Triadic Sweedler Measuring in the Arrow Topos

A calculation of witness-sensitive universal capabilities

Research study

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Abstract

We investigate how much information is already present in a Sweedler measuring triple before choosing any algebra or coalgebra. The ambient category is the arrow topos $\mathbf{Arr} = \mathbf{Set}^{\mathbf{2}}$, whose objects $p : W \to S$ may be read as witnesses over underlying states. The first calculation shows that the Sweedler hom for a pointwise unary tick monad and identity residual is precisely the pointwise stream comonad, even when tested against non-pointwise capability comonads. For any operation parameter $a : A_0 \to A_1$, we calculate the universal capability explicitly: underlying-state trees indexed by A_1^*, with independent witness slots indexed by A_0^*. The ordinary unary stream comonad and the non-pointwise binary-tree comonad with witnesses along a distinguished branch become special cases of one theorem. We give a direct terminal-coalgebra proof and separate generic witness compatibility from dynamics supplied by a coalgebra. The results are explicit specialisations of the established free-monad/cofree-comonad principle, rather than a new general Sweedler theorem.

1 Measuring before realisation

Our point of departure is a heterogeneous three-place relation

$$\mathrm{Meas}_{\mathcal{E}}(T, D; R), \tag{1}$$

where T and R are monads (demand and residual) and D is a comonad (capability) on a suitable cartesian closed category $\mathcal{E}$. A measuring consists of natural maps

$$\psi_{X,Y} : TX \times DY \longrightarrow R(X \times Y), \tag{2}$$

subject to unit and multiplication/counit and comultiplication compatibility. Explicitly, the unit condition is

$$\psi_{X,Y}(\eta^T_X \times 1) = \eta^R_{X\times Y}(1_X \times \varepsilon^D_Y), \tag{3}$$

where product reassociations are suppressed. The multiplication condition states that executing two successive T-layers with D's comultiplication agrees with executing their multiplication in T and combining the two resulting R-layers with μ^R. In the examples below, T is free and $R = \mathrm{Id}$, so these axioms become the ordinary iteration law for its generating operations. The identity-residual interaction laws originate with Katsumata–Rivas–Uustalu [2]; the residual measuring formulation is developed by McDermott–Rivas–Uustalu [1], as an instance of the duoidal Sweedler theory of López Franco–Vasilakopoulou [3].

When the relevant representing objects exist, the same measuring relation has three presentations:

$$\begin{aligned} \mathrm{Meas}_{\mathcal{E}}(T, D; R) &\cong \mathrm{Mon}(D \triangleright T, R) \\ &\cong \mathrm{Mon}(T, [D, R]) \\ &\cong \mathrm{Comon}(D, \{T, R\}_{\mathcal{E}}). \end{aligned} \tag{4}$$

The symbol $D \triangleright T$ denotes the *Sweedler copower*, not endofunctor composition or cartesian product. Formula (4) expresses three representations of one heterogeneous relation, in the manner of heteromorphisms for an ordinary adjunction. It is not a cyclic permutation of the three participants. The existence hypotheses are most safely read in the accessible-endofunctor duoidal setting of [1]; for the instances below we construct the required representing comonads directly.

Our question is precise: after changing $\mathcal{E}$ from **Set** to $\mathbf{Arr} = \mathbf{Set}^{\mathbf{2}}$, can a measuring $D \to \{T, R\}_{\mathbf{Arr}}$ already express meaningful transport of witnesses, before any D-coalgebra has been selected?

2 The arrow topos and its internal exponentials

Write $\mathbf{2} = (0 \to 1)$ and $\mathbf{Arr} = \mathbf{Set}^{\mathbf{2}}$. Its objects and morphisms are maps and commutative squares, respectively. We write an object as

$$p : W \longrightarrow S,$$

where W is a set of witness-bearing states, S the set of underlying states, and p forgets the witness. The same underlying state may have several witnesses or none. Monomorphisms p describe the special case of predicates. The interpretation is convenient but not part of the categorical definition.

The cartesian products and natural-numbers object are pointwise. Exponentiation is not generally pointwise. This difference is central to the calculation.

Lemma 2.1 (Exponential of one arrow by another)**.** *For $p : W \to S$ and $a : A_0 \to A_1$ in* **Arr***, the internal exponential p^a is canonically isomorphic to the arrow*

$$\left(W^{A_0} \times_{S^{A_0}} S^{A_1} \longrightarrow S^{A_1} \right), \tag{5}$$

where the pullback is taken over $p^{A_0} : W^{A_0} \to S^{A_0}$ and precomposition $S^a : S^{A_1} \to S^{A_0}$.

Proof. At 1, the representable is $y(1)$; at 0, $y(0) = 1$, so $y(0) \times a = a$. At 1, the exponential contains the maps from the representable at 1 times a to p, giving S^{A_1}. At 0, a map of arrows $a \to p$ is a pair $(u : A_0 \to W, v : A_1 \to S)$ satisfying $pu = va$, precisely the stated pullback. Restriction from 0 to 1 remembers v. Alternatively, checking $\mathbf{Arr}(q \times a, p) \cong \mathbf{Arr}(q, p^a)$ for arbitrary q gives the same calculation. □

For the distinguished inclusion

$$a : 1 = \{0\} \hookrightarrow 2 = \{0, 1\},$$

we obtain the particularly simple formula

$$p^a \cong \left(W \times S \xrightarrow{p \times 1_S} S \times S \right). \tag{6}$$

Thus an a-indexed pair of underlying states has a witness for its first member, but need not have a witness for the second. This elementary non-pointwise exponential is the source of the branching example.

3 The unary control: no new universal stream capability

Let $\underline{\mathbb{N}} = (\mathbb{N} \xrightarrow{1} \mathbb{N})$ be the natural-numbers object of **Arr** and put

$$T = \underline{\mathbb{N}} \times -, \qquad R = \mathrm{Id}_{\mathbf{Arr}}.$$

The monad T is free on one unary operation: a computation carries a number of ticks. For an arbitrary accessible comonad D on **Arr**, not necessarily a pointwise lift, there is a complete calculation of its measurings.

Proposition 3.1 (Generic tick)**.** *There is a natural bijection*

$$\mathrm{Meas}_{\mathbf{Arr}}(\underline{\mathbb{N}} \times -, D; \mathrm{Id}) \cong \mathrm{Nat}(D, \mathrm{Id}). \tag{7}$$

Proof. Writing $T(X) = \coprod_{n\in\mathbb{N}} X$, a natural transformation $T(X) \times DY \to X \times Y$ must be the identity on the X component (naturality in X), and is therefore a family $\alpha_{n,Y} : DY \to Y$, natural in Y. The measuring unit law gives $\alpha_0 = \varepsilon^D$. Write $\alpha_{m,Y} : DY \to Y$ for the response after m ticks. The multiplication law states literally

$$\alpha_{m+n,Y} = \alpha_{n,Y} \circ \alpha_{m,DY} \circ \delta^D_Y. \tag{8}$$

Put $a = \alpha_1$ and $u_Y = a_{DY} \circ \delta^D_Y : DY \to DY$. Naturality of a at δ_Y and coassociativity give

$$\delta_Y \circ u_Y = a_{DDY} \circ D(\delta_Y) \circ \delta_Y = a_{DDY} \circ \delta_{DY} \circ \delta_Y = u_{DY} \circ \delta_Y.$$

By induction $\alpha_{m,DY} \circ \delta_Y = u^m_Y$; here the induction uses the displayed identity and naturality. Thus (8) reduces to

$$\alpha_{m+n,Y} = \alpha_{n,Y} \circ u^m_Y, \qquad \alpha_{n,Y} = \varepsilon^D_Y \circ u^n_Y. \tag{9}$$

Conversely, these iterates satisfy the measuring axioms by the coKleisli identities. Thus ψ and a determine each other. □

Remark 3.2. The notation in (9) uses the ordinary endomorphism $u = a_D\delta^D$ on D. The correct counit calculation uses naturality of a at ε^D_Y and the *other* counit identity:

$$\varepsilon^D_Y \circ u_Y = \varepsilon^D_Y\, a_{DY}\, \delta^D_Y = a_Y\, D(\varepsilon^D_Y)\, \delta^D_Y = a_Y.$$

The coKleisli formulation is often the cleanest way to verify the iteration identity.

Let $G(X) = X^{\underline{\mathbb{N}}}$ be the stream comonad on **Arr**, with counit $s \mapsto s_0$ and comultiplication $s \mapsto (n \mapsto (m \mapsto s_{n+m}))$. Since $\underline{\mathbb{N}}$ is the constant arrow $\mathbb{N} \to \mathbb{N}$, its exponential is pointwise:

$$G(p) = (W^{\mathbb{N}} \xrightarrow{p^{\mathbb{N}}} S^{\mathbb{N}}).$$

It is the cofree comonad on the identity endofunctor: comonad maps $D \to G$ correspond to natural transformations $D \to \mathrm{Id}$.

Theorem 3.3 (Unary comparison theorem)**.** *The Sweedler hom for unary tick demand and identity residual in the arrow topos is the pointwise stream comonad:*

$$\boxed{\{\underline{\mathbb{N}} \times -, \mathrm{Id}\}_{\mathbf{Arr}} \cong (-)^{\underline{\mathbb{N}}}.} \tag{10}$$

The pointwise lift of the Set stream comonad carries the pointwise universal measuring; transposing this measuring defines a canonical comonad morphism from that lift to the arrow-topos Sweedler hom. This morphism is an isomorphism.

Proof. By Proposition 3.1 and the cofree-comonad universal property, both sides of (10) represent the functor $D \mapsto \mathrm{Nat}(D, \mathrm{Id})$ on accessible comonads. The explicit description follows from pointwise products in **Arr**. □

For $a : D \to \mathrm{Id}$, the associated comonad map $\phi : D \to G$ is explicitly

$$\phi_p(d)(n) = \varepsilon_p^D(u_p^n(d)), \qquad u_p = a_{Dp}\delta_p^D. \tag{11}$$

Both endpoint maps are defined at once by this arrow-category formula and automatically commute with p.

A non-pointwise capability still factors through streams

Let $i : \mathbf{Set} \to \mathbf{Arr}$ send W to $1_W : W \to W$. The adjunction $i \dashv \mathrm{dom}$ allows the stream comonad on **Set** to induce a comonad

$$D = i \circ (-)^{\mathbb{N}} \circ \mathrm{dom}, \qquad D(p) = (W^{\mathbb{N}} \xrightarrow{1} W^{\mathbb{N}}).$$

This D is not a pointwise lifting of a Set endofunctor, because its codomain component depends on the domain W of p. For every $k \in \mathbb{N}$, evaluation at k gives $a^{(k)} : D \to \mathrm{Id}_{\mathbf{Arr}}$, with components $s \mapsto s_k$ and $s \mapsto p(s_k)$. Formula (11) becomes

$$\phi^{(k)}_{p,0}(s)(n) = s_{nk}, \qquad \phi^{(k)}_{p,1}(s)(n) = p(s_{nk}). \tag{12}$$

Thus $k = 0$ produces constant streams, $k = 1$ the original stream, and $k = 2$ every second state. Non-pointwise capabilities exist, but the *universal* capability in this example remains pointwise.

4 Arrow-valued operation parameters

Now let $a : A_0 \to A_1$ be any object of **Arr** and write a^* for its free internal monoid (the object of finite words), computed pointwise:

$$a^* = (A_0^* \xrightarrow{a^*} A_1^*).$$

The internal writer monad $T_a = a^* \times -$ is free on the unary-operation endofunctor $P_a = a \times -$. In particular, its permitted operation labels at the witness level and at the underlying level need not coincide.

Proposition 4.1 (Parametrised generic operation)**.** *For any accessible comonad D on* **Arr** *there are natural bijections*

$$\mathrm{Meas}_{\mathbf{Arr}}(a^* \times -, D; \mathrm{Id}) \cong \mathrm{Nat}(a \times D, \mathrm{Id}) \cong \mathrm{Nat}(D, (-)^a). \tag{13}$$

Consequently, if $\mathrm{Cofree}(F)$ *denotes the cofree comonad on an accessible endofunctor F, then*

$$\boxed{\{a^* \times -, \mathrm{Id}\}_{\mathbf{Arr}} \cong \mathrm{Cofree}((-)^a).} \tag{14}$$

Proof. This is the standard free-monad case of the Sweedler calculation in [1], specialised to the arrow topos. The free-monad measuring equations say that a measuring out of T_a is uniquely specified by its action on one generating a-labelled operation. By

naturality in the free return-value variable, the resulting generator-level operation is precisely $a \times DY \to Y$, naturally in Y. The first bijection is this restriction/extension correspondence. The second is cartesian closure in **Arr**. The defining universal property of the cofree comonad then identifies the representing object on the right of (14). The following theorem calculates this cofree object for every set map a, independently of an abstract existence theorem. □

For $a : 1 \hookrightarrow 2$, a natural transformation $D \to (-)^a$ at $p : W \to S$ is, by (6), a square

$$\begin{array}{ccc} D(p)_0 & \longrightarrow & W \times S \\ \downarrow & & \downarrow{\scriptstyle p \times 1} \\ D(p)_1 & \longrightarrow & S \times S \end{array} \tag{15}$$

Its two components supply two successors of an underlying state and, at the witness level, a witness for the first successor. Commutativity requires that the chosen witness maps to that underlying successor. This is a *generic* compatibility: no particular coalgebra or state evolution has been supplied yet.

5 The general witness-tree comonad

Write A_i^* for finite words, including the empty word ϵ, and write $a^* : A_0^* \to A_1^*$ for letterwise application of a. For any arrow $p : W \to S$ define

$$H_a(p) = \left(S^{A_1^*} \times_{S^{A_0^*}} W^{A_0^*} \longrightarrow S^{A_1^*} \right), \tag{16}$$

where the first pullback leg is restriction along a^* and the other is postcomposition with p. Its elements above a tree $t : A_1^* \to S$ are exactly families $w : A_0^* \to W$ with

$$p(w(u)) = t(a^*(u)) \quad (u \in A_0^*). \tag{17}$$

Each word $u \in A_0^*$ supplies an *independent witness slot*. In particular, if $a^*(u) = a^*(v)$ with $u \neq v$, the two witnesses $w(u), w(v)$ need not coincide; they merely have the same image under p.

Theorem 5.1 (General arrow-parameter theorem)**.** *For every set map $a : A_0 \to A_1$, the endofunctor H_a of (16) has a canonical comonad structure and is the cofree comonad on $F_a = (-)^a$. Consequently,*

$$\boxed{\{a^* \times -, \mathrm{Id}\}_{\mathbf{Arr}} \cong H_a.} \tag{18}$$

The isomorphism is natural in the output arrow p.

Proof. By Proposition 4.1, it suffices to show that $H_a(p)$ is the terminal coalgebra for $q \mapsto p \times q^a$, for each $p : W \to S$. Let $q : Q_0 \to Q_1$. By Lemma 2.1, a coalgebra $q \to p \times q^a$ consists of a bottom-endpoint map

$$b : Q_1 \to S \times Q_1^{A_1}$$

and a top-endpoint map

$$c : Q_0 \to W \times Q_0^{A_0} \times_{Q_1^{A_0}} Q_1^{A_1}.$$

Write $b(z) = (o(z), (r_y(z))_{y \in A_1})$, and let the top map specify $\omega(v) \in W$, successors $r^0_x(v) \in Q_0$ for $x \in A_0$, and the underlying successors of $q(v)$. Being a map of arrows is equivalent to

$$p\omega = oq, \qquad qr^0_x = r_{a(x)}q \quad (x \in A_0),$$

together with the requirement that the unselected bottom successors are the ones prescribed by bq.

There is a unique A_1-branching output tree t_z defined by

$$t_z(\epsilon) = o(z), \qquad t_z(yv) = t_{r_y(z)}(v).$$

Similarly there is a unique witness-labelled A_0-branching tree

$$w_v(\epsilon) = \omega(v), \qquad w_v(xu) = w_{r^0_x(v)}(u).$$

Induction on the word u using the compatibility equations gives $pw_v(u) = t_{q(v)}(a^*(u))$. Hence $v \mapsto (t_{q(v)}, w_v)$ and $z \mapsto t_z$ constitute a coalgebra morphism $q \to H_a(p)$. It is unique because the root outputs and all successors recursively determine every label. This proves terminality.

For explicit structure maps, define rooted subtrees and witness subfamilies by

$$t_v(z) = t(vz) \quad (v, z \in A_1^*), \qquad w_u(z) = w(uz) \quad (u, z \in A_0^*).$$

Equation (17) implies $pw_u(z) = t_{a^*(u)}(a^*(z))$, so $(t_{a^*(u)}, w_u)$ is again a top element of $H_a(p)$. The counit $H_a(p) \to p$ evaluates the root: $(t, w) \mapsto w(\epsilon)$ above $t \mapsto t(\epsilon)$. The comultiplication $H_a(p) \to H_aH_a(p)$ sends

$$t \longmapsto (v \longmapsto t_v), \qquad (t, w) \longmapsto (v \longmapsto t_v,\ u \longmapsto (t_{a^*(u)}, w_u)).$$

(The latter pair is understood as the top element of $H_aH_a(p)$.) The two counit laws are deletion of the empty prefix, and the coassociativity law follows from $(t_v)_z = t_{vz}$ and $(w_u)_z = w_{uz}$. These maps are natural in p and yield the claimed cofree comonad. □

Corollary 5.2 (Unary ticks)**.** *For $a : 1 \xrightarrow{1} 1$, formula (16) reduces to $H_a(p) = (W^{\mathbb{N}} \xrightarrow{p^{\mathbb{N}}} S^{\mathbb{N}})$. Thus Theorem 3.3 is also a special case of Theorem 5.1.*

6 The distinguished branch and non-pointwiseness

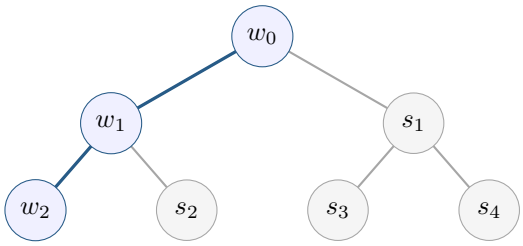
For $a : 1 \hookrightarrow 2$, with the sole domain letter sent to 0, $A_1^* = 2^*$ and $A_0^* = \mathbb{N}$. Formula (16) becomes

$$H_a(p) = \left(S^{2^*} \times_{S^{\mathbb{N}}} W^{\mathbb{N}} \longrightarrow S^{2^*}\right), \tag{19}$$

where the pullback restricts trees to the all-zero branch. Thus $t : 2^* \to S$ has a witness w_n for each $t(0^n)$, without imposing witness obligations on the other branches.

Corollary 6.1 (A non-pointwise universal capability)**.** *For $a : 1 \hookrightarrow 2$, H_a is not isomorphic to a pointwise lifting of any Set endofunctor.*

Proof. Consider arrows $p_S : 1 \to S$ whose domain is the singleton, with any choice of its image. The domain component of $H_a(p_S)$ is canonically $S^{2^* \setminus 0^*}$: values on the zero branch are fixed by the unique witness, whereas all other tree positions are free. For finite sets S of cardinality at least two, its cardinality is $|S|^{\aleph_0}$, and for $|S| = 1$ it is 1; in particular these cardinalities differ as S varies. A pointwise lift $K^{\to}$ would have domain $K(1)$ for every p_S, independent of S. Hence no natural isomorphism $H_a \cong K^{\to}$ exists. □



Blue: witnessed zero branch; grey: underlying states only.

Figure 1: The binary-tree specialisation of the general theorem.

If p is monic and is identified with a subset $W \subseteq S$, then $H_a(p)$ describes precisely those S-labelled binary trees whose all-zero branch remains in W. This is a *must on a designated branch, unconstrained elsewhere* pattern. It is a property of the capability's admissible behaviours, not a claim that a particular dynamical system exhibits them.

7 Interpretation, scope and further questions

The preceding theorem handles injective and non-injective maps uniformly. Non-injectivity creates multiple witness slots over one underlying-state node, rather than forcing their identification. If some $y \in A_1$ lies outside the image of a, the branch beginning with y is unconstrained by a witness at that step; more generally witness obligations occur at exactly the positions in the image of $a^* : A_0^* \to A_1^*$.

The explanatory order matters. The parameter a *specifies* the witness pattern; the exponential $(-)^a$ and then the cofree construction *propagate* this pattern to arbitrarily long branching behaviour. The pattern is not discovered by the measuring relation merely because that relation precedes coalgebras. A comonad map $D \to H_a$ interprets a generic capability; a coalgebra $p \to H_a(p)$ selects actual successors and hence realises a system subject to the prescribed witness obligations.

In a predicate interpretation, $p : W \hookrightarrow S$ is a subobject and the formula gives must-style constraints on selected branches. This is different from using an arrow to encode the dependence of directions on positions in a dependent Mealy machine: the underlying category may again be an arrow category, but the semantic roles of its two components are not identified without an additional comparison construction.

The free-monad/cofree-comonad correspondence is standard [2, 1], and the general duoidal setting comes from [3]. Our explicit formula specialises these established constructions to arbitrary arrow-valued unary-operation parameters; it is not a new general theorem of Sweedler theory.

The central remaining calculation is a *non-identity residual* R. Residual effects might permit a computation to fail, redirect, or weaken an obligation to supply witnesses. A second question is how far formula (16) extends to diagram toposes $\mathbf{Set}^J$, whose dependency patterns are no longer controlled by a single arrow. Both questions should be addressed at the triadic measuring level before imposing particular coalgebras or algebras.

8 Conclusion

The arrow category supports genuine witness-dependent universal capabilities once the operation parameter itself carries a nontrivial arrow. For every $a : A_0 \to A_1$, the Sweedler hom for free a-labelled unary operations and identity residual consists of A_1-branching trees together with independent witness slots indexed by A_0-words. The stream and witnessed-zero-branch examples are special cases. The triadic layer propagates the witness obligations built into a into a universal capability; coalgebras subsequently realise particular dynamics. The next potentially substantive question concerns residual effects rather than another free/cofree calculation.

References

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