the interaction act on the interfaces of these two systems by identifying the two wires labelled I:

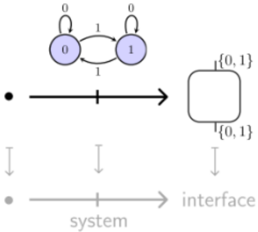


This yields the composite system from above:



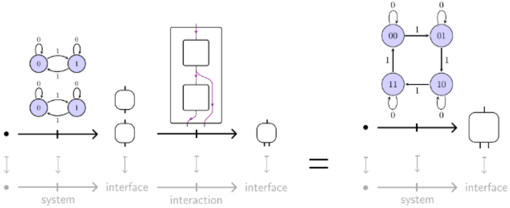
Note that this wiring is *undirected*, connecting undirected ports, to be contrasted with a *directed* form of wiring in the next example of Moore machines, which will connect input and output ports.

Here we we see an automaton which counts the parity of the number of inputs of 1:



When 0 is entered, the internal state remains unchanged, but when 1 is entered, the state flips. The output reveals the internal state.

Again, we may compose a number of systems into a more complex system by following the instructions of a wiring diagram:



Here we see in one image the effect of a wiring diagram acting on a pair of identical parity counting Moore machines to yield a Moore machine counting modulo 4.

Operad

The term '*operadic*' in DOTS points refers to the mathematical concept of an '*operad*', from algebraic topology in the 1970s.

An operad is a collection of n-ary operations (n-to-1) with associative composition from plugging the output of one operation into one of the inputs of another.

This concerns the *loose* direction, now for the *tight* direction.

Behaviour of systems

The *tight* direction is used to understand the behaviour of systems by maps between systems.

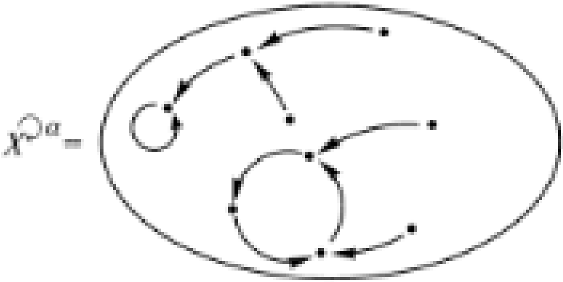
In category theory we understand objects by the arrows into them and the arrows out of them.

E.g., given an object X we can *sample* from it via a map from A and *sort* it via a map to B:

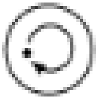
$$A \hookrightarrow X \twoheadrightarrow B$$

$$Sample \hookrightarrow Population \twoheadrightarrow Age$$

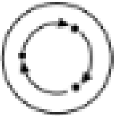
Discrete dynamical system



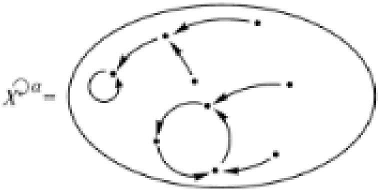
May be sampled by, say,



or

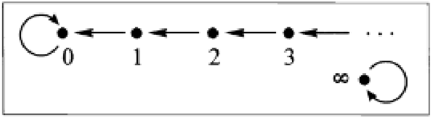


That 'internal view' of X itself may be understood as X being probed by $(\mathbb{N}, succ)$.

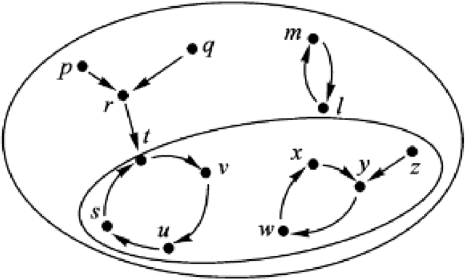


Discrete dynamical system

We might use a map from a system **to**, say,



to separate out a stable subsystem:



DOTS behaviours

DOTS allows us to do this sampling and sorting for more complicated systems, such as stochastic colored Petri nets, generalized Moore machines, etc., to locate kinds of trajectory or kinds of subsystem.

We look to represent assume-guarantee reasoning and (control, ISS)-Lyapunov functions for safe/stable behaviour.

Lyapunov theory may be construed as forming maps from systems to simple stable systems, such as $\frac{dy}{dt} = 0$.

Putting both directions together

Double categories are devised to allow simultaneous reasoning in the *tight* and *loose* directions, here: *behaviour* and *composition*.

In particular, we may be interested in safe and/or stable behaviour of the whole system in terms of these behaviours in its subsystems.

The *laxness* of some functors provides information about the degree of the failure of compositionality with regard, e.g., to *reachability* – *emergent* properties.

This approaach we expect to provide guarantees for the behaviour of composite systems, the goal of the program.