# Measuring Interactions, Realisations, and Interaction Profiles

Heterogeneous Measuring, Sweedler Representations, and Capability–Residual Exchange

Research study

4 November 2026; revised 5 November 2026

**Abstract**

We take measuring as a heterogeneous three-place relation between a demand monad, a capability comonad, and a residual monad. The Sweedler copower $D \triangleright T$, cotensor $[D, R]$, and measuring comonoid $\{T, R\}$ are its three representations, where the copower exists. This perspective distinguishes ordinary transposition in a two-variable adjunction from the additional capability–residual exchange supplied by Sweedler duality. We then recover the standard bicategory of measuring interactions from the enrichment of monads over comonads, with the ordinary universal-runner category as a reflective terminal fibre of realised interactions. We define effectful state comparisons through their action on delegated algebras, then use writer monads to identify runners with action-category cocycles and comparisons with natural transformations. For this writer subcase we prove horizontal composition and interchange, while retaining the general horizontal-coherence question. We then separate compatibility from adequacy. Delegated algebras yield a three-place evaluation of demands, realised states and residual standards. This evaluation obeys the weakest-precondition laws. It pulls standards back along composite runners, which gives a Hoare-style sequencing rule with a substructural reading, and it characterises effectful comparisons as simulations. Read as a Lambek-style residuation law, the measuring adjunction also organises tasks passed along sequences of small capabilities into languages of handing-on. It supports a triadic concept analysis of adequate interaction at the level of instances rather than theories. Garner's costructure–cosemantics theorem describes the reflexive locus on which capability–residual exchange is invertible. Finally, we pose interaction-profile detection questions, including whether reflexive residuals form a separating class of probes, and indicate an indexed heterogeneous formulation as a possible entry to dependent Sweedler theory. Established facts, conditional comparisons, and open problems are distinguished throughout.

## 1 Measuring as a heterogeneous three-place relation

Let $\mathcal{D} = [\mathsf{Set}, \mathsf{Set}]_{\mathrm{acc}}$ carry composition $\circ$ and the symmetric convolution product $\star$, with duoidal interchange

$$\zeta : (F \circ G) \star (H \circ K) \longrightarrow (F \star H) \circ (G \star K), \tag{1}$$

so that $\star$ is oplax monoidal for $\circ$, as in [1]. Consequently comonads are closed under $\star$ (oplax monoidal functors preserve comonoids), whereas the convolution of two monads need not be a monad. Write $\mathsf{Mon}$ and $\mathsf{Comon}$ for the categories of accessible monads and comonads in this duoidal setting. The starting datum is a family of sets

$$\mathsf{Meas}(T, D; R), \qquad T, R \in \mathsf{Mon}, \quad D \in \mathsf{Comon}, \tag{2}$$

functorial contravariantly in $T$ and $D$, and covariantly in $R$. A measuring is equivalently a compatible natural transformation $\lambda : D \circ T \Rightarrow R \circ D$, with the unit and multiplication

coherence required by the measuring laws [1]. We regard (2) as conceptually analogous to the heteromorphism sets $\mathsf{Het}(A, B)$ of an ordinary adjunction. In the latter case the same heteromorphisms are represented by $\mathcal{B}(FA, B)$ and $\mathcal{A}(A, UB)$; here there are three ways to represent the measuring relation.

$$\boxed{\begin{aligned}\mathsf{Meas}(T, D; R) &\cong \mathsf{Mon}(D \triangleright T, R)\\ &\cong \mathsf{Mon}(T, [D, R])\\ &\cong \mathsf{Comon}(D, \{T, R\}).\end{aligned}} \tag{3}$$

The first line is asserted only when the *Sweedler copower* $D \triangleright T$ exists in the chosen category of monads. The other two representations belong to the established Sweedler enrichment and cotensor theory in our accessible setting [1]. All three bijections must be natural with the indicated variances; they are not merely equalities of cardinalities.

**The roles of the three representations.** The copower $D \triangleright T$ is universal among residual monads $R$ admitting a measuring $(T, D; R)$; the cotensor $[D, R]$ is universal among demand monads $T$; and $\{T, R\}$ is universal among capability comonads $D$. This is a two-variable adjunction, not a cyclic symmetry of three interchangeable objects. In particular, $D \triangleright T$ is neither the comonoid tensor $D \star E$ nor the endofunctor composite $D \circ T$: $\triangleright$ denotes the representing *monad*, while $\star$ and $\circ$ are ambient monoidal products. Copowers may cease to exist after restricting to polynomial monads even if the relevant representability survives in a larger category. We shall retain $\mathsf{Meas}(T, D; R)$ when a copower is unavailable.

**A heterogeneous viewpoint.** The naturality of (2) under monad and comonad morphisms, together with the three representing universal properties, can be taken as a compact interface to Sweedler theory. It does not supplant the duoidal machinery: the machinery constructs the representations and supplies coherence for composing measurings. Nor does it imply that arbitrary three-place relations have all three representations. It does, however, suggest an order of exposition in which interaction comes before any particular representing object.

## 2 The standard enriched category of measuring interactions

The Sweedler hom-comonads give the enrichment of monads over comonads of McDermott–Rivas–Uustalu [1]. We do not claim this construction as new. From the third representation in (3), define

$$\mathsf{Meas}(T, D; R) \cong \mathsf{Comon}(D, \{T, R\}), \qquad \mathsf{Int}(T, R) = \mathsf{Comon}(\mathcal{D})/\{T, R\}. \tag{4}$$

An object $\phi : D \to \{T, R\}$ is a *measuring interaction* whose explicit capability is $D$. The enriched composition of hom-comonads yields functors

$$\mathsf{Int}(T, R) \times \mathsf{Int}(R, P) \longrightarrow \mathsf{Int}(T, P), \quad (D, \phi), (E, \psi) \longmapsto (D \star E,\ \mathrm{comp} \circ (\phi \star \psi)),$$

up to the usual bicategorical associativity and units. Thus these slice hom-categories form the standard bicategory associated to a category enriched in a monoidal category. In particular the comonoid tensor, including its comultiplication via the duoidal interchange $\zeta$, is part of the established enrichment machinery; it should not be replaced by an informal assertion that coalgebra laws are automatic.

**Strict squares.** If $f : T \to T'$, $g : R \to R'$ are monad maps and $h : D \to D'$ is a comonad map, a compatible square has the well-typed equation

$$\{T, g\}\,\phi = \{f, R'\}\,\psi\,h \tag{5}$$

(with contravariance in $T$ made explicit). A proposed double-categorical interchange for squares must be checked against enriched composition and $\zeta$; it is not proved here.

## 3 Realisations and their universal reflection

Fix $T, R$ and write $H = \{T, R\}$. The covariant change-of-comonad functor associated to $\phi : D \to H$ sends a $D$-coalgebra $\delta : S \to DS$ to the $H$-coalgebra $(\phi_S\delta) : S \to HS$. Let

$$\mathsf{Real}(T, R) = \int_{(D,\phi)\in\mathsf{Comon}/H} \mathsf{Coalg}(D)$$

be the Grothendieck construction of this assignment: a morphism $(D, \phi, S) \to (E, \psi, U)$ is a pair consisting of a map $h : D \to E$ over $H$ and a coalgebra map $h_*S \to U$.

**Proposition 3.1** (Terminal-fibre reflection)**.** *The functor $L : \mathsf{Real}(T, R) \to \mathsf{Coalg}(H)$ given by $L(D, \phi, S) = \phi_*S$ is left adjoint to the terminal-fibre inclusion $j : \mathsf{Coalg}(H) \hookrightarrow \mathsf{Real}(T, R)$. Moreover $Lj \cong 1$; hence $j$ is fully faithful.*

*Proof.* A morphism $(D, \phi, S) \to j(U)$ must lie over the unique slice arrow $\phi : (D, \phi) \to (H, 1_H)$; its remaining data are exactly a coalgebra morphism $\phi_*S \to U$. This is natural in both variables. The terminal-fibre restriction of $L$ is the identity. □

This explains what is gained by remembering $D$: the chosen realisation exhibits a runner as part of a reusable capability theory, while reflection forgets the factorisation. It does *not* by itself show that two distinct capabilities can be distinguished by a particular runner.

**A small diagnostic.** For the unary writer theory $T = \mathbb{N} \times -$, the identity capability admits $\mathsf{Meas}(T, \mathsf{Id}; T) \cong \mathbb{N}$. A coloured capability $C \times -$ gives $\mathsf{Meas}(T, C \times -; T) \cong \mathbb{N}^C$. This second identity follows from the coproduct decomposition of $C \times -$ in comonads, so it is not evidence for a new invariant of capabilities. Detecting $D$ requires varying the target pair $(T, R)$.

## 4 Effectful comparisons and delegated algebras

A coalgebra for the measuring comonad $H = \{T, R\}$ on a state set $S$ is called a $(T, R)$-*runner*. Equivalently, it determines a natural family of execution maps

$$\rho_X : S \times TX \longrightarrow R(S \times X), \tag{6}$$

compatible with the units and multiplications of $T$ and $R$. Intuitively, $\rho_X$ executes a $T$-command using state $S$, returning an $R$-computation of a new state and an answer. The unit condition is $\rho_X(s, \eta^T_X(x)) = \eta^R_{S\times X}(s, x)$; the multiplication condition says that executing a composite command agrees with sequential execution, using the canonical strength needed to thread the state through $R$. This execution description is the usual concrete form of a measuring coalgebra [1].

For any $R$-algebra $(B, b : RB \to B)$, execution delegates a $T$-algebra structure to the function set $B^S$. Writing $\mathrm{ev} : S \times B^S \to B$ for evaluation, its structure map is

$$a_{\rho,B} : T(B^S) \longrightarrow B^S, \qquad a_{\rho,B}(t)(s) = b\big(R(\mathrm{ev})(\rho_{B^S}(s, t))\big). \tag{7}$$

The runner laws imply the $T$-algebra laws for $a_{\rho,B}$, and the construction is natural in morphisms of $R$-algebras. Thus a runner is not merely a state-transition device: it transforms every $R$-algebra into a $T$-algebra on its $S$-indexed functions.

Strict morphisms between runners use functions $h : S \to S'$ preserving execution. A more general comparison may itself have an $R$-effect, and is therefore a Kleisli arrow

$$\alpha : S \longrightarrow RS'.$$

For each $R$-algebra $(B, b)$ it induces

$$\alpha^*_B : B^{S'} \to B^S, \qquad \alpha^*_B(u)(s) = b\big(Ru(\alpha(s))\big). \tag{8}$$

Given runners $\rho$ on $S$ and $\rho'$ on $S'$, call $\alpha$ an *effectful runner comparison* if, for every $R$-algebra $B$, the map $\alpha^*_B$ is a homomorphism from the delegated $T$-algebra $(B^{S'}, a_{\rho',B})$ to $(B^S, a_{\rho,B})$. Explicitly, the condition is

$$\alpha^*_B \, a_{\rho',B} = a_{\rho,B} \, T(\alpha^*_B). \tag{9}$$

It depends only on the resulting runners, even if these runners have been realised using different capability comonads $D$. Identities are given by $\eta^R_S : S \to RS$ and effectful comparisons compose by the Kleisli formula $\beta \diamond \alpha = \mu^R \, R\beta \, \alpha$. Their induced maps on $B$-powers compose contravariantly; hence Equation (9) is stable under composition. At each fixed pair $(T, R)$ this gives a category of runners and effectful comparisons.

**Question 4.1** (Effectful horizontal interchange)**.** *Beyond the writer case of Section 5, can these effectful comparison categories be assembled into a double or bicategorical structure for general monads, with horizontal composition compatible with measuring composition and the duoidal interchange $\zeta$?*

The ordinary enriched composition of measuring comonoids does not itself provide the requisite action on Kleisli arrows for arbitrary changing residuals. The writer-monoid construction in the next section gives a positive instance, with a concrete horizontal formula and interchange proof. It does not yet solve the problem in general or for capability-labelled and tight squares.

# 5 Twisted intertwiners and the writer effectful calculus

## 5.1 Twisted intertwiners as stateless triadic representations

The triadic representation of a measuring (realised on $S$) consists of a $T$-algebra $A$, an $R$-algebra $B$, and $e : A \times S \to B$ compatible with evaluation of a $T$-term against the runner. For $D = \mathsf{Id}$ and $S = 1$, a measuring is a monad map $\theta : T \to R$ and the condition reduces to

$$e \, a = b \, R(e) \, \theta_A. \tag{10}$$

Thus $e : A \to \theta^* B$ is a $T$-algebra map. For writer monads $T = G \times -$ and $R = H \times -$, the monad morphism is a monoid homomorphism $\theta : G \to H$ and

$$e \, \rho_G(g) = \rho_H(\theta(g)) \, e.$$

This is the twisted-intertwiner equation (where the usual group-theoretic definition often restricts the twist to an automorphism of one group) [6, 7]. The result identifies an established special case, not a claim of novelty.

### 5.2 Writer runners as action-category cocycles

Let $G, H$ be monoids, with multiplication $gg'$ denoting execution of $g$ followed by $g'$. Put $T = G \times -$ and $R = H \times -$.

**Proposition 5.1** (Writer-runner classification)**.** *A runner $T \rightsquigarrow R$ on $S$ is equivalent to a right $G$-action $s \mapsto s \cdot g$ and $\kappa : S \times G \to H$ satisfying*

$$\kappa_s(1) = 1, \qquad \kappa_s(gg') = \kappa_s(g)\kappa_{s\cdot g}(g'). \tag{11}$$

*It acts by $(g, x, s) \mapsto (\kappa_s(g), x, s \cdot g)$. Equivalently, $\kappa$ is a functor $S//G \to \mathbf{B}H$ from the action category, and the projection $\pi : S//G \to \mathbf{B}G$ is a discrete opfibration.*

*Proof.* Naturality of the runner in $x$ forces the stated form. The unit law yields $s \cdot 1 = s$ and $\kappa_s(1) = 1$. Comparing the two evaluations of a nested writer term yields $(s \cdot g) \cdot g' = s \cdot (gg')$ and (11); conversely these equations imply both runner laws. □

The runner is therefore a span

$$\mathbf{B}G \xleftarrow{\pi} S//G \xrightarrow{\kappa} \mathbf{B}H, \qquad \pi \text{ a discrete opfibration.}$$

This expresses the demand–capacity–residual picture: the left leg records the demanded action and its state transport, while the right leg records the resulting residual effect. For left $G$- and $H$-sets $(A, \rho_G)$ and $(B, \rho_H)$ a triadic representation is a family $e_s : A \to B$ with

$$e_s \rho_G(g) = \rho_H(\kappa_s(g)) e_{s\cdot g}. \tag{12}$$

Strictly, for the action-category arrow $g : s \to s \cdot g$, the square in (12) has the reverse direction from the usual naturality equation for covariant action functors. It can be regarded as a natural transformation after choosing the corresponding opposite/contravariant action conventions. We use the explicit equation to avoid suppressing this variance issue.

### 5.3 Effectful comparisons as natural transformations

**Proposition 5.2** (Cell classification)**.** *For runners $(S, \kappa)$ and $(U, \kappa')$ from $G$ to $H$, write $\alpha(s) = (h(s), \varphi(s)) \in H \times U$. Then (8) is a compatible effectful comparison exactly when*

$$\varphi(s \cdot g) = \varphi(s) \cdot g, \qquad h(s)\kappa'_{\varphi(s)}(g) = \kappa_s(g) h(s \cdot g). \tag{13}$$

*Thus $\varphi : S//G \to U//G$ lies over $\mathbf{B}G$, and $h$ is a natural transformation $\kappa \Rightarrow \kappa'\varphi$ (using composition of arrows in $\mathbf{B}H$ in the stated temporal order).*

*Proof.* The delegated $G$-action on $B^S$ is $(g \cdot f)(s) = \kappa_s(g) \cdot f(s \cdot g)$. Substituting $\alpha^* f(s) = h(s) \cdot f(\varphi(s))$ into the homomorphism equation gives

$$h(s)\kappa'_{\varphi(s)}(g) \cdot f(\varphi(s) \cdot g) = \kappa_s(g) h(s \cdot g) \cdot f(\varphi(s \cdot g)).$$

The laws are sufficient. To recover both laws, choose the free $H$-set $B = H \times U$ and $f(u) = (1, u)$; equality of components yields the result. □

For $S = U = 1$, (13) reads $h\kappa'(g) = \kappa(g)h$. For groups, $\kappa$ and $\kappa'$ differ by conjugation; these are the usual deformations of twists, with source/target direction fixed by the Kleisli comparison. For $S = U$ and $\varphi = 1$, it is the nonabelian cocycle coboundary equation. For monoids, $h$ need not be invertible.

### 5.4 Horizontal composition and interchange

Let $\rho : G \rightsquigarrow H$ on $S$ with cocycle $\kappa$, and $\sigma : H \rightsquigarrow K$ on $U$ with cocycle $\lambda$. Their composite has state $S \times U$ and

$$(s,u) \cdot g = (s \cdot g, u \cdot \kappa_s(g)), \qquad (\lambda \diamond \kappa)_{(s,u)}(g) = \lambda_u(\kappa_s(g)). \tag{14}$$

Its action category is the pullback of the discrete opfibration $U//H \to \mathbf{B}H$ along $\kappa : S//G \to \mathbf{B}H$, with the evident projection to $\mathbf{B}G$.

Suppose $(h,\varphi) : \rho \to \rho'$ and $(k,\psi) : \sigma \to \sigma'$ satisfy (13), with source and target state spaces $S, S'$ and $U, U'$ respectively. Put $u_1 = u \cdot h(s)$. Define

$$(k,\psi) \odot (h,\varphi) : (s,u) \longmapsto \big(\lambda_u(h(s))k(u_1),\ (\varphi(s), \psi(u_1))\big). \tag{15}$$

The first comparison's $H$-effect is executed in the second runner before the second comparison is applied. This transport uses precisely the lifting property of the discrete opfibration.

**Proposition 5.3** (Writer effectful horizontal composition)**.** *Formula* (15) *defines a compatible effectful comparison between the composite runners. It preserves identity comparisons, is associative up to the canonical product-state reassociation, and satisfies interchange with vertical Kleisli composition.*

*Proof.* We first check the two equations (13). Write $a = \kappa_s(g)$, $a' = \kappa'_{\varphi(s)}(g)$, $s_g = s \cdot g$, and $v = u \cdot h(s)$. The comparison law for $(h,\varphi)$ gives $h(s)a' = ah(s_g)$. Thus

$$(u \cdot a) \cdot h(s_g) = u \cdot h(s)a' = v \cdot a'.$$

Equivariance of $\psi$ now shows that the new target state after $g$ is the target state of (15) after the composite $G$-action. For the residual law, the source-side expression is

$$\begin{aligned}
&\lambda_u(h(s))k(v)\lambda'_{\psi(v)}(a') \\
&\quad = \lambda_u(h(s))\lambda_v(a')k(v \cdot a') = \lambda_u(h(s)a')k(v \cdot a') \\
&\quad = \lambda_u(ah(s_g))k((u \cdot a) \cdot h(s_g)) \\
&\quad = \lambda_u(a)\lambda_{u \cdot a}(h(s_g))k((u \cdot a) \cdot h(s_g)),
\end{aligned}$$

which is precisely the composite-runner residual $\lambda_u(a)$ followed by the comparison residual at $(s_g, u \cdot a)$. Here we used the $(k,\psi)$ comparison equation and the cocycle law for $\lambda$.

The horizontal identity comparison has $h(s) = 1_H$, $\varphi = 1_S$ and hence (15) gives residual $\lambda_u(1_H)1_K = 1_K$ and unchanged state. For associativity with a third writer runner, both parenthesisations execute the residual $h(s)$ through the second runner, then its output through the third, and finally the successive comparison residuals; applying the cocycle laws twice and reassociating $S \times U \times V$ yields the same $K$-successor state and final residual. Equivalently, this is associativity of pullback transport for the three discrete opfibrations.

For interchange, take vertically composable first-stage cells $(h,\varphi)$, $(h',\varphi')$ and second-stage cells $(k,\psi)$, $(k',\psi')$. Let $x = h'(\varphi(s))$ and $v = u \cdot h(s)$. Horizontal composition after vertical composition has residual

$$\lambda_u(h(s)x)\, k(v \cdot x)\, k'(\psi(v \cdot x)).$$

Vertical composition after horizontal composition has residual

$$\lambda_u(h(s))\, k(v)\, \lambda'_{\psi(v)}(x)\, k'(\psi(v) \cdot x).$$

Their equality follows from $k(v)\lambda'_{\psi(v)}(x) = \lambda_v(x)k(v \cdot x)$, equivariance $\psi(v) \cdot x = \psi(v \cdot x)$, and $\lambda_u(h(s))\lambda_v(x) = \lambda_u(h(s)x)$. State components agree by the same equivariance. The vertical identity/unit laws are those of the writer Kleisli category. □

**Remark 5.4** (Precise scope)**.** *The proposition concerns the 2-dimensional calculus with monoids as objects, writer runners as horizontal arrows and effectful comparisons as 2-cells. Its coherence can be organised by spans* $\mathbf{B}G \leftarrow X \rightarrow \mathbf{B}H$ *with discrete-opfibration left legs and lax comparison data on right legs. It does* not *establish horizontal composition for general monads, for specified capability factorizations, or for tight monad-map squares. Nor does it identify a new Sweedler enrichment. The original intertwiners note also reported finite enumeration checks; the proof above supplies the missing comparison-law calculation.*

# 6 From measuring to adequacy

A measuring, or a runner realising it, shows that demands can be *coherently interpreted* through a capability into a residual. It does not show that they are interpreted *well*. The smallest example is decisive. For $T = \mathbb{N} \times -$ and $D = R = \mathsf{Id}$ we have $\mathsf{Meas}(T, \mathsf{Id}; \mathsf{Id}) \cong \mathsf{Mon}(T, \mathsf{Id})$, a singleton, and the induced runner on $S = 1$ ignores every tick. It satisfies all the runner laws, because nothing in the writer theory requires a tick to have an observable effect. We therefore separate *compatibility*, meaning the existence of measurings and runners, from *adequacy*, meaning how well a realised runner meets given standards. This section shows that the delegated algebras of Section 4 already supply the required evaluation. The resulting adequacy relation is three-place at the level of *instances* (demands, realised states, and standards), in contrast with the theory-level two-variable adjunction of (3).

## 6.1 Delegated evaluation

Fix a $(T, R)$-runner $\rho$ on $S$ with execution maps (6), an $R$-algebra $(B, b)$, and a *postcondition* $q : S \times X \to B$. For $t \in TX$ and $s \in S$ define

$$\langle t \mid s \mid q \rangle_{\rho,B} = b\big(R(q)(\rho_X(s,t))\big) \in B. \tag{16}$$

In words, we execute $t$ from state $s$ and then assess the residual computation of final state and answer, using the standard $b$ and the postcondition $q$. This is the familiar pattern of generic weakest-precondition semantics, in which Eilenberg–Moore algebras serve as algebras of truth values, qualitative or quantitative [8]. What the runner adds is the transport of demands through state into residual effects.

**Proposition 6.1** (Delegated evaluation)**.** *Write* $q^\sharp : X \to B^S$ *for* $q^\sharp(x)(s) = q(s,x)$.

*(i)* $\langle t \mid s \mid q \rangle_{\rho,B} = a_{\rho,B}(T(q^\sharp)(t))(s)$. *In particular the delegated algebra (7) is the case* $X = B^S$, $q = \mathrm{ev}$.

*(ii)* Unit: $\langle \eta^T_X(x) \mid s \mid q \rangle_{\rho,B} = q(s,x)$.

*(iii)* Sequencing: *for* $k : X \to TY$ *and* $q : S \times Y \to B$,

$$\langle \mu^T_Y(Tk(t)) \mid s \mid q \rangle_{\rho,B} = \langle t \mid s \mid q_k \rangle_{\rho,B}, \qquad q_k(s', x) = \langle k(x) \mid s' \mid q \rangle_{\rho,B}.$$

*(iv)* Naturality: *for an* $R$*-algebra map* $m : B \to B'$, $m\langle t \mid s \mid q \rangle_{\rho,B} = \langle t \mid s \mid mq \rangle_{\rho,B'}$.

*(v)* Free triadic representations: *write the triadic-representation condition of Section 5 for a* $T$*-algebra* $(A, a)$ *explicitly as* $e(a(\tau), s) = b(R(e)(\rho_A(s,\tau)))$ *for* $\tau \in TA$, *where* $R(e)$ *is applied after the evident reordering* $S \times A \cong A \times S$. *Then triadic representations* $e : TX \times S \to B$ *on the free* $T$*-algebra correspond bijectively to postconditions* $q : S \times X \to B$, *via* $e \mapsto e(\eta^T_X(-), -)$ *and* $q \mapsto \langle - \mid - \mid q \rangle_{\rho,B}$.

*Proof.* (i) follows from naturality of $\rho$ in $X$ together with $\mathrm{ev} \circ (S \times q^\sharp) = q$. (ii) follows from the runner unit law and $b\,\eta^R_B = 1_B$. For (iii), naturality gives $\rho_{TY}(s, Tk(t)) = R(S \times k)\rho_X(s,t)$. The runner multiplication law gives $\rho_Y(s, \mu^T_Y \tau) = \mu^R\, R(\rho_Y)\, \rho_{TY}(s,\tau)$. The algebra law

$b\,\mu_B^R = b\,R(b)$ then rewrites the left-hand side as $b\,R(b\,R(q)\,\rho_Y\,(S\times k))\rho_X(s,t)$, which is the right-hand side. (iv) holds because $m$ commutes with $b$. For (v), $\langle - \mid - \mid q\rangle$ satisfies the compatibility condition by (iii) with $k = 1_{TX}$, and it restores $q$ by (ii). Conversely, a compatible $e$ satisfies $e(t,s) = e(\mu^T T\eta^T t, s) = b\,R(e(\eta^T -, -))\rho_X(s,t)$. □

Item (iii) is the weakest-precondition law for sequential composition, and item (v) shows that delegated evaluation is the free case of the triadic representations already introduced. We also checked (iii) by exhaustive evaluation on randomly generated small writer runners over $(\mathbb{N}, +)$ with finite $\mathbb{N}$-set standards.

### 6.2 Standards pull back along runners

Let $\sigma : R \rightsquigarrow P$ be a runner on $U$, and let $(C, c)$ be a $P$-algebra. We use the sequential composite $\sigma \circ \rho : T \rightsquigarrow P$ on $S \times U$, whose execution sends $((s,u),t)$ to $\sigma_{S\times X}(u, \rho_X(s,t)) \in P(U \times S \times X)$, reordered to $P((S\times U)\times X)$. In the writer case this is (14). We take this concrete composite as given; its identification with the enriched composition of Section 2 is the standard runner–coalgebra correspondence, and we do not prove it again here.

**Proposition 6.2** (Pullback of standards)**.** *Give $C^U$ the delegated $R$-algebra structure $a_{\sigma,C}$. For $q : (S\times U)\times X \to C$ define $\tilde{q} : S \times X \to C^U$ by $\tilde{q}(s,x)(u) = q((s,u),x)$. Then*

$$\langle t \mid (s,u) \mid q\rangle_{\sigma\circ\rho,\,C} = \langle t \mid s \mid \tilde{q}\rangle_{\rho,\,C^U}(u).$$

*Proof.* By definition the right-hand side is $c\,P(\mathrm{ev})\,\sigma_{C^U}(u, R(\tilde{q})\rho_X(s,t))$. Naturality of $\sigma$ rewrites this as $c\,P(\mathrm{ev}\circ(U\times\tilde{q}))\,\sigma_{S\times X}(u,\rho_X(s,t))$, and $\mathrm{ev}\circ(U\times\tilde{q}) = q$ up to the reordering of state components. □

This is an assume–guarantee principle. Assessing the composite against a $P$-standard is the same as assessing $\rho$ against the pulled-back $R$-standard $C^U$. An adequacy guarantee for $\rho$ relative to a fixed family of $R$-standards therefore transfers to the composite only for those $P$-standards whose pullbacks lie in that family. The proposition says nothing about whether the family needed downstream is the family anyone has checked upstream. (The identity was also confirmed numerically in the writer case.)

**Corollary 6.3** (Delegation is contravariantly functorial)**.** *Let $\kappa : C^{S\times U} \cong (C^U)^S$ be currying. Then $\kappa$ is an isomorphism of $T$-algebras*

$$(C^{S\times U}, a_{\sigma\circ\rho,C}) \;\cong\; ((C^U)^S, a_{\rho,\,(C^U,a_{\sigma,C})}).$$

*That is, delegating a $P$-algebra along $\sigma\circ\rho$ is, up to currying, delegating it first along $\sigma$ and then along $\rho$.*

*Proof.* Apply Proposition 6.2 with $X = C^{S\times U}$ and $q = \mathrm{ev}$, so that $\tilde{q}^\sharp = \kappa$. Then use Proposition 6.1(i) on both sides. □

**Remark 6.4** (A logical reading of sequencing)**.** *Proposition 6.2 and Corollary 6.3 have the form of a Hoare sequencing rule. Write $\mathrm{wp}_\sigma$ for the passage from a $P$-standard to the pulled-back $R$-standard. Adequacy of $\sigma\circ\rho$ against $C$ is then adequacy of $\rho$ against $\mathrm{wp}_\sigma(C)$, and $\mathrm{wp}_{\sigma\circ\rho} \cong \mathrm{wp}_\rho\,\mathrm{wp}_\sigma$. In substructural terms, a runner behaves as a witness of an implication from standards on its residual to standards on its demand. Sequential composition of runners is then* cut *on such implications, not a tensor of specifications. A rule of the form "$\rho \Vdash \Sigma$ and $\sigma \Vdash \Theta$ imply $\sigma\circ\rho \Vdash \Sigma\otimes\Theta$" therefore has the wrong shape for sequential delegation.*

*Three further points locate this reading. Each is a proposal, not a result.*

(a) Ternary frames. *In the relational semantics of substructural logics, the multiplicative connectives are interpreted through a ternary combination relation, which induces a quantale on predicates by convolution [12]. Runner composition is defined only when the residual of one runner matches the demand of the next. Any frame for it must therefore be many-sorted, in promonoidal or profunctorial form, rather than a single set of worlds. The measuring relation* $\mathsf{Meas}(T, D; R)$ *is not itself such a frame: it is heterogeneous and proof-relevant.*

(b) Two products. *In the bicategory of Section 2, the composite of interactions* $(D, \phi)$ *and* $(E, \psi)$ *has capability* $D \star E$*, where* $\star$ *is Day convolution. Capabilities therefore combine multiplicatively and commutatively, while standards chain by pullback along* $\circ$*. This suggests a bunched reading [13]. It would have a non-commutative, Lambek-style implication for sequential delegation and a commutative multiplicative for pooling capabilities.*

(c) Exchange. *Atkey and Kokke's semantics for the deep-inference systems BV and MAV [18] combines a commutative parallel product* $⅋$ *with a non-commutative, self-dual sequence connective* $\triangleleft$*. Their* frames *satisfy*

$$(w \triangleleft x) ⅋ (y \triangleleft z) \leq (w ⅋ y) \triangleleft (x ⅋ z).$$

*Reading* $⅋$ *as* $\star$*,* $\triangleleft$ *as* $\circ$*, and* $\leq$ *as an arrow, this is exactly the orientation of* (1)*, with no reversal. Their algebras, like concurrent Kleene algebra [14], carry the duoidal law the other way round. Atkey and Kokke read frames as describing the* evolution *of processes, and algebras and CKA as describing their* properties*. On this reading* $[\mathsf{Set}, \mathsf{Set}]_{\mathrm{acc}}$ *with* $\zeta$ *has the orientation of a categorified frame. Runners are things that evolve, while standards of assessment belong on the dual, algebra side. Compatibility of adequacy judgements with* $\zeta$ *then becomes the soundness of an exchange rule, and Question 4.1 acquires a logical counterpart. This is a match of shape:* $[\mathsf{Set}, \mathsf{Set}]_{\mathrm{acc}}$ *is not posetal, and the frames of [18] are single-sorted, unlike runner composition (point (a)).*

### 6.3 Effectful comparisons as simulations

The effectful comparisons of Section 4 are defined by quantifying over all $R$-algebras. Delegated evaluation shows that this amounts to a contextual simulation relation, with $R$-standards as tests.

**Proposition 6.5** (Comparisons transport evaluation)**.** *Let* $\rho, \rho'$ *be* $(T, R)$*-runners on* $S, S'$ *and* $\alpha : S \to RS'$*. Then* $\alpha$ *is an effectful runner comparison if and only if, for all* $X$*, all* $R$*-algebras* $(B, b)$*, all* $q' : S' \times X \to B$*, all* $t \in TX$ *and all* $s \in S$*,*

$$b\,R(\langle t \mid - \mid q' \rangle_{\rho',B})(\alpha(s)) = \langle t \mid s \mid q \rangle_{\rho,B}, \qquad q(s_1, x) = b\,R(q'(-, x))(\alpha(s_1)). \tag{17}$$

*Proof.* By Proposition 6.1(i), the left-hand side of (17) is $\alpha_B^*(a_{\rho',B}(T(q'^\sharp)t))(s)$. The right-hand side is $a_{\rho,B}(T(\alpha_B^* q'^\sharp)t)(s)$, since $q^\sharp = \alpha_B^*\, q'^\sharp$. Hence (9) implies (17). Conversely, taking $X = B^{S'}$ and $q' = \mathrm{ev}$, so that $q'^\sharp = 1$, recovers (9). □

In words: starting $\rho'$ from the effectful image of $s$ and assessing gives the same verdict as running $\rho$ from $s$ and assessing the effectful image of its final state. Restricting the class of standards gives coarser contextual preorders, which are the natural objects when only some standards matter. The effectful horizontal interchange problem (Question 4.1) and the transfer of adequacy along composites (Proposition 6.2) are therefore two aspects of the same question: how delegated algebras are transported along composite runners.

### 6.4 Triadic adequacy contexts

An *$R$-standard on $X$* is a tuple $\varsigma = (B, b, q, \mathrm{Acc})$ consisting of an $R$-algebra, a postcondition $q : S \times X \to B$ (or a family of these indexed compatibly over realisations), and an acceptance subset $\mathrm{Acc} \subseteq B$. Fix a set $\mathcal{P}$ of realised runner states. For a fixed realisation $(D, \phi, \gamma)$ these are the states of $\phi_*\gamma$. Fix also a set $\Sigma$ of standards. The *adequacy context* is the triadic context in the sense of Lehmann and Wille [9]

$$\mathbb{K} = (TX,\ \mathcal{P},\ \Sigma,\ Y), \qquad (t, s, \varsigma) \in Y \iff \langle t \mid s \mid q \rangle_{\rho,B} \in \mathrm{Acc}.$$

Its triadic concepts are maximal triples $(A_1, A_2, A_3)$ with $A_1 \times A_2 \times A_3 \subseteq Y$: a set of demands, a set of capacities, and a set of standards such that every capacity meets every demand under every standard. Wille's basic theorem identifies the resulting structures with complete trilattices [10]. The derivation operators close one component given the other two; they are triadic Galois connections in Guitart's sense [11]. This is the precise sense in which adequacy is three-place. The three sorts are instances, namely terms of the free $T$-algebra, states of realised capabilities, and $R$-algebras with postconditions. They are not the theories $T$, $D$, $R$, whose relationship remains the two-variable adjunction of (3).

When $B$ is a complete lattice or a commutative quantale $V$, one may drop Acc and take the incidence to be $V$-valued, $Y(t, s, \varsigma) = \langle t \mid s \mid q \rangle_{\rho,V}$. This is where enrichment enters the adequacy question. It enters the *value of the incidence*, not the hom-objects $\mathsf{Meas}(T, D; R)$, and an enriched hom would only compare measuring laws without supplying any standard of success. The $V$-valued analogue of the triadic Galois connections is proposed here, not developed.

### 6.5 The writer example

Take $T = R = \mathbb{N} \times -$ and $D = \mathsf{Id}$. By Section 3, $\mathsf{Meas}(T, \mathsf{Id}; T) \cong \mathbb{N}$; the runner on $S = 1$ for $k \in \mathbb{N}$ is $(n, x) \mapsto (kn, x)$. A writer algebra is an $\mathbb{N}$-set. Use the demands $t_n = (n, n) \in T\mathbb{N}$, which record in their answer how many ticks were requested. Consider the following standards.

- *Budget:* $B = \mathbb{N}$, $b(n, v) = n + v$, $q = 0$, $\mathrm{Acc} = \{v \leq m\}$. Then $Y(t_n, k, \cdot)$ holds iff $kn \leq m$. The stuttering runner $k = 0$ meets every demand under every budget. One-sided, safety-like standards reward doing nothing.
- *Honesty:* $B = \mathbb{Z}$ with $b(n, v) = v + n$, $q(x) = -x$, $\mathrm{Acc} = \{0\}$. Then $Y(t_n, k, \cdot)$ holds iff $(k - 1)n = 0$. For $n \geq 1$ only $k = 1$ qualifies. The postcondition reads the requested count from the answer, so the reference interpretation is expressed by the standard rather than built into the theory.

In the adequacy context with these standards, the triadic concepts separate stuttering, honest and over-ticking runners, and Proposition 6.5 confirms that no effectful comparison relates distinct $k$. Directly, for $S = U = 1$, Proposition 5.2 requires $h + kn = k'n + h$ in $\mathbb{N}$ for all $n$. With $R = \mathsf{Id}$, by contrast, every $\mathsf{Id}$-algebra is a bare set and $\langle t \mid * \mid q \rangle = q(x)$. No standard can depend on the ticks *executed*, as opposed to values *returned*. The verdict on the stuttering runner of the opening example is not "inadequate" but "not assessable": the residual determines what can be assessed.

**Remark 6.6** (What standards cannot see)**.** *An $R$-standard can only assess what $R$ records. Latency, resource use or quality of outcome are invisible to every standard unless the demand or residual theory is enriched to record them, for instance by graded, temporal or cost monads. Delegated evaluation does not repair an underspecified demand. It makes precise which specification is missing.*

**Question 6.7** (Adequacy under composition)**.** *Given a class $\Sigma_P$ of $P$-standards, characterise the $R$-standards obtained from it by pullback along runners $R \rightsquigarrow P$. Determine when the*

*triadic concepts of an adequacy context for $\sigma \circ \rho$ are determined by those for $\rho$ (relative to pulled-back standards) and for $\sigma$. Is there a sound program logic of adequacy, many-sorted in the sense of Remark 6.4(a), with a sequencing rule given by Corollary 6.3 and an exchange rule given by $\zeta$? For reflexive $R$, does the capability–residual exchange of Section 8 relate adequacy contexts, exchanging standards and capabilities?*

### 6.6 Libraries, derivations and residuation

So far a single capability has carried a demand all the way to its residual. In practice a task is often passed through a sequence of small capabilities, each handing on what remains. This subsection records the resulting grammatical structure.

**Libraries and derivations.** A *library* $\mathcal{L}$ is a set of entries $\ell = (D_\ell, T_\ell, R_\ell, \phi_\ell)$ with $\phi_\ell \in \mathsf{Meas}(T_\ell, D_\ell; R_\ell)$. A *derivation* of $T \rightsquigarrow R$ from the word $w = \ell_1 \cdots \ell_n$ consists of monad maps

$$f : T \to T_{\ell_1}, \qquad g_i : R_{\ell_i} \to T_{\ell_{i+1}} \ (1 \le i < n), \qquad g_n : R_{\ell_n} \to R.$$

These are the coercions supplied by the functoriality of $\mathsf{Meas}$, which is contravariant in the demand and covariant in the residual. Transporting each $\phi_{\ell_i}$ along its coercions and composing in the bicategory of Section 2 yields a measuring

$$\phi_w \in \mathsf{Meas}\big(T,\ D_{\ell_1} \star \cdots \star D_{\ell_n};\ R\big),$$

up to the associativity of $\star$. The capability of a derivation is a team, with state space the product of the members' state spaces; it is not a nesting. Write $L_{\mathcal{L}}(T, R) \subseteq \mathcal{L}^*$ for the set of words admitting a derivation. This is the language of handing-on sequences that accomplish the task. It profiles the task $(T, R)$ relative to the available capabilities, which reverses the roles in the interaction profile $\mathcal{P}_D$ of Section 10.

**Residuation.** The three representations (3),

$$\mathsf{Mon}(D \triangleright T, R) \cong \mathsf{Mon}(T, [D, R]) \cong \mathsf{Comon}(D, \{T, R\}),$$

have the form of Lambek's residuation law $a \cdot b \le c \iff b \le a\backslash c \iff a \le c/b$ [15]. The calculus here is two-sorted: comonads act on monads. The copower $D \triangleright T$ is the residue of $T$ once $D$ has acted. The cotensor $[D, R]$ is the most demanding theory that $D$ can carry to $R$, so it is literally *what remains to be done*. The comonad $\{T, R\}$ is the type of capabilities carrying $T$ to $R$. Sequential handing-on is non-commutative because each residual must serve as the next demand. That is why the analogy is with the Lambek calculus rather than with commutative linear logic.

**Proposition 6.8** (Forward and backward chaining)**.** *Assume cotensor associativity $[D \star E, Q] \cong [D, [E, Q]]$, natural in its arguments, as in Proposition 8.1. Then*

$$\mathsf{Meas}(T, D_1 \star \cdots \star D_n; R) \cong \mathsf{Mon}(T, [D_1, [D_2, \cdots [D_n, R] \cdots]]).$$

*If moreover the copowers involved exist, then $(D \star E) \triangleright T \cong E \triangleright (D \triangleright T)$, and*

$$\mathsf{Meas}(T, D_1 \star \cdots \star D_n; R) \cong \mathsf{Mon}\big(D_n \triangleright \cdots \triangleright D_1 \triangleright T,\ R\big).$$

*Proof.* The first statement is (3) followed by repeated cotensor associativity. For the second, $\mathsf{Mon}((D \star E) \triangleright T, R) \cong \mathsf{Mon}(T, [D, [E, R]]) \cong \mathsf{Mon}(D \triangleright T, [E, R]) \cong \mathsf{Mon}(E \triangleright (D \triangleright T), R)$, naturally in $R$, so the Yoneda lemma applies. □

The two formulas are backward and forward chaining. One either computes from the goal $R$ what each earlier stage must deliver, or computes from $T$ what remains after each handing-on. They also show where a library genuinely constrains the problem. If intermediate theories were unrestricted, the cotensors $[D_{i+1}, [\cdots [D_n, R]]]$ would always be available as intermediate stages, and membership of a word would reduce to non-emptiness of the single set $\mathsf{Meas}(T, D_w; R)$. We expect, though we have not checked against the enriched composition of Section 2, that every measuring of $D \star E$ then factors through the evaluation measuring of $E$ at $[E, R]$. A library matters because it fixes the intermediate theories, which represent what each next party is prepared to accept, and the specified measurings $\phi_\ell$.

**Residuation among instances.** The residuation above concerns theories. A second residuation concerns runners. Fix theories $T, M, R$. For a set $A$ of runners $T \rightsquigarrow M$ and a set $B$ of runners $M \rightsquigarrow R$, write $A; B = \{\sigma \circ \rho \mid \rho \in A,\ \sigma \in B\}$, with runners taken up to the canonical reassociation of state spaces. For a set $C$ of runners $T \rightsquigarrow R$ define

$$A \Rightarrow C = \{\sigma \mid \sigma \circ \rho \in C \text{ for all } \rho \in A\}, \qquad C \Leftarrow B = \{\rho \mid \sigma \circ \rho \in C \text{ for all } \sigma \in B\}.$$

Then $A; B \subseteq C \iff B \subseteq A \Rightarrow C \iff A \subseteq C \Leftarrow B$, which is Lambek's law with its two slashes. Here $A \Rightarrow C$ is the set of runners that complete the task $C$ whichever runner in $A$ goes first, so it is *what remains to be done* at the level of instances, alongside the theory-level cotensor $[D, R]$. Taking $C$ to be the runners meeting a standard links this to the adequacy contexts above. If $C$ is downward closed under effectful comparison and composition is monotone in comparisons, then the residuals are downward closed as well. This mirrors the Day construction on lower sets in [18]. Monotonicity is established here only in the writer case, by Proposition 5.3.

**The writer illustration.** Let every entry have $T_\ell = R_\ell = \mathbb{N} \times -$ and $D_\ell = \mathsf{Id}$, with $\ell_k$ the measuring $k \in \mathbb{N}$ of Section 3. Since $\mathsf{Id}$ is the unit of Day convolution, $\mathsf{Id} \star \mathsf{Id} \cong \mathsf{Id}$. Composites of measurings out of $\mathsf{Id}$ are composites of monad maps, so the word $\ell_{k_1} \cdots \ell_{k_n}$ yields the measuring $k_1 \cdots k_n$. Under the honesty standard of the writer example above, the adequate words over $\{\ell_0, \ell_1, \ell_2\}$ are exactly $\ell_1^*$. Under any budget standard, every word containing $\ell_0$ is adequate. More generally, Corollary 6.3 pulls a standard on $R$ back one handing-on at a time, so the *adequate sublanguage* $L^{\varsigma}_{\mathcal{L}}(T, R)$ for a standard $\varsigma$ can be checked stepwise along a derivation.

**Concurrency.** The convolution $\star$ pools capabilities, and comonads are closed under it because $\star$ is oplax for composition (1). Demands are not closed under it: the convolution of two monads need not be a monad. Splitting a demand into concurrent parts therefore needs other combinations of monads. The natural candidates are the sum $T_1 + T_2$, which interleaves operations, and the tensor $T_1 \otimes T_2$, whose operations commute [17]. We expect that runners $T_i \rightsquigarrow R_i$ on $S_i$ combine to a runner $T_1 + T_2 \rightsquigarrow R_1 + R_2$ on $S_1 \times S_2$, each component handling its own operations, and similarly for tensors when the residual effects commute. Neither claim is checked here.

**Question 6.9** (Grammars of handing-on)**.** *For a finite library of polynomial entries over a finite set of intermediate theories, is membership in $L_{\mathcal{L}}(T, R)$ decidable, and is $L_{\mathcal{L}}(T, R)$ context-free? Pentus showed that Lambek grammars generate exactly the context-free languages [16]. The coercions through* Mon *and the two-sortedness of the present calculus may break this analogy, and so may concurrent combination by sums or tensors. Pentus's argument works with a sequent calculus, but the logic BV of sequential and parallel composition has no sequent calculus and needs deep inference [19, 18]. Once concurrency is admitted,*

*context-freeness is therefore not the expected answer. Which sublibraries suffice to generate a given task, and how does $L_{\mathcal{L}}(T, R)$ change as the library grows?*

## 7 Transpose versus exchange

The three isomorphisms in (3) are *transpositions*: they present one and the same measuring as a monad morphism or a comonad morphism. They do not change the measuring triple $(T, D, R)$. The construction in the next section has a different logical form. It sends a measuring with triple $(T, D, R)$ to one with triple $(T, R^\circ, D^\vee)$, replacing the capability and residual by their duals. Thus the exchange is not an automatic fourth representation of a fixed heteromorphism relation. Its invertibility needs an additional reflexivity hypothesis. This distinction is essential to any claim of symmetry or triality.

## 8 Sweedler duality and the capability–residual exchange

Fix the dualising residual $\mathsf{Id}$ and write

$$R^\circ = \{R, \mathsf{Id}\} \quad \text{and} \quad D^\vee = [D, \mathsf{Id}].$$

The measuring adjunction yields an adjunction between comonads and opposite monads, with the monad-side unit

$$\eta_R : R \longrightarrow (R^\circ)^\vee = [R^\circ, \mathsf{Id}]. \tag{18}$$

**Proposition 8.1** (Capability–residual exchange)**.** *Assume the cotensor associativity $[D \star E, Q] \cong [D, [E, Q]]$ is an isomorphism of monads, natural in its arguments, and that the symmetry $D \star E \cong E \star D$ is compatible with the comonad monoidal structure. Then, for monads $T, R$ and comonad $D$, there is a natural map*

$$\Xi_{T,D,R} : \mathsf{Meas}(T, D; R) \longrightarrow \mathsf{Meas}(T, R^\circ; D^\vee). \tag{19}$$

*If $\eta_R$ is an isomorphism, then $\Xi_{T,D,R}$ is a bijection for all $T, D$.*

*Proof.* Under the cotensor representation in (3), the map is the composite

$$\begin{aligned} \mathsf{Mon}(T, [D, R]) &\xrightarrow{[D,\eta_R]_*} \mathsf{Mon}(T, [D, [R^\circ, \mathsf{Id}]]) \\ &\cong \mathsf{Mon}(T, [D \star R^\circ, \mathsf{Id}]) \\ &\cong \mathsf{Mon}(T, [R^\circ \star D, \mathsf{Id}]) \\ &\cong \mathsf{Mon}(T, [R^\circ, [D, \mathsf{Id}]]). \end{aligned}$$

All steps after postcomposition with $[D, \eta_R]$ are isomorphisms under the hypotheses. The first is also an isomorphism when $\eta_R$ is. □

Call $R$ *reflexive* if $\eta_R$ is invertible. The exchange is an authentic symmetry of measuring *sets* for reflexive residuals while holding demand fixed. It is not, as stated, a symmetry of the entire category of realised interactions or of effectful 2-cells. Nor does it assert an $S_3$ symmetry of the triple $(T, D, R)$.

**Finite-dimensional analogy.** For finite-dimensional algebras $A, B$ and a finite-dimensional coalgebra $C$ over a field, the familiar duality gives

$$\mathsf{Meas}(A, C; B) \cong \mathsf{Meas}(A, B^*; C^*), \qquad g(\beta, a)(c) = \beta(f(c, a)).$$

This is a useful model of the exchange, but finite-dimensional linear duality is stronger than anything available for arbitrary accessible Set-endofunctors.

## 9 Garner's theorem and the reflexive locus

Polynomiality of $R^\circ$ is not an additional conjecture: it is substantially covered by Garner's costructure–cosemantics adjunction [2]. His cosemantics construction takes an accessible monad to a presheaf comonad determined by its behaviour category; the adjunction is idempotent and its fixed points are precisely presheaf monads and comonads. These are established results, not new claims of this study.

**Proposition 9.1** (Identification with costructure–cosemantics; comparison)**.** *Under the standard identification of the Sweedler dual $R^\circ = \{R, \mathsf{Id}\}$ with the comonad classifying Set-comodels of $R$, Garner's cosemantics comonad is $R^\circ$. Under the corresponding identification of $D^\vee = [D, \mathsf{Id}]$ with his costructure monad, the units and counits agree. Consequently $R^\circ$ is a presheaf (polynomial) comonad for every accessible monad $R$, and the duality is idempotent, with reflexive monads precisely the presheaf monads and reflexive comonads precisely the presheaf comonads.*

*Explanation of the comparison.* The identification on the comonad side is characterised by the equivalence of coalgebras over $\mathsf{Set}$ with $R$-comodels, which determines the comonad from the forgetful functor and its right adjoint. The dual left-hand adjoint is then determined (up to canonical isomorphism) by the adjunction; comparison of the explicit cotensor description $[D, \mathsf{Id}]$ with Garner's costructure construction fixes the orientation. Garner establishes the presheaf-image and idempotence statements [2]. A convention-level expansion of the last comparison, especially for monoid opposites, remains a desirable appendix calculation. □

Here *presheaf monad* is Garner's term for the monad induced by a small category; with the conventions below it is the dependent update monad $U_c = [c, \mathsf{Id}]$. The corresponding comonad is the polynomial (directed-container) comonad

$$cX = \coprod_{x \in \mathrm{Ob}(c)} X^{c[x]}, \qquad c[x] = \{f : x \to y \text{ in } c\}. \tag{20}$$

An explicit calculation is helpful, independently of the classification theorem.

**Proposition 9.2** (Dependent-update comodels)**.** *Let $c$ be a small category and let $U_c = [c, \mathsf{Id}]$. Assuming the duoidal-induced multiplication on $U_c$ is the standard Ahman–Uustalu dependent-update multiplication, comodels of $U_c$ on a set $Q$ are precisely discrete opfibrations $Q \to c$, equivalently copresheaves $c \to \mathsf{Set}$ with total set $Q$. Hence $\{U_c, \mathsf{Id}\} \cong c$, and $U_c$ is reflexive.*

*Proof.* Let $O = \mathrm{Ob}(c)$ and $\Sigma = \prod_{x \in O} c[x]$. Yoneda gives $U_c X \cong \Sigma \times X^O$. A comodel $Q \times U_c X \to X \times Q$ is a pair of maps $o, u : Q \times \Sigma \to O, Q$, acting by

$$(q, \sigma, \xi) \longmapsto (\xi(o(q, \sigma)),\ u(q, \sigma)).$$

The unit is $\eta(x) = (\sigma_{\mathrm{id}}, \mathrm{const}_x)$, and the multiplication, writing $f; g$ for composition, has first component $x \mapsto \sigma(x); \tau^x(\mathrm{cod}\, \sigma(x))$, with continuation $x \mapsto \xi^x(\mathrm{cod}\, \sigma(x))$. Varying independently the continuation functions $\xi^x$ in the comodel multiplication law shows that $o(q, \sigma)$ is independent of $\sigma$; put $p(q) = o(q, \sigma)$. The same law with outer section the identity forces $u(q, \sigma)$ to depend only on $\sigma(p(q))$, so define $f \cdot q = u(q, \sigma)$ when $f = \sigma(p(q))$. Every $f$ extends to a section since each $c[x]$ contains an identity. The remaining equations become

$$p(f \cdot q) = \mathrm{cod}(f), \quad \mathrm{id}_{p(q)} \cdot q = q, \quad (f; g) \cdot q = g \cdot (f \cdot q).$$

These are exactly the action axioms for a copresheaf. Conversely they define a comodel, and morphisms agree. Thus $\mathsf{Coalg}(\{U_c, \mathsf{Id}\}) \simeq [c, \mathsf{Set}] \simeq \mathsf{Coalg}(c)$ over $\mathsf{Set}$, whence the two comonads agree. The duality triangle identity then gives reflexivity of $U_c$. □

Proposition 9.2 states the multiplication-identification qualification explicitly, since the computation requires that identification. The classification is independently supported by Garner's theorem.

**Examples and collapse.** Writer monads, reader monads and state monads belong to the reflexive class, with suitable one-object, discrete and codiscrete categories. In contrast, a theory with a nullary operation has only the empty Set-comodel: a cooperation would require $S \to \varnothing$. Hence exceptions with nonempty exception set and standard finite nondeterminism are non-reflexive; their cosemantics comonad is the initial comonad, and the double dual is the terminal monad. Their exchange map can collapse all distinctions. This illustrates the strong base-dependence of comodel semantics; pointed or partial bases require a new existence analysis.

## 10 Interaction profiles and restricted detection

Define the interaction profile of a capability $D$ by

$$\mathcal{P}_D(T, R) = \mathsf{Meas}(T, D; R) = \mathsf{Comon}(D, \{T, R\}). \tag{21}$$

This is a restricted Yoneda-type test: the probes are just the Sweedler hom-comonads as $T, R$ vary. To determine $D$ from its profile, one needs a suitably coherent family of bijections, not mere equalities of cardinalities at each $(T, R)$.

**Coarse probes.** Let $T_P$ be the free monad on $PX = \coprod_{i \in I} X^{A_i}$. At residual $R = \mathsf{Id}$, the cofree-comonad adjunction gives

$$\mathcal{P}_D(T_P, \mathsf{Id}) \cong \mathsf{Nat}\Big(D, \prod_{i \in I}(A_i \times -)\Big) \cong \prod_{i \in I}(\mathsf{Set}(D1, A_i) \times \mathsf{Nat}(D, \mathsf{Id})). \tag{22}$$

Each such test sees only limited functorial data. This observation alone does not prove that all identity-residual probes fail jointly. For the unary writer demand $T = \mathbb{N} \times -$, the generic-operation description simplifies to $\mathcal{P}_D(T, R) \cong \mathsf{Nat}(D, R)$; varying the residual therefore adds substantial probes.

**Corollary 10.1** (Exchange formula for profiles)**.** *Under Proposition 8.1, if $R$ is reflexive then there is a bijection, natural in the relevant variables,*

$$\boxed{\mathcal{P}_D(T, R) \cong \mathcal{P}_{R^\circ}(T, D^\vee).} \tag{23}$$

This relates a test of arbitrary $D$ against a reflexive residual to a test of the *polynomial* capability $R^\circ$ against residual $D^\vee$. It does not say that $D$ itself is reflexive, or that the exchanged realisation categories are equivalent. One must also avoid an unjustified codensity inference: the right-hand side involves $D^\vee$, so it may itself have lost information about $D$.

**Question 10.2** (Principal detection question)**.** *Does the coherent restricted profile $\mathcal{P}_D(T, R)$, with $R$ ranging only over reflexive (dependent-update) monads, distinguish accessible comonads $D$ up to isomorphism? If not, exhibit non-isomorphic capabilities with identical restricted profiles, and identify a minimal larger class of residual probes that separates them.*

A sharper version asks whether the family $\{\{T, R\} : T$ any monad, $R$ reflexive$\}$ is codense in the category of accessible comonads, or in a restricted category of capabilities. The issue of fullness for the resulting nerve is separate from object separation and faithfulness. In

addition, profiles of *realised* interactions may detect finer data than bare measuring sets; effectful comparisons from Sections 4–5 should then be incorporated, and, relative to a chosen class of standards, the adequacy contexts of Section 6. In the writer case they form a category whose objects are action-category cocycles and whose morphisms are equivariant state maps equipped with natural transformations. To turn such a construction into a profile of a particular capability $D$, however, one must retain its measuring map $D \to \{T, R\}$ and $D$-coalgebra realisation: the effectful runner category alone forgets $D$.

## 11 Towards dependent heterogeneous measuring

The three-place viewpoint suggests a possible entry point to dependent Sweedler theory. For each context $\Gamma$, one may first seek a family

$$\mathsf{Meas}_\Gamma(T, D; R)$$

of suitably indexed measurings, with substitutions $\sigma : \Delta \to \Gamma$ acting by reindexing. Because commands and residuals may be transported differently in slices, one should *not* posit a single strict equality $\sigma^*\mathsf{Meas}_\Gamma = \mathsf{Meas}_\Delta$ without specifying the transported theories and a comparison map. A tentative target is a base-change transformation between measuring relations, subject to pseudofunctorial coherence and Beck–Chevalley conditions in appropriate pullback squares. One may then ask separately which of the three representing objects $D \triangleright T$, $[D, R]$, and $\{T, R\}$ exist fibrewise, and whether their base-change comparison morphisms are invertible, merely lax, or absent. This proposal is a *research question*, not a construction of dependent Sweedler theory. In particular, its variance data, relevant transport functors, and interaction with effectful state comparisons need to be established. It avoids unnecessarily imposing at the outset a single indexed duoidal ambient in which all three representations commute strictly with substitution.

## 12 Research priorities and limits of the analogy

Two principal research problems remain, joined by a third arising from Section 6. The first is to extend the writer effectful horizontal calculus of Proposition 5.3 to more general monads, preserving interchange, tight monad-map squares, explicit capability factorisations, and the duoidal interchange $\zeta$. The proved writer case suggests isolating the transport property of action-category discrete opfibrations and testing its extension from one-object categories to category actions and dependent-update theories. The second is a detection theorem or counterexample for the restricted profile of Question 10.2; a practical starting point is to compare writer, reader, exception, and powerset residuals on small polynomial capabilities. A third problem is Question 6.7: the behaviour of adequacy contexts under runner composition, including the $V$-valued case. Because effectful comparisons are exactly simulations relative to all standards (Proposition 6.5), progress on it should bear directly on Question 4.1. A possible route to a logic of adequacy follows the frame-to-algebra construction of [18]. There, lower sets with Day convolution, which preserves duoidality, are followed by $+$-closure and the Chu construction. Here one would take runners up to effectful comparison as a frame, with sequential composition as the sequence connective and pooling of capabilities as the parallel one. Lower sets of runners would then carry the instance-level residuals of Section 6.6, and the extents of triadic adequacy concepts would live there. Two obstacles are known. Runner composition is typed, so a promonoidal, many-sorted version of the construction is needed. Monotonicity of composition in comparisons is known only in the writer case. Within this frame, intermediate theories built from the goal, namely the cotensors $[D_{i+1}, [\cdots [D_n, R]]]$

of Proposition 6.8, resemble the analytic intermediates of a cut-free derivation, while an intermediate imposed by a library resembles a cut formula. Whether this resemblance supports anything like cut elimination is open. Similarly, Atkey and Kokke extend the Chu construction to a self-dual sequence connective. It is a question, not a claim, whether the capability–residual exchange of Section 8 could supply dualising structure of that kind on the reflexive locus. In parallel, the conventions identifying Garner's costructure with $[D, \mathsf{Id}]$, and the precise induced multiplication for dependent updates, should be checked against the duoidal formulas.

**Triality as a diagnostic only.** The three measuring transposes in (3) are an ordinary two-variable adjunction, not a cyclic three-way duality. Species (one comonad versus two monads), polarity (two inputs versus one output), and transpose position yield different $1 + 2$ splittings. The exchange in Proposition 8.1 is a more concrete, restricted symmetry, but gives no $S_3$ action. A Chu or polycategorical symmetrisation would require extra dualising structure not supplied here. The heterogeneous presentation isolates the representability structure needed for the present questions without assuming an additional cyclic duality. A genuinely three-place structure, with triadic Galois connections in Guitart's sense, does appear one level down. It relates instances, namely demands, realised states and residual standards, through the adequacy relation of Section 6. Its three-placeness is that of a triadic context, not a symmetry of the theories $(T, D, R)$.

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