2-Covering spaces

David M Roberts
University of Adelaide

October 1, 2009

Topological groupoids and functors

A *topological groupoid* **X** consists of:

- A space X_0 ('objects') and a space X_1 ('arrows' or 'morphisms')
- Maps $s, t \colon X_1 \to X_0$ ('source' and 'target') $\mathrm{id}_{(-)} \colon X_0 \to X_1$ ('unit' or 'identity')
- Composition $X_1 \times_{X_0} X_1 \to X_1$ and inverse $(-)^{-1} \colon X_1 \to X_1$

Often will denote a groupoid **X** by

$$X_1 \rightrightarrows X_0$$

Topological groupoids and functors

A *functor* between topological groupoids, $f\colon \mathbf{X} \to \mathbf{Y}$ consists of

- Maps $f_0\colon X_0 \to Y_0$ and $f_1\colon X_1 \to Y_1$ ('object component' and 'arrow component')

such that

- $sf_1 = f_0 s$, $tf_1 = f_0 t$ (respects source and target)
- $f(\gamma\eta) = f(\gamma)f(\eta)$, $f(id_x) = id_{f(x)}$ and $f(\gamma^{-1}) = f(\gamma)^{-1}$ (respects composition, identities and inverses)

We thus have a category TG of topological groupoids and functors.

Examples

- There is a functor $Top \hookrightarrow TG$ sending a space to the groupoid with no non-trivial arrows.
- There is a full subcategory $Gpd \hookrightarrow TG$ of groupoids with the discrete topology - these will be referred to as t-d (topologically discrete) groupoids.
- Let X be a locally connected, semilocally simply-connected topological space. The fundamental groupoid

  $$\Pi_1(X) := (X^I/\sim \rightrightarrows X)$$

  can be given a topology such that $\Pi_1(X) \to X \times X$ is a covering space.

Examples

Let $p : E \to X$ be a map, and $vert(E^I) \subset E^I$ the subset of *vertical* paths. Denote by $\sim_v$ the equivalence relation 'vertically homotopic rel endpoints'.
Then

$$\Pi_1/_X(E) := (vert(E^I)/\sim_v \rightrightarrows E^\delta)$$

is a (t-d) groupoid equipped with a functor

$$\Pi_1/_X(E) \to X.$$

When p is a fibre bundle, or even locally homotopically trivial, and the fibres have universal covering spaces, we can make $\Pi_1/_X(E)$ a topological groupoid.

Fibres are discrete. . . ish

- The fibres of a covering space are sets, or more accurately, spaces in the essential image of the functor $Set \to Top$.
- We think of a covering space as being a family of sets parameterised by the base space. The idea is that a 2-covering space is a family of groupoids, all of which are in the essential image of

$$Gpd \to TG'$$

where TG' is a category (actually bicategory) of groupoids which is just the naïve (2-)category TG with some formal inverses thrown in.

Brief detour into TG'

Definition

A *weak equivalence* (Everaert-Kieboom-van der Linden) $f : \mathbf{X} \xrightarrow{\sim} \mathbf{Y}$ of topological groupoids is a functor which satisfies:

- $$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ X_0 \times X_0 & \xrightarrow[f_0 \times f_0]{} & Y_0 \times Y_0 \end{array}$$

  is a pullback (f is *fully faithful*), and

- $$\begin{array}{ccccc} X_0 & \longleftarrow & X_0 \times_{Y_0} Y_1 & \xrightarrow{\rho} & Y_0 \\ {\scriptstyle f_0}\downarrow & & \downarrow & & \\ Y_0 & \xleftarrow[s]{} & Y_1 & \xrightarrow[t]{} & Y_0 \end{array},$$

  ρ admits local sections (f is *essentially epi*).

Brief detour into TG'

- There is a 2-category (also denoted TG) of groupoids, functors and *natural transformations* (Ehresmann). Equivalences in this 2-category are weak equivalences but the converse is not true.
- The bicategory TG' is 'the' universal construction such that the functor $q\colon TG \to TG'$ sends weak equivalences to actual equivalences (Pronk). We can construct TG' such that q includes TG as a *strict* sub-bicategory with the same objects as TG' ([R.], drawing on Makkai, Bartels).
- 1-arrows in this bicategory, which are spans $\mathbf{X} \xleftarrow{\simeq} \mathbf{X}[U] \to \mathbf{Y}$, will be denoted $\mathbf{X} \,\mathrel{\mapstochar\longrightarrow}\, \mathbf{Y}$ and equivalences will be denoted $\mathbf{X} \overset{\sim}{\mathrel{\mapstochar\longrightarrow}} \mathbf{Y}$.

Fibres are discrete. . . ish

Definition

A groupoid $\mathbf{P}$ is *weakly discrete* if it is equivalent (in TG') to a t-d groupoid. That is, there is an equivalence $\mathbf{D} \overset{\sim}{\longrightarrow\!\!\!\!\!+\!\!\!\!\to} \mathbf{P}$.

If we let $\mathbf{X}^\delta$ denote a groupoid considered with the discrete topology, we have the following pleasing result:

Fibres are discrete. . . ish

Lemma

A groupoid $\mathbf{P}$ *is weakly discrete if and only if the canonical functor* $\mathbf{P}^\delta \to \mathbf{P}$ *is a weak equivalence.*

The fundamental groupoid $\Pi_1(X)$ of a (locally nice) space is weakly discrete. If X is locally badly behaved this can fail to be so, or even fail to be topological e.g. Hawaiian earring (P. Fabel).

2-covering spaces

Definition

Let $\mathbf{Z} \to X$ be a functor. $\mathbf{Z}$ is a *2-covering space* of X if there is an open cover $\{U_i\}$ of X such that for each pullback groupoid

$$\mathbf{Z}_{U_i} := (Z_1 \times_X U_i \rightrightarrows Z_0 \times_X U_i)$$

2-covering spaces

Definition

Let $\mathbf{Z} \to X$ be a functor. $\mathbf{Z}$ is a *2-covering space* of X if there is an open cover $\{U_i\}$ of X such that for each pullback groupoid

$$\mathbf{Z}_{U_i} := (Z_1 \times_X U_i \rightrightarrows Z_0 \times_X U_i)$$

there is an equivalence $\phi_i \colon \mathbf{Z}_{U_i} \xrightarrow{\sim} U_i \times \mathbf{D}_i$ commuting with the obvious maps to U_i,

2-covering spaces

Definition

Let $\mathbf{Z} \to X$ be a functor. $\mathbf{Z}$ is a *2-covering space* of X if there is an open cover $\{U_i\}$ of X such that for each pullback groupoid

$$\mathbf{Z}_{U_i} := (Z_1 \times_X U_i \rightrightarrows Z_0 \times_X U_i)$$

there is an equivalence $\phi_i \colon \mathbf{Z}_{U_i} \xrightarrow{\sim} U_i \times \mathbf{D}_i$ commuting with the obvious maps to U_i, and where each $\mathbf{D}_i$ is t-d.

2-covering spaces

Definition

Let $\mathbf{Z} \to X$ be a functor. $\mathbf{Z}$ is a *2-covering space* of X if there is an open cover $\{U_i\}$ of X such that for each pullback groupoid

$$\mathbf{Z}_{U_i} := (Z_1 \times_X U_i \rightrightarrows Z_0 \times_X U_i)$$

there is an equivalence $\phi_i \colon \mathbf{Z}_{U_i} \xrightarrow{\sim} U_i \times \mathbf{D}_i$ commuting with the obvious maps to U_i, and where each $\mathbf{D}_i$ is t-d.

2-covering spaces equipped with an equivalences like the $\mathbf{Z}_{U_i}$ are will be called trivialisable.

An example

Let A be an abelian topological group. Recall (Murray) that an A-bundle gerbe (P, Y) on a space X consists of a map $Y \to X$ admitting local sections, a principal A-bundle $P \to Y \times_X Y$ together with a 'product' map

$$p_{12}^* P \otimes p_{23}^* P \to p_{13}^* P$$

of bundles over $Y^{[3]} := Y \times_X Y \times_X Y$ which is associative over $Y^{[4]}$.

Theorem

An A-bundle gerbe (P, Y) on X for discrete A determines a 2-covering space

$$(P \rightrightarrows Y) \to X.$$

An example, continued

As a concrete example, let (E, S^3) be the lifting bundle gerbe associated to the $U(1)$-bundle $S^3 \to S^2$ and the central extension $\mathbb{Z} \to \mathbb{R} \to S^1$.

$$E \to S^3 \times_{S^2} S^3$$

is a principal $\mathbb{Z}$-bundle and $E \rightrightarrows S^3$ is a 2-covering space of S^2. It can be shown that this 2-covering space is not trivialisable.

Properties of 2-covering spaces

Let us now consider a general 2-covering space $\mathbf{Z} \to X$. We find the following results

Theorem

- *The map $Z_0 \to X$ admits local sections,*
- *$Z_1 \to Z_0 \times_X Z_0$ is a covering space,*
- *Each fibre Z_x is a weakly discrete groupoid,*
- *For path-connected X, all the fibres are equivalent objects in TG'.*

Note that for the second point we need to allow fibres to be empty, and if $Z_0 \times_X Z_0$ is not path-connected then the fibres need not all be isomorphic.

Generic example

Let us return to our example from earlier, $\Pi_1/_X(E) \to X$. We need to use some easy properties of the functor $\Pi_1 : \mathit{Top}/X \to \mathit{TG}/X$:

- Π_1 commutes with pullbacks:
  $\Pi_1/_X(E) \times_X A \simeq \Pi_1/_A(E \times_X A)$,
- $\Pi_1/_X(X \times F) \simeq \Pi_1(F) \times X$,
- Vertically homotopic maps are sent to isomorphic functors

The properties are used to *define* the topology on $\Pi_1/_X(E)$.

Generic example

Assume (wlog) X is path-connected.

- If $E \to X$ is a fibre bundle then as it is locally trivial, $\Pi_1/_X(E)$ is locally trivialisable. All that is necessary for $\Pi_1/_X(E)$ to be a 2-covering space is that the typical fibre has a universal covering space.
- If $E \to X$ is locally homotopy trivial, and the typical fibre (defined up to homotopy equivalence) has a universal covering space, then $\Pi_1/_X(E)$ is again a 2-covering space.
- As a more specific example of this last case, assume that X is paracompact and has a nhd basis of inessential open sets (i.e. $U \hookrightarrow X$ is null-homotopic). Then any Hurewicz fibration over X is locally homotopy trivial (Dold).

Low-dimensional homotopy properties

- Recall that for a covering space $p\colon \widetilde{X} \to X$, $\pi_1(p)$ is injective. We would like a similar result for 2-covering spaces, but now we do not consider the fundamental group, but the fundamental *2-group*. This is best explained with an example, which we shall need to use later
- Consider the fundamental groupoid of a (based) loop space, $\Pi_1(\Omega X)$. The H-space structure on ΩX induces an up-to-homotopy associative multiplication on the objects and arrows of the groupoid, and reversing paths gives up-to-homotopy inverses.

Low-dimensional homotopy properties

- $\Pi_2(X, *) := \Pi_1(\Omega X)$ is called the *fundamental 2-group* of X (e.g. Baez-Lauda). Groupoids with such a group-like structure are called 2-groups (they have a history under different names going back to the 1960s). Note that this construction is functorial for pointed spaces and maps.
- We do not have at present a completely satisfactory loop 'space' of a groupoid, but it is still possible [R.] to define the fundamental 2-group $\Pi_2(\mathbf{X}, *)$ of a pointed groupoid $\mathbf{X}$ by other means.
- This reduces (up to equivalence) to the original construction when $\mathbf{X}$ is a space, and is a functor of pointed groupoids and basepoint-preserving functors.

Low-dimensional homotopy properties

There are analogues of path- and homotopy-lifting theorems (more like Dold fibrations than Hurewicz or Serre). These help us to prove:

Theorem

Given a 2-covering space $\mathbf{Z} \to X$ *(and basepoints), the induced functor* $\Pi_2(\mathbf{Z}, *) \to \Pi_2(X, *)$ *is faithful.*

From work of M. Dupont we see that this should be taken as the definition of a sub-2-group.

A 2-connected cover

We say a groupoid $\mathbf{X}$ is 2-connected if $\Pi_2(\mathbf{X}, *)$ is equivalent to the trivial groupoid.
Let (X, x) be a pointed space with a nhd basis of inessential open sets (not assuming X is paracompact) and let $PX \to X$ be the path fibration.

Theorem
$\mathbf{X}^{(2)} := \Pi_1/_X(PX) \to X$ is a 2-covering space and the groupoid $\mathbf{X}^{(2)}$ is 2-connected.

Furthermore, this is functorial: a map $X \to Y$ gives a commuting square

$$\begin{array}{ccc} \mathbf{X}^{(2)} & \longrightarrow & \mathbf{Y}^{(2)} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Comments

- The local contractibility condition on the base is too strong, but this is the best we can do with the current definition of 2-covering space. This points toward replacing TG' with a different localisation TG'' (replacing open covers with another Grothendieck topology).
- We can form (weak) quotients of $\mathbf{X^{(2)}}$ by sub-2-groups of $\Pi_2(X, *)$ – these will also be 2-covering spaces, giving us a functor from the 2-category of such sub-2-groups to that of (pointed, path-connected) 2-covering spaces. This is conjectured to be an equivalence of 2-categories.
- 2-covering spaces give us lots of explicit examples of 2-bundles (T. Bartels) which previously have been hard to come by.

References

R. D.M. Roberts, *Fundamental bigroupoids and 2-covering spaces*, PhD thesis. Draft available from http://ncatlab.org/david+roberts/show/HomePage (Chapters 1 and 2).