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Duality as a category-theoretic concept

David Corfield

Department of Philosophy, SECL, University of Kent, Canterbury CT2 7NZ, UK

A R T I C L E I N F O

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A B S T R A C T

In a paper published in 1939, Ernest Nagel described the role that projective duality had played in the reformulation of mathematical understanding through the turn of the nineteenth century, claiming that the discovery of the principle of duality had freed mathematicians from the belief that their task was to describe intuitive elements. While instances of duality in mathematics have increased enormously through the twentieth century, philosophers since Nagel have paid little attention to the phenomenon. In this paper I will argue that a reassessment is overdue. Something beyond doubt is that category theory has an enormous amount to say on the subject, for example, in terms of arrow reversal, dualising objects and adjunctions. These developments have coincided with changes in our understanding of identity and structure within mathematics. While it transpires that physicists have employed the term 'duality' in ways which do not always coincide with those of mathematicians, analysis of the latter should still prove very useful to philosophers of physics. Consequently, category theory presents itself as an extremely important language for the philosophy of physics.

1. Introduction

Phenomena covered by the term *duality* have long fascinated mathematicians. While the classification of the five platonic solids is recorded in Book XIII of Euclid's Elements, in what is sometimes called 'Book XV', but believed to be written much later in the 6th century AD by Isidore of Miletus, or perhaps his student, a cube is inscribed in an octahedron and an octahedron inscribed in a cube. This pattern continues, of course, to the other Platonic solids, where the dodecahedron and icosahedron are found to be dual to each other, and the tetrahedron self-dual.

By the middle of the nineteenth century, various 'algebraic' approaches to logic had been developed, and it had been observed that a logical duality obtained on switching propositions and their negations at the same time as switching 'and' and 'or'. For example, De Morgan duality asserts that

- $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$,
- $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$.

Meanwhile in analysis it had been found that problems involving solutions to differential equations could be transformed by Fourier analysis, where the transform of a product of functions is equal to the convolution of the individual transforms, and the transform of the convolution of two functions is the product of the individual transforms.

However, the pinnacle of the nineteenth century interest in duality was reached with projective duality in geometry. Texts would be laid out in parallel columns showing the proofs of dual theorems, with the necessary exchange of 'point' and 'line', 'collinear' and 'concurrent', and so on. For example, we have the following dual theorems, attributed to Pascal and Brianchon:

- Given a hexagon inscribed in a conic section, each of the three pairs of opposite sides determines a point, and these three points are collinear.
- Given a hexagon circumscribed on a conic section, each of the three pairs of opposite vertices determines a line, and these three lines are concurrent.

Duality also came to fascinate physicists through this century. Maxwell understood topics in optics from the perspective of projective geometry, but a more significant manifestation appeared in electromagnetism. Already Faraday had seen that one could anticipate new phenomena by the interchange of electric and magnetic terms. If a fluctuating magnetic field could produce a current in a wire, a fluctuating current should move the needle of a

E-mail address: d.corfield@kent.ac.uk

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nearby compass, indicating a generated magnetic field. This duality was present in Maxwell's own equations for electromagnetism in a vacuum, which reveal invariance under the exchange $E \to B$ and $B \to -E$.

What is notable about these initial appearances of duality is their tendency to broaden, deepen and merge. The duality of the Platonic solids would lead to dual complexes in Poincaré's *analysis situs*, and hence to Poincaré duality, relating aspects of a space in complementary dimensions. Logical duality would lead to inversion of order structures such as lattices, and there merge with similar ideas coming from projective geometry. Pontrjagin dual groups would later be devised to understand Poincaré and other dualities in algebraic topology, and in turn would explain the duality of Fourier analysis. Meanwhile in physics, in his theory of special relativity, Einstein would exploit Maxwellian symmetry which would come to be understood as electric–magnetic duality. In the 1920s Fourier analysis was seen to underlie wave-particle duality of quantum mechanics via the transformation between position space and momentum space. Later in 1931, Dirac seeking a quantum version of electromagnetism was led by electric–magnetic duality to predict the existence of magnetic monopoles.

Right up to the present day, mathematicians' and physicists' fascination with duality shows no sign of abating, from the pure realm of number theory to theoretical physics. For example, we hear that

> It has long been suspected that the Langlands correspondence is somehow related to various dualities observed in quantum field theory and string theory. Both the Langlands correspondence and the dualities in physics have emerged as some sort of non-abelian Fourier transforms. (Geometric Langlands Program Project, 2007)

This is part of an intense interaction between theoretical physicists, mathematical physicists and pure mathematicians, in particular work in the field of 'geometric representation theory'. Analogies between number theory and quantum field theory are widespread, resting on such observations as Michael Atiyah's from the 1970s that the Montonen–Olive dual charge group coincides with the Langlands dual group, and leading to Witten and Kapustin's identification of one side of homological mirror symmetry with one side of the categorical Langlands correspondence, itself understood as a consequence of S-duality (see Frenkel, 2009). Dualities lie at the core of each side of the analogy.

If cutting-edge physics and mathematics have converged on similar structures, what might philosophers of each discipline achieve if they bring their respective backgrounds to think about manifestations of duality? Philosophers of physics have a long-standing interest in situations where two apparently different theories deliver the same empirical predictions. While with gauge equivalent theories it does not seem unreasonable to treat them as variations of the 'same' theory, this appears less plausible in the case of dual string theories. Since mathematics treats dualities between apparently different kinds of mathematical entity, we might expect philosophers of mathematics to be able to be of some service here. However, a search through *The Oxford Handbook of Philosophy of Mathematics and Logic* (Shapiro, 2005) reveals that the phenomenon of duality has made very little impression on the discipline in the Anglophone world. On the other hand, from the perspective of the *philosophy of mathematical practice* (see Mancosu, 2008), if we are to describe the nature of current mathematics, such a central, thematic concept as duality deserves treatment, and, together with Ralf Krömer, I have begun this task (Krömer & Corfield, 2014). That we have an audience in the philosophy of physics should give us great encouragement.

At the very least, from the mathematical side there should be some attempt to convey what kind of thing mathematical duality is, whether it is a circumscribable concept about which it may be possible to forge a general mathematical theory, or rather a much looser, family resemblance kind of notion. A glance at *The Princeton Companion to Mathematics* entry for *duality* may incline us to the latter viewpoint:

> Duality is an important general theme that has manifestations in almost every area of mathematics … Despite the importance of duality in mathematics, there is no single definition that covers all instances of the phenomenon. (Gowers, Barrow-Green, & Leader, 2008, p. 187)

So does mathematical duality shape up to be an exhaustively definable concept, or will it retain an elusive quality, which allows it to manifest itself from time to time in Protean fashion in different portions of mathematics? Well, even if not exhaustible, there is already a theoretical framework in which it is possible to draw together much of what is designated as duality. That framework is provided by category theory, and a major thrust of this paper is to support the idea that the ability to formulate results at such a high level of generality indicates how category theory may provide indispensable insights into the subject matter of mathematics. Set theoretic resources are far too weak in this regard.

Category theory will also provide insight into another notable aspect of the mathematical treatment of duality. While many early forms that we have seen related things of a similar nature – points–lines, functions–functions, groups–groups, logical expressions–logical expressions – later dualities expanded to allow different kinds of entity to be related: theories–models, spaces–quantities. Some have looked to subsume these different faces under the so-called 'Isbell duality' which governs many relationships between geometry and algebra.

As we proceed, we will have to come to understand the differences between physicists' and mathematicians' uses of the word 'duality'. It transpires that these diverge considerably, and yet this should not stand in the way of a dialogue. On the one hand, there is interesting physics to be found employing genuine mathematical duality, while on the other, even if on occasions a case of physical duality is better described as a case of mathematical *equivalence*, we should find that the constructions I describe here are still useful. In particular, there are indications of a close resemblance between string dualities and the so-called 'Morita equivalence' (see Okada, 2009). In his paper, Morita (1958) treated both equivalences ('isomorphisms') of module categories, but also 'dualities' of such categories. They arise through similar constructions where a 'bimodule' mediates between two settings.

I will return to this matter below. First, however, let us set the scene to see what philosophy has had to say about duality in mathematics until now.

2. Philosophers on mathematical duality

To date philosophers have found surprisingly little to say about this feature of mathematics, especially in the Anglophone world. One notable exception was Ernest Nagel who in his 1939 paper explained how the discovery of duality in projective geometry liberated mathematics from the idea that it was dealing with specific elements bearing a set of defining properties. Before we come to look more closely at this paper, it is worth noting that Nagel is dealing here merely with one episode in the history of mathematics' treatment of duality, an episode that had run its course decades earlier. With the further advantage of hindsight three-quarters of a century after Nagel, we should expect new issues to have arisen.

The key to understanding Nagel's paper is to recall that he was a full-blooded logical empiricist. As such he wanted to use the history of mathematics to support some of that doctrine's tenets: the reduction of mathematics to logic, the denial of geometry as *synthetic a priori*, the sharp separation of mathematical from physical geometry, etc.

> It is a fair if somewhat crude summary of the history of geometry since 1800 to say that it has led from the view that geometry is the apodeictic science of space to the conception that geometry, in so far as it is part of natural science, is a system of “conventions” or “definitions” for ordering and measuring bodies.
> The object of the present essay is to trace in part the development of the shift in point of view just indicated. This change owes next to nothing to the speculations of professional philosophers and logicians, and is the outcome of technical needs and advances of mathematics proper. Nevertheless, it has had a profound influence upon modern conceptions of logic and methodology … (Nagel, 1939, p. 143)

We find here then a justification of Nagel's beliefs about the place of logic within mathematics as arising from the internal struggles of the field. Philosophy is to come to hold such views through a study of the practice of mathematics, a lesson that was largely forgotten over succeeding decades.

Regarding geometry in particular, he claims

> The liberation of geometrical terms from their usual but narrow interpretation first required a thoroughgoing denial of the need for absolute simples as the foundation for a demonstrative geometry. Such a liberation was in large measure the consequence of the discovery of the principle of duality and of the manifold extensions and applications which were made of it. (Nagel, 1939, p. 179)

So where Euclid saw the need to define the point (“that which has no parts”) and the straight line (“that which lies evenly on its points”), Nagel observes that we need not see geometry as resting on any such ‘absolute simples’. The possibility of interchanging ‘point’ and ‘line’ while retaining truth indicates that mathematical geometry is not the study of some specific spatial entities, but is merely the investigation of a body of theories given in the (then) modern axiomatic way.

His paper culminates in three theses:

- “The distinction between a pure and an applied mathematics and logic has become essential for any adequate understanding of the procedures and conclusions of the natural sciences.”
- “Familiarity with the techniques of implicit definitions of terms and the method of their constructive explication is of equal importance for comprehending scientific method and contemporary discussions of it.”
- “And the concepts of structure, isomorphism, and invariance, which have been fashioned out of the materials to which the principle of duality is relevant, dominate research in mathematics, logic, and the sciences of nature.” (Nagel, 1939, p. 217)

From the logical empiricist viewpoint, philosophers of physics would appear to have a more interesting role to play since there is the important task of understanding the *coordinating principles* relating these structures to the world. On the other hand, even accepting Nagel's division of labour, there would have been gainful employment for philosophers of mathematics to think hard about his third point. Unfortunately, this chance was lost by the continued failure to keep abreast of the most important developments in theorising about the “concepts of structure, isomorphism, and invariance.” Arriving at an account of mathematical duality as currently treated forces us back to this task.

Turning briefly away from the Anglophone world, I should mention a French attempt to deal with duality, namely, in the work of Albert Lautman. Indeed, I treated this case in a paper which discusses his work (Corfield, 2010). Lautman was working at the same time as Nagel, but threw himself into the intricacies of contemporary mathematics. Unlike Nagel, he believed mathematics is about something, or rather that there is something ‘above’ mathematics which realises itself in the unfolding of mathematical theories, and elsewhere. The main point of my essay was to remark that there is no need to see these high level ideas as existing outside of mathematics. Since the time of Lautman's tragic early death, and due in no small part to the efforts of many of the mathematicians he knew, a mathematical language did emerge which could treat these high-level thematic ideas. Let us turn to this new language now.

3. General category theoretic approaches to duality

Before we begin on the category theoretic treatment of duality, it is worth pointing out that from the viewpoint of mathematicians, physicists display a certain looseness when it comes to using terms ‘duality’, ‘symmetry’ and ‘reciprocity’. Sometimes duality is used merely to designate a non-trivial equivalence. Indeed it is possible to understand homological mirror symmetry, a case of T-duality, as simply an equivalence of A_∞-categories (Kontsevich, 1995). In my view it is preferable to retain ‘duality’ for kinds of involution with some form of structural reversal. For example, where one set, A, forms a part of a larger set, B, via an injection, the lattice of subsets of B projects onto that of A. Sets and the kinds of lattice that are formed by their subsets thus present us with a duality in the proper mathematical sense.

Naturally, even within mathematics itself there may be different views on whether something merits the term. The role of the Langlands ‘dual’ ${}^LG$ in the number-theoretic Langlands program is not as symmetric as in the geometric Langlands program, which is in turn governed by the physicists' S-duality, since it essentially serves as an ingredient for the construction of automorphic L-functions from Galois representations in G. In a comment to a message to Sarnak, Langlands (2014), who did not himself coin the phrase ‘dual group’ remarks

> This duality [electric–magnetic duality/S-duality] is quite different than the functoriality and reciprocity introduced in the arithmetic theory [ordinary Langlands]. (Langlands, 2014)

The exchange of root data which govern this operation convinced someone to call this a ‘dual’, where we might prefer the term ‘reciprocal’.

Let us first turn to some indisputable forms of mathematical duality of relevance to physics. These occur when we rely on a pairing $A \times B \to C$, and use maps from B to C to represent elements of A and maps from A to C to represent elements of B. A typical case involves some collection of entities, and then real or other valued functions on that collection. Then we may be able to reconstruct an element of the collection from information of where it is sent by the functions. Cases of this situation include the classic algebraic coordinatisation of a space, where a point is given by a tuple of coordinates. A more elaborate version, where C is allowed to vary, includes the reconstruction of an integer from all of its remainders modulo the primes, an idea at the heart of modern algebraic geometry.

This form of duality is what Shahn Majid calls representational duality, and is what he intends in his claim “as physics improves its structures tend to become self-dual” (Majid, 2012, p. 117). It also underlies the algebra–geometry duality of the following table (nLab, *duality in physics*):

Algebra	**Geometry**
Poisson algebra	Poisson manifold
deformation quantisation	geometric quantisation
algebra of observables	space of states
Heisenberg picture	Schrödinger picture
AQFT	FQFT
Higher Algebra	**Higher Geometry**
Poisson n-algebra	n-plectic manifold
E_n-algebras	higher symplectic geometry
BD-BV quantisation	higher geometric quantisation
factorisation algebra of observables	extended quantum field theory
factorisation homology	cobordism representation

Note that while by default "deformation quantisation" refers to formal deformation quantisation and may appear not to be able to provide a proper dual for geometric quantisation, there is a "full" C^*-algebraic version, although details of the full duality have not been completely worked out. A case which is thought to be amenable to such treatment is the geometric quantisation of a Poisson manifold as the holographic boundary theory of a 2d Poisson–Chern–Simons theory which it seems gives a geometric-dual analogue of the interpretation by Cattaneo and Felder of Kontsevich deformation quantisation as the boundary of the perturbative Poisson sigma-model.[1]

Looking now for a more systematic framework for duality, the first point to make is that you cannot begin to understand its categorical treatment without the fundamental notion of an *adjunction*. Taken in its ordinary and original sense, this is as a weakened form of inverse to a functor between categories. Say I have a functor $F:\mathcal{C}\to\mathcal{D}$. I want to know how the image, $F(x)$, of an object, x, of $\mathcal{C}$, behaves in $\mathcal{D}$ insofar as maps out of it are arranged. Say I have an object, y, in $\mathcal{D}$. Now, arrows from $F(x)$ to y can be understood from the perspective of the original category $\mathcal{C}$ if there is a right adjoint to F, let us say G, which means that $Hom_{\mathcal{D}}(F(x),y)\cong Hom_{\mathcal{C}}(x,G(y))$. It is no accident that this (natural) isomorphism bears a strong resemblance to the defining property of adjoint operators on Hilbert spaces, where if A is a continuous linear operator on a space $\mathcal{H}$, then its adjoint A^* satisfies the equation $\langle Ax,y\rangle=\langle x,A^*y\rangle$, for $x,y\in\mathcal{H}$. Indeed, there is a setting in 'enriched' category theory in which adjoint operators and adjoint functors coincide (Baez, 1996).

A standard example to give now is a free-underlying adjunction between the category of sets and that of some algebraic structure, such as groups:

$$Hom_{Set}(x, Underlying(y))\cong Hom_{Group}(Free(x),y)$$

On the other hand, in the case of the underlying functor from topological spaces to sets, $U:Top\to Sets$, this functor has both a left adjoint, placing the discrete topology on a set, and a right adjoint, placing the codiscrete topology on a set.

Adjunctions are almost everywhere in mathematics, as Saunders Mac Lane told us (1971, p. 103), and are at the heart of the category theoretic view of mathematics.

> Essentially everything that makes category theory nontrivial and interesting beyond groupoid theory can be derived from the concept of adjoint functors. (nLab, *adjunctions*)

In view of the new foundational program known as homotopy type theory (Univalent Foundations Program, 2014), which takes groupoids and their higher versions to be the basic forms of mathematical entities, this is a powerful statement.

Now any adjunction restricted to the *fixed points* of each category results in an adjoint equivalence.[2] An object of a category is fixed by the adjunction if the result of applying the two functors in turn is isomorphic to the original object. This may not yield anything interesting, especially if there are no fixed points as in the free group example above. In the case of the adjoints to the underlying functor of topological spaces, on the other hand, we find equivalences between discrete spaces and sets and between codiscrete spaces and sets.

To bring duality in here, we need an adjunction to operate between one category and the *dual* or *opposite* of another, that is, the category with arrows reversed. Then this will induce a dual equivalence on fixed points. For example, we can see Pontrjagin duality between the category of abelian groups and that of compact topological abelian groups in this way. We will see a method to generate interesting dualities of this kind below.

As often happens in category theory, this construction is found to have a much broader setting. Here the definition of adjunction can be formulated in such a way that it can be 'internalised' within any (weak) 2-category, a device with objects, 1-morphisms between them and 2-morphisms between these (see Lack, 2010, Section 2.1). The ordinary setting is where we consider the 2-category of categories, functors and natural transformations. But there are some simple kinds of (one object) 2-category, which amount to categories with a form of multiplication, a monoidal category. An easy example here is the category of vector spaces, which considered as a 2-category has a single object, morphisms labelled by vector spaces so that composition is equivalent to taking the tensor product, and linear transformations as 2-morphisms. In this setting an object with a right adjoint (which by the symmetry of tensor products here is also a left adjoint) works out to be an object, A, with maps $1\to A\times D(A)$ and $D(A)\times A\to 1$, satisfying relevant equations. This structure is present for a finite vector space and its dual. Note that in the representation of such situations by string diagrams, as employed in Bob Coecke's pictorial approach to quantum mechanics, the relevant maps are cups and caps (Coecke, 2005).

We can tell a similar story at a higher level, namely with monoidal 2-categories. An example of this is the 2-category of categories, profunctors and profunctor morphisms, which parallels the shift from the category of sets and functions to that of sets and relations. A profunctor between two categories, $\mathcal{C}$ and $\mathcal{D}$ is a functor $\mathcal{D}^{op}\times\mathcal{C}\to Set$, similar to the definition of a relation between two sets X and Y as a map $X\times Y\to 2$. In the same spirit as with adjoint duals of vector spaces, we find here that each category has a dual, namely its opposite. The existence of such a dual is a very important feature of categories. Indeed for the category of categories this is the only proper symmetry, or autoequivalence.

One way to generate the kinds of adjunction that lead to interesting dualities is a construction known as the *nucleus of a profunctor* (see Willerton, 2015, Section 3 for details). This yields a dual adjunction between $Set^{\mathcal{C}^{op}}$ and $Set^{\mathcal{D}}$. The profunctor closely resembles the kernel in an integral transform, transforming functions on one side (here presheaves) to functions on the other (here copresheaves), so is very much like a *categorified* Fourier transform.

[1] My thanks to Nick Teh for raising the issue of the extent of the quantisation duality in the table, and to Urs Schreiber for providing me with details for this response (see his MathOverflow answer http://mathoverflow.net/a/135928/447).

[2] Lambek et al. (1982) give a list of examples, illustrating what he calls the 'Hericlitian principle of the unity of opposites'.

To make clearer sense of this construction it is easier to move to what is known as *enriched* category theory. The idea here is that in ordinary category theory the maps between two objects form a set, $Hom_{\mathcal{C}}(x, y) \in Set$. But the definition of a category can easily be modified to allow objects of other categories to be these *Hom*-objects. Monoidal categories with their product to allow composition and a tensor unit to allow an identity fit the bill.

Surprisingly, choosing to enrich with the very simple poset of truth values generates interesting examples. A truth value-enriched category is a poset. Now we can treat sets as discrete posets so that given a relation between two sets, we can set up an adjunction between their power sets. The resulting dualities include famous cases such as

1. *Algebraic geometry*: the classic duality between affine varieties and radical ideals.
2. *Number theory*: The correspondence from Galois theory between intermediate field extensions and subgroups of the Galois group.
3. *Linear algebra*: duality between linear subspaces and annihilators.
4. *Logic*: duality between sets of sentences which are closed under logical consequence and the set of models of the theory.
5. *Convex geometry*: duality between closed convex sets and convex hulls.
6. *Analysis*: lower closed subsets and upper closed subsets, in other words, Dedekind cuts, real numbers (together with $\pm\infty$).[3]

Some of these cases may be better described by relying on the partial order structure of each side. Other than truth values, an important choice is to enrich in *Ab*, the category of abelian groups. For a very nonstandard example, it is possible to enrich in the extended real numbers, where enriched categories here are a kind of (non-symmetric) metric space in which points may be infinitely far apart. In this way Fenchel–Legendre duality can be generated (see Willerton, 2015).

Returning to the standard enriching category *Set*, and the Hom-profunctor for a category, $Hom : (c, d) \mapsto Hom(c, d)$, we have an adjunction between the category of presheaves on the category and the opposite of the category of its copresheaves. Examples of this kind are often referred to as *Isbell conjugates*. If $\mathcal{C}$ is chosen as a category of test spaces, we find here an adjunction between spatial entities probed by members of $\mathcal{C}$, and algebraic things co-probed by them. We can also restrict to subcategories of these presheaves, for instance, functors preserving various limits, or sheaves, often finding dual equivalences.

This brings us close to a notion which goes by various names, including the picturesque 'objects keeping summer and winter homes', where a dualising object, *V*, belongs to two categories, and gives rise to a dual adjunction by taking maps into it in the respective categories.

Many dualities arise from such a *V* being describable in two ways:

- 2 as a space (with discrete topology) and as a Boolean algebra: Stone duality between Stone spaces and Boolean algebras
- $\mathbb{R}/\mathbb{Z}$ as a compact Hausdorff topological abelian group and as a plain abelian group: Pontrjagin duality (Fourier duality a special case).
- A ground field as a vector space and as a linearly compact vector space over itself: Lefschetz duality.

The category *Set* itself possesses a huge amount of structure and so can be seen as belonging to a variety of different 2-categories. For example, it is a category with finite products, and it is also a category with all limits, filtered colimits, and regular epimorphisms. This sets up a logical duality between first-order theories and their collections of models:

$$\text{Models} = \text{Hom}_{fp-Cat}(\text{Algebraic theory, Set})$$
$$\text{Algebraic theory} = \text{Hom}_{lfcre-Cat}(\text{Models, Set})$$

which is established by Forssell in his thesis (2008). He writes there

> …instances of the algebra–geometry duality can be seen to manifest a syntax-semantics duality between an algebra of syntax and a geometry of semantics. (Forssell, 2008, p. 2)

This example is rather like Tannaka–Krein duality (a noncommutative extension of Pontrjagin duality) which allows one to recover a group from its category of representations (and underlying functor to vector spaces).

About this kind of dualising object, Lawvere and Rosebrugh wrote

- *Formal* duality concerns mere arrow reversal in the relevant diagrams.
- *Concrete* duality, on the other hand, occurs in situations where a new diagram is formed from an old one by exponentiating each object with respect to a given dualising object, e.g., *X* becomes V^X, with *V* being the dualising object. The arrows are naturally reversed in the new diagram.

They continue

> Not every statement will be taken into its formal dual by the process of dualising with respect to *V*, and indeed a large part of the study of mathematics
>
> space vs. quantity
>
> and of logic
>
> theory vs. example
>
> may be considered as the detailed study of the extent to which formal duality and concrete duality into a favorite *V* correspond or fail to correspond. (Lawvere & Rosebrugh, 2003, p. 122).

As mentioned in the introduction, there are commonalities between the setting of some of the duality constructions we have seen and that of what is known as Morita equivalence. At the heart of each is something playing a mediating role between two worlds, whether dualising object or bimodule.

Finally, on the mathematical side, I would like to mention another favourite topic of Lawvere's. We saw above that the underlying functor from topological spaces to sets had an adjoint on each side. This induces an adjoint pair on *Top*. Such induced adjoint pairs generate a range of dual constructions: product/coproduct, universal/existential quantifiers, idempotent monads and comonads (representing modalities such as possibility and necessity), etc. Ways of capturing geometric forms of space through the concept of 'cohesion' involve a chain of four adjoints, so inducing a triple of adjoints on a category (Corfield, forthcoming; Schreiber, 2013), and are suggesting ways to see why there is something 'geometric' and even 'differential' going on in number theory, as the Langlands Program suggests.

4. Philosophical reflections and conclusions

We moved from a network of instances of duality in the introduction to a network of categorical constructions that cover most examples of interest. All constructions revolve around the

[3] Thanks to Willerton (2013) for these examples.

fundamental concept of adjunction, and yet there is unlikely to be a single monolithic account of duality. Be that as it may, we should note the power of the resources of category theory to capture such a high level concept. Someone who cares only for the ability of a 'foundational' language to capture 'in principle' all pieces of mathematical reasoning may reassure themselves that set theory could in principle speak about particular cases of duality, but I would find hopelessly implausible any claim that it can be done in anything like as 'natural' (Corfield, 2003, Section 9.8) and systematic a way as with category theory. This is because fundamentally at stake in many cases of mathematical duality is the existence of one nontrivial autoequivalence of the 1-category, and indeed $(\infty, 1)$-category, of categories, which sends a category to its opposite.

The right way to treat 'sameness' between categories is the notion of equivalence. Should this incline us to identify opposite categories? Well, no, since they are not equivalent so long as they are not self-dual. Consider now cases where we can find 'concrete' descriptions of the two sides of a duality. This means providing a certain kind of underlying functor to the category of sets, which means we are taking these entities to be a certain kind of structured set. The opposite of such a functor goes from the opposite category to the opposite of the category of sets, which can be described as the category of complete atomic Boolean algebras. A category of entities which may be viewed as sets with a simple structure is dual to one whose objects may be viewed as complete atomic Boolean algebras with simple structure, so perhaps quite complicated as structured sets. This failure of self-duality in the category of sets can be seen from the behaviour of two of its objects $\varnothing$ and 1, the singleton set. As initial and terminal objects, were *Set* self-dual, they would behave in a dual way. However, maps into $\varnothing$ amount to the identity from $\varnothing$. Maps out of 1 amount to the elements of any set.

Singling out *Set* as part of the criterion for what makes a category 'concrete' introduces symmetry-breaking into the category of categories. I considered this fundamental asymmetry in an article comparing the 'coalgebra' of unfolding and decomposition to the 'algebra' of construction and composition (Corfield, 2011). What makes the category of sets so special that we take it as a default? Certainly it has many pleasant properties, or better universal characterisations. On the other hand, we may want to place ourselves in more intrinsically 'dual' settings such as the self-dual category of sets and relations, *Rel*.[4] In John Baez's paper 'Quantum Quandaries' (Baez, 2004) we find exposition of the observation that many categories involved in physics, such as the various cobordism categories of topological quantum field theory, and the category of Hilbert spaces, are monoidal but not cartesian. As we saw above in the case of the category of vector spaces, this means there is a form of product of two objects not generally equipped with projections, one manifestation of this in physics being that the state of two interfering quantum systems cannot generally be given by a state of each. We can see this non-cartesian flavour already in the shift from *Set* to *Rel*, and also as we move from functions to spans or correspondences, right up to the appearance of monoidal (∞, n)-categories with full duals as targets for extended topological quantum field theories (Lurie, 2009). This is a central part of the appearance of genuine mathematical duality in physics.

Turning now to the theme of other contributions to this journal, physical dualities in string theory, let us see how close they are to mathematical duality proper. Well, T-duality in its topological form is related to Fourier–Mukai duality, which concerns integral transforms through a kernel, so resembles a 'categorified' Fourier transform. However, there are extra choices to be made and any given string background may have none, one or more than one "T-duals". Mirror symmetry does arise through a genuine $\mathbb{Z}_2$ action on the Hodge diamond, but then a $\mathbb{Z}_2$-action by itself doesn't need to be called a duality. For example, reflecting a plane in a line, we don't speak of a duality between reflected points. Electric–magnetic duality can be thought of as a $\mathbb{Z}_2$-action on the parameter space of (super-)Yang-Mills theory and hence is perhaps closer in some ways to being justifiably called a duality, but on the other hand, it is a form of S-duality which is a vestige of a larger symmetry by a $SL(2, \mathbb{Z})$ action.

It seems that at stake is the complexity of local Lagrangian gauge QFT, whose "moduli space" contains some obvious and some rather subtle equivalences. The issue seems to be one of understanding notions of sameness and difference when dealing with such complicated moduli spaces of field theories which may involve orbifolds, stacks, higher gauge groups (in the sense of one object n-groupoids) and other such constructions from current geometry. Morita theory is lurking behind the scenes here, and can be found associated to several items mentioned in the paragraph above, such as the Fourier–Mukai transform. We should have every expectation then that the resources provided by natural language or the philosopher's traditional tool, predicate logic, are far from optimal to deal with these situations.

Before embarking on such a project it would be important to understand the corresponding treatment of covariance in general relativity, where for a spacetime Σ and an object of field values **Fields** one constructs a configuration space by forming the 'action groupoid' for the action of $Diff(\Sigma)$ on the space of functions $[\Sigma, \mathbf{Fields}]$ (Schreiber, 2013). This is the right way to retain the relevant identifications, where forming the simple quotient loses important information. As a simple illustration of the difference between a quotient and corresponding action groupoid, take a finite set X and the action of $Sym(X)$ upon it. Since the action is clearly transitive, the quotient is trivial. The action groupoid by contrast has elements of X as objects and morphisms between any x and y in X labelled by permutations sending x to y. This retains important structure at a point, for example, the isotopy group at x is the stabiliser there.

It seems plausible that similar considerations will help us with QFT. We will have spaces of geometric data – such as manifolds, Riemannian structures, and torus bundles – providing charts for an atlas of the space of QFTs. Understanding how dualities are ways to identify charts in this picture will be key. It seems in the case of T-duality what may be governing the identification structure is the smooth T-duality 2-group (nLab, *smooth T-duality 2-group*), an idea due to Thomas Nikolaus.

Rather than predicate logic, the philosopher of physics of the future looking for a formalism to treat such subtle issues of equivalence may well need to learn homotopy type theory (Univalent Foundations Program, 2014), along with its 'cohesive', 'linear', and 'directed' variants, which is supremely well-adapted to express a properly structuralist notion of sameness and difference. It has also been employed by Urs Schreiber to formalise Lagrangian quantum field theory (Schreiber, 2014). Schreiber has proposed that we consider duality in the sense of QFT in terms of a 'homotopified' equivalence relation, known as an 'effective epimorphism', imposed on the collection of Lagrangian data (nLab, *duality in physics*). This would suggest we not only think of dualities relating examples of Lagrangian data which are quantised to equivalent QFTs, but also consider higher 'dualities of dualities', for example, equivalences between equivalences of Calabi–Yau manifolds in homological mirror symmetry.

[4] For an interesting discussion of the relative importance of *Set* and *Rel* and the possibility of combining them see http://mathoverflow.net/q/121031/447.

Taking the ‘principle of duality’ to refer to the full range of constructions treated in this article, Nagel's claim, quoted in Section 2, that

> …the concepts of structure, isomorphism, and invariance, which have been fashioned out of the materials to which the principle of duality is relevant, dominate research in mathematics, logic, and the sciences of nature. (Nagel, 1939, p. 217)

would appear to hold every bit as true today. As in the case of projective geometry and the rise of axiomatic geometry in the Hilbertian mode, described by Nagel, through the internal demands of their discipline, many involving dualities, mathematicians have forged deeper understandings of invariance and higher equivalence. Making sense of such understandings, philosophers of mathematics have a role to play in creating a dialogue with philosophers of physics who are looking to interpret the dualities and equivalences found in current physics.

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