Homotopy type theory and its modal variants

David Corfield

Philosophy, University of Kent, Canterbury, UK

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Philosophy and 'current' mathematics

Philosophy and mathematics have kept close company over the centuries.

- Plato, Aristotle – Euclidean geometry
- Descartes – analytic geometry
- Leibniz – differential calculus
- (Reaction to) Kant – non-Euclidean geometry
- Frege, Peano, Poincaré, Russell, Hilbert – Foundations
- ...

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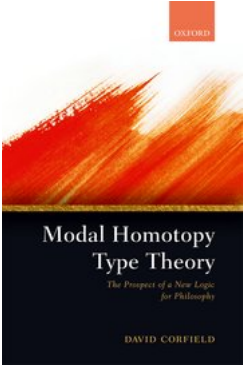
The development of mathematics has never stopped.

Unfortunately what is generally taken as ‘philosophical’ has tended to be restricted to what logicians and set theorists have discovered, rather than the work of mathematicians (e.g., Noether, Mac Lane, Grothendieck, Lurie...)

At last we're seeing signs that there's something to satisfy all parties, a blend of:

- Categorical logic of William Lawvere: Adjointness in foundations.
- (Constructive) intensional type theory of Per Martin-Löf.
- Homotopical mathematics of Lurie and others.

The prospect of a new logic for philosophy



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- $x, y : A$, then $(x =_A y)$ is a proposition.

We ask *whether* two elements are the same, not *how* they are the same.

However, arising from the needs of current geometry and current physics, we find that having solely such a basic shape is a restriction. Beyond sets we need

- *Homotopy types* or *n-groupoids*: points, reversible paths between points, reversible paths between paths, ...

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But from a different perspective, they may appear to be the basic entities, and sets will have to be picked out by some specification.

The internal view

For any two elements of a collection or *type* we can ask whether they are the same or not.

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But then we can iterate:

- From $x =_A y$, and $p, q : x =_A y$, we form $p =_{(x =_A y)} q$.

Drop the 'Uniqueness of Identity proofs'

We need not insist that any two proofs of the sameness of entities are themselves the same.

We reject the axiom that claims this is the case, or in other words we don't insist that the following type is necessarily inhabited:

$$p =_{(x =_A y)} q \,.$$

Hierarchy of *homotopy* types

We have a hierarchy of kinds of types to be treated uniformly:

...	...
2	2-groupoid
1	groupoid
0	set
-1	mere proposition
-2	contractible type

The external view

- Gathering together all sets results in a collection which behaves nicely: a *topos*.
- Gathering together all homotopy types results in a collection which behaves *extremely* nicely: an $(\infty, 1)$-*topos*.

We may tell a justificatory story running at least from Grothendieck to Lurie.

($(\infty, 1)$-toposes are a particularly nice environment for cohomology:
https://ncatlab.org/nlab/show/cohomology)

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- *Homotopy type theory* as (homotopy type) theory is a synthetic theory of homotopy types or ∞-groupoids. It is modelled by spaces (but also by lots of other things).
- *Homotopy type theory* as homotopy (type theory) is the internal language of ∞-toposes. It is a type theory in the logical sense, and may be implemented on a computer.

Homotopy (type theory)

HoTT is a constructive dependent type theory

- Elements of types correspond to proofs of propositions correspond to programs carrying out specified tasks.
- Types may depend on other types, tasks may depend on the way other tasks can be fulfilled: $x : A \vdash B(x) : Type$
- Note a type of types (indeed an infinite series) $Type_i$.
- Type formation: $\mathbf{0}$, $\mathbf{1}$, sum type $A + B$, product type $A \times B$, function type $[A, B]$, ...
- Two important constructions are dependent sum (pair/co-product), $\sum_{x:A} B(x)$ and dependent product (function), $\prod_{x:A} B(x)$.
- Identity types: $A : Type, a, b : A \vdash Id_A(a, b) : Type$

(Homotopy type) theory

Synthetic treatment of abstract spatial structure – homotopy types.

- A structurally invariant theory of ∞-groupoids, structure emerging from iterated identity types.
- Dependent types correspond to spaces sitting over another space.
- Dependent sum corresponds to the *total* space.
- Dependent product corresponds to the type of *sections*
- Physics: principal bundles, gauge-of-gauge transformations.

(Cf. Mike Shulman's Homotopy type theory: the logic of space)

Dependent types

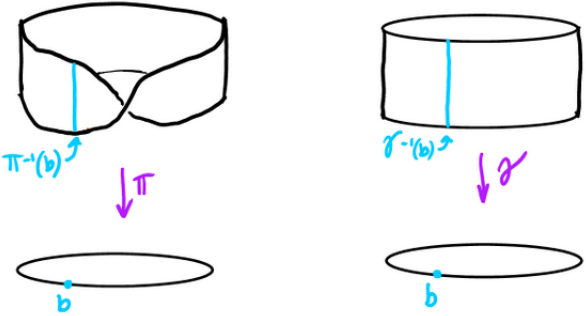
An important part of Martin-Löf type theory is the notion of a *dependent* type, denoted

$$x : A \vdash B(x) : Type.$$

Here the type $B(x)$ *depends* on an element of A, as in

- $Days(m)$ for $m : Month$
- $Players(t)$ for $t : Team$

It's helpful to have in mind the imagery of spaces fibred over other spaces:



Realising *n*-types as spaces, such spaces over other spaces are everywhere in mathematics and physics, fibre bundles and gauge fields.

Two central constructions we can apply to these types are **dependent sum** and **dependent product**: the total space and the sections.

In general we can think of this **dependent sum** as sitting 'fibred' above the base type A, as one might imagine the collection of league players lined up in fibres above their team name.

Likewise an element of the **dependent product** is a choice of a player from each team, such as $Captain(t)$.

Dependent sum	Dependent product
$\sum_{x:A} B(x)$ is the collection of pairs (a, b) with $a : A$ and $b : B(a)$	$\prod_{x:A} B(x)$, is the collection of functions, f, such that $f(a) : B(a)$
When A is a set and $B(x)$ is a constant set B: The product of the sets.	When A is a set and $B(x)$ is a constant set B: The set of functions from A to B.
When A is a proposition and $B(x)$ is a constant proposition, B: The conjunction of A and B.	When A is a proposition and $B(x)$ is a constant proposition, B: The implication $A \to B$.

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When A is a set and $B(x)$ is a varying proposition: Existential quantification.	When A is a set and $B(x)$ is a varying proposition: Universal quantification.

As Lawvere taught us, these are left and right adjoints.

The bottom line is that homotopy type theory for the lower levels of the hierarchy encapsulates:

- Propositional logic
- (Typed) predicate logic
- Structural set theory

Considering the full type theory, the line between logic and mathematics has blurred – homotopy groups of the spheres, group actions,...

Structural inference - univalence

HoTT is a structural theory *par excellence*. Especially when we ensure *univalence*

Univalence Axiom: $Equiv(A, B) \simeq A =_U B$

If A and B are equivalent types, then whatever we can establish about A may be transferred to B.

(Ways around what is non-computational about UA: cubical HoTT, and now Higher Observational TT.)

What's the point?

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Cf. recently announced The Fermat's Last Theorem Project

> *Well before the project is finished, Lean will understand the concepts of automorphic forms and representations, Galois representations, potential automorphy, modularity lifting theorems, the arithmetic of varieties, class field theory, arithmetic duality theorems, Shimura varieties and many other concepts used in modern algebraic number theory.*

Dependent types to present ordinary mathematics

Let k be a field, V a finite-dimensional vector space over k, and f an endomorphism of V. Then define $E(V, k, f)$, the eventual image of f, as the vector space which is the intersection of all $f^n(V)$. Show $f(E) = E$.

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$k : \mathit{Field}, V : \mathit{FinVect}(k), f : \mathit{Endo}(V, k) \vdash g : (f(E) = E)$

More natural than most formalisms, perhaps since natural language appears to use dependent types.

Why HoTT beyond Lean?

Lean relies on UIP (uniqueness of identity proofs), so no higher level types.

By contrast, HoTT allows us:

- Synthetic homotopy theory
- Heuristic guidance in constructing mathematical theories

Synthetic homotopy theory

Anything that proved in HoTT may be interpreted any ∞-topos.

- $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ (proof)
- Blakers-Massey theorem (nLab)
- Much more here

Modality

Philosophers and computer scientists have sought *modal* variants of propositional and predicate logic.

It was natural then to expect a *modal* HoTT.

Modalities are kinds of monad and comonad, operators arising from adjunctions, used in computer science to treat *effects* and *context dependence*.

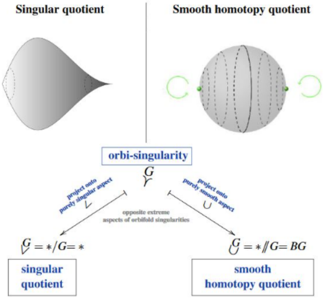
Heuristic guidance

- Hisham Sati, Urs Schreiber:

  Proper Orbifold Cohomology

  download article:

  - pdf (v2)
  - arXiv:2008.01101 (v1)



Abstract. The concept of *orbifolds* should unify differential geometry with equivariant homotopy theory, so that orbifold cohomology should unify differential cohomology with proper equivariant cohomology theory. Despite the prominent role that orbifolds have come to play in mathematics and mathematical physics, especially in string theory, the formulation of a general theory of orbifolds reflecting this unification has remained an open problem. Here we present a natural theory argued to achieve this. We give both a general abstract axiomatization in higher topos theory ("singular cohesion"), as well as concrete models for ordinary as well as for super-geometric and for higher-geometric orbifolds. Our first main result is a fully faithful embedding of the 2-category of orbifolds into a

This article looks to achieve this combination guided by a composition of 'modalities'.

Heuristic guidance

We provide a synthetic treatment of topological and geometric stucture via another of Lawvere's discoveries: an account of cohesion via *modalities*.

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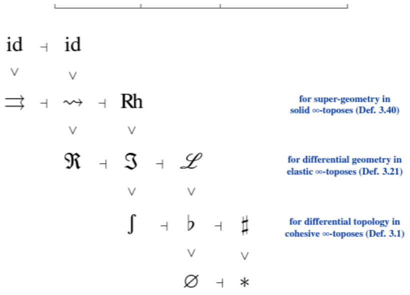
Consider these as systems of adjunctions

1. The geometric aspect of orbifold theory. In order to formulate, internal to suitable ∞-toposes, the **(a)** differential topology, **(b)** differential geometry, and **(c)** super-geometry of orbifolds (hence of manifolds, super-manifolds, super-orbifolds, etc.) in their smooth guise as étale ∞-stacks (18), we consider a corresponding progression of adjoint modalities (20), which starts out in the form of the "axiomatic cohesion" of [La07], on to a second layer that contains a "de Rham shape" operation $\Im$ as considered in [Si96] [ST97], and then to a third layer which captures super-geometry in a new axiomatic way.



Heuristic guidance

To this we add modalities for singularities

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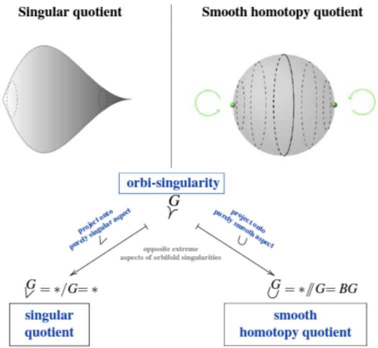
2. The singular aspect of orbifold theory. Envision the picture of an orbifold singularity $\curlyvee$ and a mathematical magnifying glass held over the singular point. Under this magnification, one sees resolved the singular point as a *fuzzy fattened point*, to be denoted $\overset{G}{\curlyvee}$. Removing the magnifying glass, what one sees with the bare eye depends on how one squints:

(i) The physicists (see, e.g., [BL99, §1.3]) and the classical geometers (see, e.g., [IKZ10][Wat15]) say that they see an actual singular point, such as the tip of a cone $\vee$. This is the *plain quotient* $\overset{G}{\vee} := */G = *$, a point.

(ii) The higher geometers (see, e.g., [MP97] [CPRST14]) say that they see the smooth G-action around that point, hence a smooth stacky geometry $\cup$. This is the *homotopy quotient* $\overset{G}{\cup} := */\!\!/G = BG = K(G,1)$ (16).



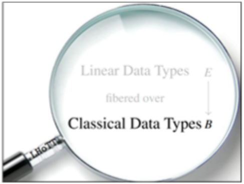
Linear HoTT

For parameterized spectra/generalized twisted cohomology we need *linear* homotopy type theory, using a further modality related to *linear* logic.

Some fascinating treatment of quantum computation and quantum physics more generally is expressible in this language.

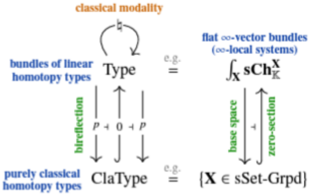
Cf. Sati and Schreiber, The Quantum Monadology

Concretely, LHoTT enhances the syntactic rules of classical HoTT by further type formations which serve to exhibit every (homotopy) type E of the language as secretly consisting of an underlying classical (intuitionistic) base type $B \equiv \natural E$ equipped, in a precise sense, with a microscopic (infinitesimal) halo of linear/quantum data. As such, LHoTT may neatly be thought of as the formal logical expression of a microscope that resolves quantum aspects on structures that macroscopically appear classical. This way LHoTT embeds quantum logic into classical logic in a way reminiscent of Bohr's famous dictum[2] that all quantum phenomena must be expressible in classical language.



Quantum halos. Formally this is achieved by adjoining to classical HoTT an *ambidextrous* modal operator ♮ [RFL21] (an *infinitesimal cohesive modality* [Sch13, Def. 3.4.12, Prop. 4.1.9]), whose modal types (Lit. 1.14) are the *purely classical* (ordinary) homotopy types, embedded *bi-reflectively* (157) among all data types (see §2.1):

The presence of the ♮-modality exhibits general types E : Type as microscopic/infinitesimal *halos* around their underlying purely classical type $\natural E$: ClaType. It is a profound fact (146) of ∞-topos theory that models for such *infinitesimal cohesion* (see Lit. 1.21) are provided by parameterized module spectra, in particular by flat ∞-vector bundles ("∞-local systems", see [SS23-EoS]) which, in their 0-sector (Rem. 1.22), accommodate quantum circuit semantics (cf. §2.4) in indexed sets of vector spaces (cf. §2.1) such as known from the Proto-Quipper quantum language (Lit. 1.5).



(1)

Motivic Yoga. LHoTT witnesses these quantum halos as *linear types* (24) equipped with a closed tensor product $\otimes$ and compatible base change operations which satisfy the rules of Grothendieck's "motivic yoga of six operations" in Wirthmüller style (Def. 2.18, cf. [Ri22a, §2.4][SS23-EoS, §3.3]). It is this "motivic" structure from which the structure of quantum physics derives, as originally observed in [Sch14a] and here brought out in §2.1.

Linear/Quantum Data Types

Characteristic Property	**1.** Their cartesian product blends into the co-product:	**2.** A tensor product appears & distributes over direct sum	**3.** A linear function type appears adjoint to tensor
Symbol	$\bigoplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for W : ClaType$^{\mathrm{fin}}$)	cart. product $\prod_W \mathcal{H}_w \simeq \bigoplus \mathcal{H}_w \simeq \coprod_W \mathcal{H}_w$ co-product; direct sum	$\mathscr{V} \otimes \big(\bigoplus_{w:W} \mathcal{H}_w \big) \simeq \bigoplus_{w:W} (\mathscr{V} \otimes \mathcal{H}_w)$	$(\mathscr{V} \otimes \mathcal{H}) \multimap \mathscr{K} \simeq \mathscr{V} \multimap (\mathcal{H} \multimap \mathscr{K})$
AlgTop Jargon	biproduct, stability, ambidexterity	Frobenius reciprocity	mapping spectrum
		Grothendieck's Motivic Yoga of 6 oper. (Wirthmüller form)	
Linear Logic	additive disjunction	multiplicative conjunction	linear implication
Physics Meaning	parallel quantum systems	compound quantum systems	qRAM systems

Modal HoTT

1. HoTT: synthetic language to describe structure

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2. *Cohesive* HoTT: synthetic language for differential and equivariant structure, differential cohomology of (higher) gauge theory.
3. *Linear* HoTT: synthetic language for 'linear' structure (infinitesimal, tangent, abelian, stable, etc.), quantum information

((1) Shulman; (2) Sati-Schreiber; (3) Myers-Sati-Schreiber)

Philosophical leads

Philosophers of mathematics should already have been persuaded by the success of category theory, and by now be ready to hear about the successes of higher category theory.

Although HoTT is very young, and modal HoTT even younger, at last we have an opportunity to bring *real* mathematics to the attention of philosophy, and not just to the tiny domain of philosophy of mathematics.

Philosophical leads

- Logicism, constructivism, structuralism, formalism
- Computational trinitarianism
- Husserl, ...
- Metaphysics: Types, identity, modal types...
- Natural language
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But never forget the place of category theory here.

- HoTT and $(\infty, 1)$-toposes go hand in hand.
- Modal HoTT is about functors between $(\infty, 1)$-toposes

Physics with Modal HoTT

Schreiber

Introduction to Higher Supergeometry

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- Urs Schreiber

  Introduction to Higher Supergeometry

  lecture at *Higher Structures in M-Theory*

  Durham Symposium

  August 2018

Abstract. Due to the existence of a) gauge fields and b) fermion fields, the *geometry of physics* is higher supergeometry, i.e. super-geometric homotopy theory. This is made precise via Grothendieck's functorial geometry implemented in higher topos theory. We give an introduction to the higher topos of higher superspaces and how it accomodates higher Lie theory of super L-∞ algebras. We close by indicating how geometric homotopy theory reveals that the superpoint emerges "from nothing", and that core structure of M-theory emerges out of the superpoint via a sequence of invariant universal higher central extensions. This will be discussed in more detail in other talks in the meeting.

Additional reading

Mike Shulman

- Homotopy type theory: the logic of space, arXiv:1703.03007
- Homotopy Type Theory: A synthetic approach to higher equalities, arXiv:1601.05035

Modal developments

- Brouwer's fixed-point theorem in real-cohesive homotopy type theory, arXiv:1509.07584
- Cartan Geometry in Modal Homotopy Type Theory, arXiv:1806.05966
- Sketch given for Noether's theorem
- Cohesive Covering Theory (Extended Abstract), https://hott-uf.github.io/2018/abstracts/HoTTUF18_paper_15.pdf