Graded modal logics

Past, present, and a possible future

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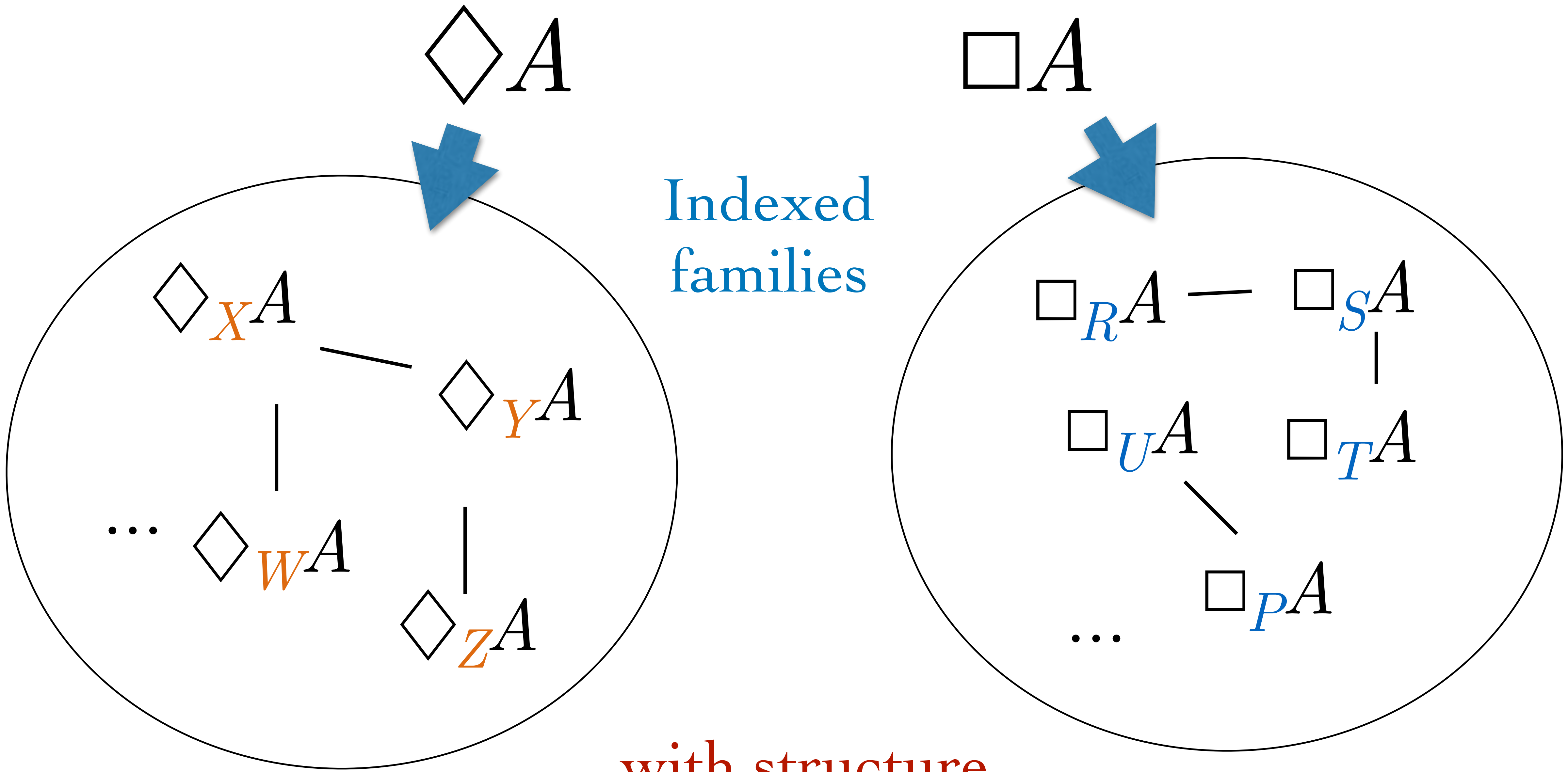
Graded modalities (informally)



with structure

matching the shape of proofs/programs or a semantics

The past

Gradation and degrees (1970s)

- Quine (1953) "Three grades of modal involvement"
- Goble (1970) "Grades of modality"
  - distinguish propositions by "*degrees or grades of modality*"
- Fine (1969/1972) "In So Many Possible Worlds"
  - quantitative spectrum between $\exists$ and $\forall$ in Kripke models

Grades of Modality (Goble 1970) $\mathscr{L}_k$

Indexed family of necessity modalities $\{N_i A\}_{i \in [1..k]}$

e.g. $N_2 A$ "*more*" necessery than $N_1 A$

- Gradation $(j \leq i) \;\Rightarrow\; N_i A \to N_j A$
- K $N_i(A \to B) \to N_i A \to N_i B$
- T $N_i A \to A$
- Nec $\vdash A$ **implies** $\vdash N_i A$

Semantics: based on indexed sets of formulas

In So Many Possible Worlds (Fine 1972)

Indexed family of necessity modalities $\{\Diamond^{\geq i} A\}_{i \in |\mathscr{W}|}$

A is true in at least i worlds

- Gradation $(j \leq i) \Rightarrow \Diamond^{\geq i} A \rightarrow \Diamond^{\geq j} A$
- Base $\Diamond^{\geq 0} A \leftrightarrow \mathsf{T}$
- (???) $\Diamond^{\geq k} A \leftrightarrow \bigvee_{i=0}^{k} (\Diamond^{\geq i}(A \wedge B) \wedge \Diamond^{\geq k-i}(A \wedge \neg B))$

Semantics: Kripke +

$$w \vDash \Diamond^{\geq i} A \ \textbf{ iff } \ |\{v \in \mathscr{W} \mid w R v \wedge v \vDash A\}| \geq i$$

General: pre-order graded modality

- Indexing set I
- Indexed family of (K-)modalities $\{F_i : \mathbb{P} \to \mathbb{P}\}_{i \in I}$ $F_i(A \to B) \to F_i A \to F_i B$
- Pre-order $(\leq) \subseteq I \times I$
- Gradation $(j \leq i) \;\Rightarrow\; F_i A \to F_j A$

Compact definition: *contravariant functor (to closed-endofunctor category)*

$$F : (I, \preceq)^{\mathsf{op}} \to [\mathbb{P}, \mathbb{P}]$$

Subexponentials (Danos et al. 1993 ... Nigam+Miller 2009)

Indexed family of exponential modalities $!^a$

$a \in I$ + structure $\langle I, \preceq, W, C \rangle$

$$\frac{\Gamma, !^a A, !^a A \vdash B \quad a \in C}{\Gamma, !^a A \vdash B} \qquad \frac{\Gamma \vdash B \quad a \in W}{\Gamma, !^a A \vdash B}$$

$$\frac{!^{a_1}A_1, \ldots, !^{a_n}A_n \vdash B \quad a \preceq a_i}{!^{a_1}A_1, \ldots, !^{a_n}A_n \vdash !^a B} R \qquad \frac{\Gamma, A \vdash B}{\Gamma, !^a A \vdash B} L$$

Subexponentials are pre-order graded modalities

$! : (I, \preceq)^{\mathsf{op}} \rightarrow [\mathbb{P}, \mathbb{P}]$

with gradation

$$R\frac{L\dfrac{\overline{A \vdash A}}{!^i A \vdash A}}{!^i A \vdash !^j A} j \leq i$$

Bounded Linear Logic (Girard, Scedrov, Scott 1992)

$\mathbb{N}$-indexed family of modalities $!_n A$ meaning A is used at most n times

$$\frac{!_{r_1}A_1,\ \ldots,\ !_{r_n}A_n \vdash B}{!_{s*r_1}A_1,\ \ldots,\ !_{s*r_n}A_n \vdash\ !_s B}\,pr \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\,der$$

$$\frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\,contr \qquad \frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\,weak$$

Axiomatically:

$!_r (A \to B) \to\ !_r A \to\ !_r B$

gradation $\quad !_s A \to\ !_r A \quad where\ r \leq s$

pr $\quad !_{r*s} A \to\ !_r !_s A$

$!_1 A \to A \quad$ der

contr $\quad !_{r+s} A \to\ !_r A \wedge !_s A$

$!_0 A \to \top \quad$ weak

Recent past

Coeffects (2013/14)

Idea: generalise BLL to an arbitrary semiring

$\Box_r A \;\; \textit{where} \; r \in (R, *, 1, +, 0)$

Axioms:

$$\Box_{r*s} A \to \Box_r \Box_s A$$

$$\Box_0 A \to 1$$

$$\Box_1 A \to A$$

$$\Box_{r+s} A \to \Box_r A \wedge \Box_s A$$

$$\Box_r (A \to B) \to \Box_r A \to \Box_r B$$

$$\Box_s A \to \Box_r A \quad \textit{where } r \leq s$$

Result: semiring graded necessity

Model: exponential graded comonad

(2013) Petricek, O, Mycroft - Coeffects: Unified Static Analysis of Context-Dependence
(2014) Ghica, Smith - Bounded linear types in a resource semiring
(2014) Brunel, Gaboardi, Mazza, Zdancewic - A Core Quantitative Coeffect Calculus

Coeffects: Linear types with semiring-graded necessity

$$\text{ax}\ \frac{}{x : A \vdash x : A} \qquad \text{abs}\ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B} \qquad \text{app}\ \frac{\Gamma \vdash t : A \to B \quad \Delta \vdash t' : A}{\Gamma + \Delta \vdash t\, t' : B}$$

Resource accounting: $r, s \in (\mathcal{R}, *, 1, +, 0, \leq)$

Propositions: $A, B ::= A \to B \mid \Box_r A \mid c$

Contexts: $\Gamma, \Delta ::= A, \Gamma \mid \underbrace{\boxdot_r A}, \Gamma \mid \cdot$

non-linear variable (discharged modality)

Contraction:

$(x : A, \Gamma) + (x : A, \Delta)$ *is ill-formed*

$$(x : \boxdot_r A, \Gamma) + (x : \boxdot_s A, \Delta) \quad = \quad x : \boxdot_{r+s} A, (\Gamma + \Delta)$$

Weakening: $$\frac{\Gamma \vdash t : A}{\Gamma, \boxdot_0 \Delta \vdash t : A}$$

Coeffects: Linear types with semiring-graded necessity

Shift linear variable to modal: **(derelection)**

$$\text{der}\frac{\Gamma, x : A \vdash t : B}{\Gamma, x : \boxdot_1 A \vdash t : B}$$

Propagate grading: **(promotion)**

$$\text{pr}\frac{\boxdot\Gamma \vdash t : B}{r * \boxdot\Gamma \vdash [t] : \Box_r B}$$

Composition/cut:

$$\text{let}\Box\frac{\Gamma \vdash t_1 : \Box_r A \qquad \Delta, x : \boxdot_r A \vdash t_2 : B}{\Gamma + \Delta \vdash \textbf{let } [x] = t_1 \textbf{ in } t_2 : B}$$

Coeffects: A Calculus of Context Dependence [Petricek, O, Mycroft, '14]

A core quantitative coeffect calculus [Brunel et al.'14]

Possibility/monads

$\Diamond A \quad := \quad A$ "possibly" true, A true in some worlds

S4 axioms

	$\Diamond(A \to B) \to \Diamond A \to \Diamond B$	*distributivity*
	$A \to \Diamond A$	*reflexivity*
	$\Diamond\Diamond A \to \Diamond A$	*transitivity*

Natural deduction

$$intro\ \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A} \qquad cut\ \frac{\Gamma \vdash \Diamond A \quad \Gamma, A \vdash \Diamond B}{\Gamma \vdash \Diamond B}$$

Type theory/programming

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \mathbf{pure}\ e : \Diamond A} \qquad \frac{\Gamma \vdash e_1 : \Diamond A \quad \Gamma, x : A \vdash e_2 : \Diamond B}{\Gamma \vdash \mathbf{let}\ \Diamond x = e_1\ \mathbf{in}\ e_2 : \Diamond B}$$

Graded possibility/monads

$\Diamond_x A \qquad x \in (X, \oplus, I)$ is a monoid

S4 axioms

$$\Diamond_x (A \to B) \to \Diamond_x A \to \Diamond_x B \qquad \textit{distributivity}$$

$$A \to \Diamond_I A \qquad \textit{reflexivity}$$

$$\Diamond_x \Diamond_y A \to \Diamond_{x \oplus y} A \qquad \textit{transitivity}$$

Natural deduction

$$intro\ \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond_I A} \qquad cut\ \frac{\Gamma \vdash \Diamond_x A \quad \Gamma, A \vdash \Diamond_y B}{\Gamma \vdash \Diamond_{x \oplus y} B}$$

Type theory/programming

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \mathbf{pure}\ e : \Diamond_I A} \qquad \frac{\Gamma \vdash e_1 : \Diamond_x A \quad \Gamma, v : A \vdash e_2 : \Diamond_y B}{\Gamma \vdash \mathbf{let}\ \Diamond v = e_1\ \mathbf{in}\ e_2 : \Diamond_{x \oplus y} B}$$

2008 Durov - New approach to Arkelov geometry

Smirnov - Graded monads and rings of polynomials

2013 Petricek, O, Mycroft - Coeffects: Unified Static Analysis of Context-Dependence

2014 Ghica, Smith - Bounded linear types in a resource semiring

Brunel, Gaboardi, Mazza, Zdancewic - A Core Quantitative Coeffect Calculus

Katsumata - Parametric effect monads and semantics of effect systems.

O, Petricek, Mycroft - The semantic marriage of effects and monads

Petricek, O, Mycroft - Coeffects: a calculus of context-dependent computation.

2016 Gaboardi, Katsumata, O, Breuvart, Uustalu - Combining effects & coeffects via grading

2019 O, Liepelt, Eades - Quantitative program reasoning with graded modal types

The essence of graded modalities

- 1970s/subexpontentials: pre-ordered graded K-modalities
- BLL: $\mathbb{N}$-semiring graded necessity
- Coeffects: (Pre-ordered) semiring graded necessity
- Effects/graded monads: monoid graded possibility

Indexed family with index structure matching proof rules

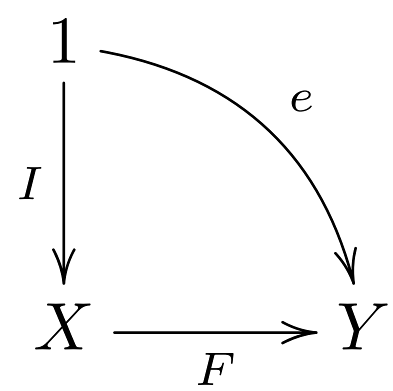
↓

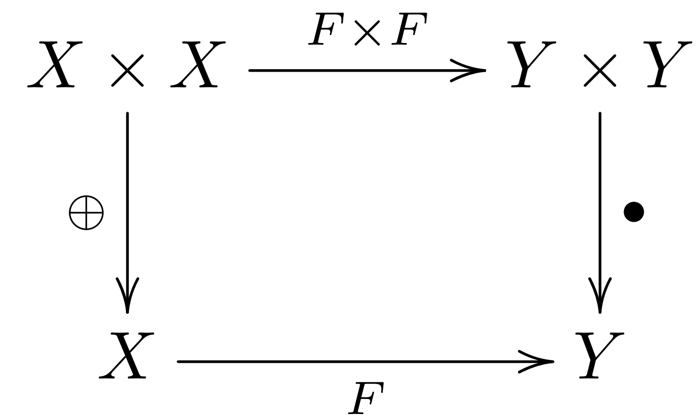
a homomorphism, i.e., a functor

$$F : I \to [\mathbb{P}, \mathbb{P}]$$

The essence of graded modalities

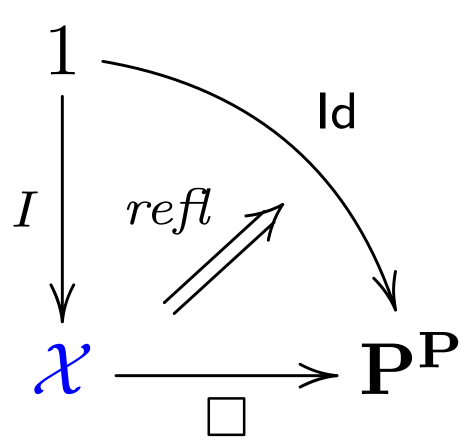
Recall, monoid homomorphism $(X, \oplus, I) \xrightarrow{F} (Y, \bullet, e)$

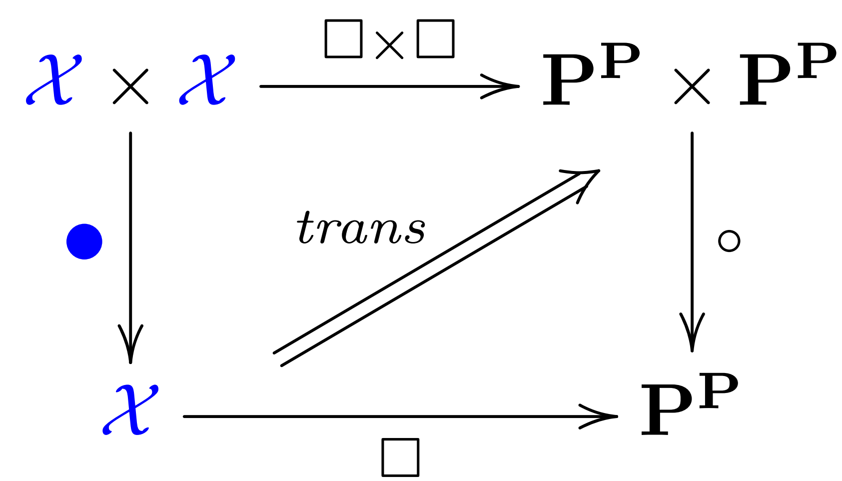




$(\mathcal{X}, \bullet, I)$-graded necessity is a **lax** monoid homomorphism

$$(\mathcal{X}, \bullet, I) \xrightarrow{\Box} (\mathbf{P}^{\mathbf{P}}, \circ, \mathsf{Id})$$





$\Box_I A \xrightarrow{refl} A$ $\qquad$ $\Box_{x \bullet y} A \xrightarrow{trans} \Box_x \Box_y A$

Combining Effects and Coeffects via Grading [Gaboardi, Katsumata, O, Uustalu, Breuvart.'16]

The present

Granule (O, Liepelt, Eades 2019)

Quantitative program reasoning with graded modal types, ICFP 2019

Track information
for quantitative reasoning

Goal: extensible

Graded modal types ⇄ SMT solver (e.g. Z3)

Be strict about
"data as a resource"

Linear types

Precision

Goal: full dependent types

Indexed types

https://granule-project.github.io/

Grades of Modality (Goble 1970) $\mathscr{L}_k$

Indexed family of necessity modalities $\{N_i A\}_{i \in [1..k]}$

- Gradation $(j \leq i) \Rightarrow N_i A \to N_j A$
- 4 $N_i A \to N_i N_i A$
- T $N_i A \to A$

Pre-ordered graded necessity

Ordering induces $([1..k], \sqcup, 1)$-monoid graded necessity

$$N_{i \sqcup j} A \to N_i N_j A$$
$$N_1 A \to A$$

In So Many Possible Worlds (Fine 1972)

$\Diamond^{\geq i} A$ A is true in at least i worlds

- Gradation $(j \leq i) \Rightarrow \Diamond^{\geq i} A \rightarrow \Diamond^{\geq j} A$
- Model: $w \vDash \Diamond^{\geq i} A$ **iff** $|\{v \in \mathscr{W} \mid w\, R\, v \wedge v \vDash A\}| \geq i$

Pre-ordered graded necessity

- Transitive and reflexive accessibility R

$$\Longrightarrow ([1..k], *, 1)\text{-monoid graded possibility}$$

$$\Diamond^{\geq i} \Diamond^{\geq j} A \rightarrow \Diamond^{\geq i*j} A$$

$$A \rightarrow \Diamond^{\geq 1} A$$

Other graded modal logics?

Temporality

Structure: $(\mathbb{N}, +, 0)$

Family: $\Box_n A = X^n A$ (from linear temporal logic)

Axioms:

$$\Box_{r+s} A \to \Box_r \Box_s A$$

$$\Box_0 A \to A$$

$$\Box_r (A \to B) \to \Box_r A \to \Box_r B$$

Constructivity / provability

relation to proof relevance?

Structure: $(\{\bot, \top\}, \wedge, \top)$

Family: $\Box_\bot A \triangleq$ true without proof (classical)

$\Box_\top A \triangleq$ true with proof (constructive)

(some) Axioms:

$$\Box_r (\neg\neg A) \to \Box_\bot A$$

$$\Box_\top A \to A$$

cf. explicit provability logics (Artemov '95, '01)

The future?

Consider

$$\mathsf{lem} : \forall(n : \mathbb{N}) \,.\, \forall(m : \mathbb{N}_{\leq n}) \,.\, (n + m \leq n + n)$$

where:

lem consumes its n and m linearly

but:

$$+ : \mathbb{N} \multimap \mathbb{N} \multimap \mathbb{N}$$

consumes its arguments linearly

consumes n three times in lem proposition

m also depends on n (linearly)

How many times does n get used?

Reconciling linearity and dependent types

Dual context approach:

(can depend only on Γ)

$$\Gamma \mid \Delta \vdash t : A$$

Cartesian context (cannot depend on Δ) ↗

↖ Linear context (can depend on Γ)

e.g. $n : \mathbb{N} \mid x : \mathsf{Fin}(n) \vdash x : \mathsf{Fin}(n)$

but $n : \mathbb{N} \mid x : \mathsf{Fin}(n) \nvdash (x, \mathsf{refl}) : \Sigma y : \mathsf{Fin}(n) \,.\, y \equiv x$

Quickly get into trouble: mostly fall-back to non-linear

A Linear Dependent Type Theory (Luo, Zhang, 2016)
A Categorical Semantics for Linear Logical Frameworks, (Vakar, 2015)
Integrating Linear and Dependent Types, (Krishnaswami et al., 2015)
Dual Intuitionistic Linear Logic (Barber, 1996)

Reconciling linearity and dependent types (2)

Quantitative Type Theory

$$x_1 :^{\rho_1} A_1, \ \dots \ , x_n :^{\rho_n} A_n \vdash t : B$$

$\rho_1, \dots, \rho_n$ mark number of usages (cf. BLL)

0 denotes (potential) use at types (contemplation) but not terms (computation)

e.g. $n :^{0} \mathbb{N}, x :^{1} \mathsf{Fin}(n) \vdash x : \mathsf{Fin}(n)$

but doesn't allow clear reasoning about type-level computation

Syntax and Semantics of Quantitative Types (Atkey, 2018)
I Got Plenty o' Nuttin' (McBride, 2016)

Roadmap

- Capture more examples from the literature (in Granule)
- Develop GMDTT and its implementation (Granule 2.0)
- Internally-extensible dependent graded modal type system

Other instances of graded modalities

- Contextual Model Type Theory (Nanevski et al. '08)

  $\Box_{\Gamma} A$ – A is true under closure of Γ

- Hardware schedules (Ghica et al. '14)
- Explicit provability logics (Artemov '95, '01)
- Multi-stage programming (generalising Pfenning & Davies)
- Costs (cf. Cicek et al. 17)
- Robustness / sensitivity (Gaboardi et al., Pierce et al.)
- Provenance
- Probabilistic programming (forwards / backwards)
- Type state (stateful protocols)

The Granule Project

https://granule-project.github.io/

Thanks!