INTERNAL CATEGORIES, ANAFUNCTORS AND LOCALISATIONS

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This article serves two purposes: to review Bartel's extension of Makkai's anafunctors from Set *sans* Choice to more general sites, and to show that the localisation of a 2-category of internal categories, functors and transformations at the class of essential equivalences is calculated by a bicategory with the same objects and anafunctors for 1-arrows.

CONTENTS

1 INTRODUCTION

Pronk, in her work on stacks [Pro96], introduced the concept of localising a bicategory at a class of 1-arrows. She gave axioms that are analogues of the familiar Gabriel-Zisman axioms for a category of fractions [GZ67]. All this was in order to prove that a certain 2-category of topological stacks is a localisation of a certain 2-category of topological groupoids. In the same paper parallel results on differentiable and algebraic stacks also appear. This work refined Moerdijk's [Moe88].

In the years since, a number of papers (e.g. [Lan01, Noo05, Ler08, Car09, Vit10]) have appeared dealing with localising 2-categories of internal groupoids at a class of *weak equivalences*. Weak equivalences, in this sense, were introduced by Bunge and Paré [BP79] for groupoids in a regular category (e.g. a topos), and are an internal version of fully faithful, essentially surjective functors between internal categories. Since 'surjective' only makes sense in concrete categories, and even then it is not always useful, we need to introduce a class E of maps with which to replace the surjection part of 'essentially surjective'. For example, in the literature on Lie groupoids, surjective submersions are universally used. With the class E specified, we refer to weak equivalences as E-equivalences.

Other examples of early work on localising 1-categories of groupoids are [Pra89, HS87]

In this paper we generalise the half of Pronk's result that says a full sub-2-category $\mathbf{Cat}'(S) \subset \mathbf{Cat}(S)$ of categories in S admits a localisation at the class W_E of E-equivalences. More formally, let $\mathbf{Cat}'(S)$ be a full sub-2-category of $\mathbf{Cat}(S)$ with objects internal categories such that all pullbacks of the source and target maps exist.

Theorem 1.1. *If $\mathbf{Cat}'(S)$ admits weak pullbacks and admits base change along arrows in E, a class of admissible maps in S, then $\mathbf{Cat}'(S)$ admits a calculus of fractions for W_E.*

See definitions 2.15 and 7.1 for details on base change and admissible maps respectively.

The construction in [Pro96], while canonical, is not very efficient, as 2-arrows are equivalence classes of diagrams, and the hom-categories are *a priori* large in the technical sense. While largeness of its own is not detrimental, it would be desirable to show that the hom-categories are at least essentially small. This is one motivation for our second result, which we shall shortly describe.

In the case that maps belonging to E are refined by covers from a subcanonical singleton pretopology J, then we can compare the localisation from theorem 1.1 to the bicategory $\mathbf{Cat}'_{ana}(S, J)$ with the same objects as $\mathbf{Cat}'(S)$ and J-*anafunctors* for 1-arrows ([Mak96, Bar06], see definition 5.1). Put simply, anafunctors are spans

A singleton Grothendieck pretopology is one where all the covering families consist of a single map.

$$X \leftarrow X[U] \xrightarrow{f} Y$$

of internal categories where the left 'leg' is a resolution of X by taking the base change $X[U]$ along a J-cover $U \to X_0$. Anafunctors will not be completely unfamiliar beasts, in that when X is an object of S and Y is a group object in S, considered as a groupoid with one object, anafunctors from the former to the latter are precisely Čech cocycles, and maps of anafunctors are coboundaries.

The second main result of this paper is the following. Note that $\mathbf{Cat}'(S)$ is a sub-bicategory of both $\mathbf{Cat}'_{ana}(S, J)$ and $\mathbf{Cat}'(S)[W_E^{-1}]$.

Theorem 1.2. *Let $\mathbf{Cat}'(S)$ and E be as in theorem 1.1 and let J be a subcanonical singleton pretopology on S which is cofinal in E. Then there is an equivalence of bicategories*

$$\mathbf{Cat}'_{ana}(S, J) \simeq \mathbf{Cat}'(S)[W_E^{-1}]$$

which is, up to equivalence, the identity on $\mathbf{Cat}'(S)$.

When a weak size axiom holds for J-covers of each object in S, then it is easy to show that $\mathbf{Cat}'_{ana}(S, J)$ is locally essentially small. This axiom holds for any reasonable category of geometric objects, for example manifolds, spaces, schemes, topoi with enough projectives.

We now outline the contents of the paper, which is intended to be self-contained. Sections one and two cover necessary background on internal categories and Grothendieck pretopologies, all of which would be familiar to experts. Section three covers weak equivalences between internal categories, while section four reviews the theory of internal anafunctors from [Bar06]. Section five covers the localisation theory for bicategories from [Pro96], before section six proves the main results of the paper.

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2 INTERNAL CATEGORIES

Internal categories were introduced by Ehresmann [Ehr63], starting with differentiable and topological categories (i.e. internal to **Diff** and **Top** respectively). We collect here the necessary definitions, terminology and notation. For a thorough recent account, see [BL04] or [Bar06].

Fix a category S. It will be referred to as the *ambient category*. We will assume throughout that S has binary products.

Definition 2.1. An *internal category* X in a category S is a diagram

$$X_1 \times_{X_0} X_1 \xrightarrow{m} X_1 \overset{s,t}{\rightrightarrows} X_0 \xrightarrow{e} X_1$$

in S such that the *multiplication* m is associative, the *unit map* e is a two-sided unit for m and s and t are the usual *source* and *target*. An *internal groupoid* is an internal category with an involution

$$(-)^{-1} \colon X_1^{iso} \to X_1^{iso}$$

satisfying the usual diagrams for an inverse.

Since multiplication is associative, there is a well-defined map $X_1 \times_{X_0} X_1 \times_{X_0} X_1 \to X_1$, which will also be denoted by m. The pullback in the diagram in definition 2.1 is

$$\begin{array}{ccc} X_1 \times_{X_0} X_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow{\scriptstyle s} \\ X_1 & \xrightarrow[t]{} & X_0 \, . \end{array}$$

This, and pullbacks like this (where source is pulled back along target), will occur often. If confusion can arise, the maps in question will be explicity written, as in $X_1 \times_{s,X_0,t} X_1$.

Often an internal category will be denoted $X_1 \rightrightarrows X_0$, the arrows m, s, t, e (and $(-)^{-1}$) will be referred to as *structure maps* and X_1 and X_0 called the object of arrows and the object of objects respectively. For example, if $S =$ **Top**, we have the space of arrows and the space of objects, for $S =$ **Grp** we have the group of arrows and so on.

The category of internal groupoids is a coreflective subcategory of the category of internal categories (morphisms are internal functors, see definition 2.5), and so for every internal category $X_1 \rightrightarrows X_0$ there is a subobject $X_1^{iso} \hookrightarrow X_1$ such that $X_1^{iso} \rightrightarrows X_0$ is an internal groupoid.

The object of isomorphisms of an internal category can be constructed from finite limit data when S is finitely complete. In the more general case we need to specify a subobject of isomorphisms.

Example 2.2. If $X \to Y$ is an arrow in S admitting iterated kernel pairs, there is an internal groupoid $\check{C}(X)$ with $\check{C}(X)_0 = X$, $\check{C}(X)_1 = X \times_Y X$, source and target are projection on first and second factor, and the multiplication is projecting out the middle factor in $X \times_Y X \times_Y X$.

Example 2.3. Let S be a category. For each object $A \in S$ there is an internal groupoid $\mathrm{disc}(A)$ which has $\mathrm{disc}(A)_1 = \mathrm{disc}(A)_0 = A$ and all structure maps equal to id_A. Such a category is called *discrete*. We have $\mathrm{disc}(A \times B) \simeq \mathrm{disc}(A) \times \mathrm{disc}(B)$.
There is also an internal groupoid $\mathrm{codisc}(A)$ with

$$\mathrm{codisc}(A)_0 = A, \ \mathrm{codisc}(A)_1 = A \times A$$

and where source and target are projections on the first and second factor respectively. The unit map is the diagonal and composition is projecting out the middle factor in $\mathrm{codisc}(A)_1 \times_{\mathrm{codisc}(A)_0} \mathrm{codisc}(A)_1 \simeq A \times A \times A$. Such a groupoid is called *codiscrete*. Again, we have $\mathrm{codisc}(A \times B) \simeq \mathrm{codisc}(A) \times \mathrm{codisc}(B)$.

Example 2.4. The codiscrete groupoid is obviously a special case of example 2.2, which is called the Čech groupoid of the map $X \to Y$. The origin of the name is that in **Top**, for maps of the form $\coprod_I U_i \to Y$, the Čech groupoid $\check{C}(\coprod_I U_i)$ appears in the definition of Čech cohomology.

Definition 2.5. Given internal categories X and Y in S, an *internal functor* $f \colon X \to Y$ is a pair of maps

$$f_0 \colon X_0 \to Y_0 \quad \text{and} \quad f_1 \colon X_1 \to Y_1$$

called the object and arrow component respectively. The map f_1 restricts to a map $f_1 \colon X_1^{iso} \to Y_1^{iso}$ and both components commute with all the structure maps.

Example 2.6. If $A \to B$ is a map in S, there are functors $\mathrm{disc}(A) \to \mathrm{disc}(B)$ and $\mathrm{codisc}(A) \to \mathrm{codisc}(B)$.

Example 2.7. If $A \to C$ and $B \to C$ are maps admitting iterated kernel pairs, and $A \to B$ is a map over C, there is a functor $\check{C}(A) \to \check{C}(B)$.

Definition 2.8. Given internal categories X, Y and internal functors $f, g\colon X \to Y$, an *internal natural transformation* (or simply *transformation*)

$$a\colon f \Rightarrow g$$

is a map $a\colon X_0 \to Y_1$ such that $s \circ a = f_0$, $t \circ a = g_0$ and the following diagram commutes

$$\begin{array}{ccc} X_1 & \xrightarrow{(g_1, a\circ s)} & Y_1 \times_{Y_0} Y_1 \\ {\scriptstyle (a\circ t, f_1)}\downarrow & & \downarrow{\scriptstyle m} \\ Y_1 \times_{Y_0} Y_1 & \xrightarrow{m} & Y_1 \end{array} \qquad (1)$$

expressing the naturality of a. If a factors through Y_1^{iso}, then it is called a *natural isomorphism*. Clearly there is no distinction between natural transformations and natural isomorphisms when Y is an internal groupoid.

We can reformulate the naturality diagram above in the case that a is a natural isomorphism. Denote by $-a$ the composite arrow

$$X_0 \xrightarrow{a} Y_1^{\mathrm{iso}} \xrightarrow{(-)^{-1}} Y_1^{\mathrm{iso}} \hookrightarrow Y_1.$$

Then the diagram (1) commuting is equivalent to this diagram commuting

$$\begin{array}{ccc} X_0 \times_{X_0} X_1 \times_{X_0} X_0 & \xrightarrow{-a \times f_1 \times a} & Y_1 \times_{Y_0} Y_1 \times_{Y_0} Y_1 \\ {\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle m} \\ X_1 & \xrightarrow[g_1]{} & Y_1 \end{array} \qquad (2)$$

a fact we will use repeatedly.

Example 2.9. If X is a category in S, A is an object of S and $f, g : X \to \mathrm{codisc}(A)$ are functors, there is a unique natural isomorphism $f \overset{\sim}{\Rightarrow} g$.

Internal categories (resp. groupoids), functors and transformations form a 2-category $\mathbf{Cat}(S)$ (resp. $\mathbf{Gpd}(S)$) [Ehr63]. There is clearly a 2-functor $\mathbf{Gpd}(S) \to \mathbf{Cat}(S)$. Also, disc and codisc, described in examples 2.3 and 2.6 are 2-functors $S \to \mathbf{Gpd}(S)$, whose underlying functors are left and right adjoint to the functor

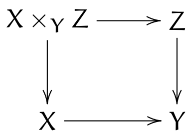
$$\mathrm{Obj}\colon \underline{\mathbf{Gpd}(S)} \to S, \qquad (X_1 \rightrightarrows X_0) \mapsto X_0.$$

Here $\underline{\mathbf{Gpd}(S)}$ is the 1-category underlying the 2-category $\mathbf{Gpd}(S)$. Hence for an internal category X in S, there are functors $\mathrm{disc}(X_0) \to X$ and $X \to \mathrm{codisc}(X_0)$, the arrow component of the latter being $(s, t) : X_1 \to X_0^2$.

Definition 2.10. An *internal* or *strong equivalence* of internal categories is an equivalence in this 2-category: an internal functor $f\colon X \to Y$ such that there is a functor $f'\colon Y \to X$ and natural isomorphisms $f \circ f' \Rightarrow \mathrm{id}_Y$, $f' \circ f \Rightarrow \mathrm{id}_X$.

Many constructions involving internal categories require pullbacks of the source and target maps. To this end, we shall be interested in a full sub-2-category $\mathbf{Cat}'(S)$ consisting of objects – internal categories $X_1 \rightrightarrows X_0$ – such that all pullbacks of s and t exist.

The strict pullback of internal categories



in a category with pullbacks is the internal category with objects $X_0 \times_{Y_0} Z_0$, arrows $X_1 \times_{Y_1} Z_1$, and all structure maps given componentwise by those of X and Z.

Definition 2.11. (see, e.g. [EKvdL05]) The *isomorphism category* of an internal category X is the internal category denoted $X^{\mathbf{I}}$, with

$$X_0^{\mathbf{I}} = X_1^{iso}, \qquad X_1^{\mathbf{I}} = (X_1 \times_{s,X_0,t} X_1^{iso}) \times_{X_1} (X_1^{iso} \times_{s,X_0,t} X_1).$$

where the fibred product over X_1 arises by considering the composition maps

$$X_1 \times_{s,X_0,t} X_1^{iso} \to X_1$$
$$X_1^{iso} \times_{s,X_0,t} X_1 \to X_1.$$

Composition in $X^{\mathbf{I}}$ is the same as commutative squares in the case of ordinary categories. There are two functors $\mathbf{s}, \mathbf{t} \colon X^{\mathbf{I}} \to X$ which have the usual source and target maps of X as their respective object components.

This construction is an internal version of the functor category $\mathbf{Cat}(\mathbf{I}, C)$, since the groupoid $\mathbf{I} = (\circ \xrightarrow{\simeq} \bullet)$ does not always exist internal to S.

Remark 2.12. There is an isomorphism $X_1^{\mathbf{I}} \simeq X_1^{iso} \times_{t,X_0,t} X_1 \times_{s,X_0,t} X_1^{iso}$ given by projecting out the last factor in

$$(X_1 \times_{s,X_0,t} X_1^{iso}) \times_{X_1} (X_1^{iso} \times_{s,X_0,t} X_1).$$

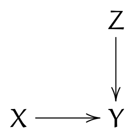
It is easy to see in this form that this pullback exists given our assumptions on pullbacks of the source and target maps.

The following lemma is a simple exercise in keeping track of pullbacks.

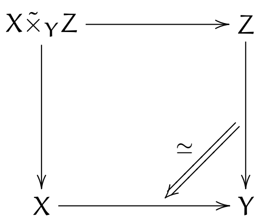
Lemma 2.13. *If* X *is an object of* $\boldsymbol{Cat}'(S)$*, then so is* $X^{\mathbf{I}}$.

The astute reader will recognise the following as an internalisation of the usual notion of weak pullback

Definition 2.14. The *weak pullback* $X\tilde{\times}_Y Z$ of a diagram of internal categories



is given, if it exists, by the strict pullback $X \times_{Y,\mathbf{s}} Y^{\mathbf{I}} \times_{\mathbf{t},Y} Z$. There is a 2-commutative square



If the ambient category S has pullbacks, then all weak pullbacks exist in **Cat**(S) and **Gpd**(S). However it is not immediate that all weak pullbacks exist in $\mathbf{Cat}'(S)$, so we will have to posit it as an additional hypothesis. Lemma 2.13 ensures that if strict pullbacks exist in $\mathbf{Cat}'(S)$, then so do weak pullbacks.

Recall that there is a functor Obj: $\underline{\mathbf{Cat}'(S)} \to S$, sending an internal category to its object of objects. Given a category X and a map $p : M \to X_0$ in S, a cartesian lift of p is, amongst other things, a functor with object component p.

Definition 2.15. For a category X and a map $p : M \to X_0$ in S, the domain $X[M]$ of a cartesian lift $X[M] \to X$ of p will be called the *base change of* X *along* p.

If the base change along any map in a given class K of maps exists for all objects of $\mathbf{Cat}'(S)$, then we say $\mathbf{Cat}'(S)$ admits base change along maps in K. We can calculate the base change by taking the strict pullback

$$\begin{array}{ccc} X[M] & \longrightarrow & X \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathrm{codisc}(M) & \longrightarrow & \mathrm{codisc}(X_0) \end{array} \tag{3}$$

Note that codisc may not land in $\mathbf{Cat}'(S)$, *so we work in* $\mathbf{Cat}(S)$, *then check if the pullback is in* $\mathbf{Cat}'(S)$. *See example* **??** *for cases when this happens.*

in **Cat**(S), when this pullback exists. The canonical functor in the top row has p as its object component. If desired we can choose a cartesian lift for each map in S (using Choice) and get a weak 2-functor with object component $(X, M \to X_0) \mapsto X[M]$.

It follows immediately from the definition that given maps $N \to M$ and $M \to X_0$, there is a canonical isomorphism

$$X[M][N] \simeq X[N]. \tag{4}$$

with object component the identity map.

Remark 2.16. If we agree to follow the convention that $M \times_N N = M$ is the pullback along the identity arrow id_N, then $X[X_0] = X$. This also simplifies other results of this paper, so will be adopted from now on.

One consequence of this assumption is that the iterated fibre product

$$M \times_M M \times_M \ldots \times_M M,$$

bracketed in any order, is *equal* to M. We cannot, however, equate two bracketings of a general iterated fibred product; they are only canonically isomorphic.

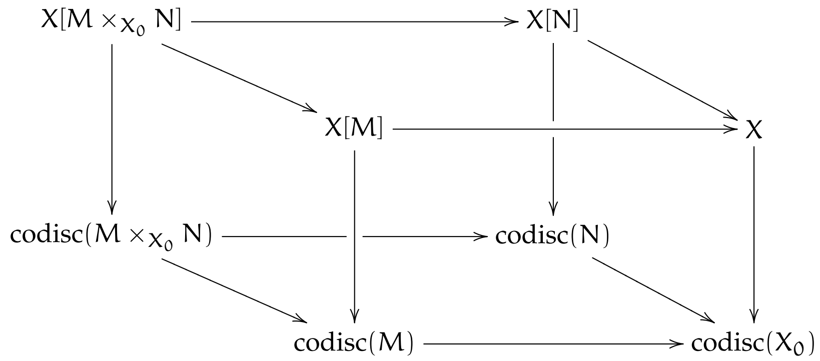
In all that follows, 'category' will mean object of $\mathbf{Cat}'(S)$ and similarly for 'functor' and 'natural transformation/isomorphism'.

Lemma 2.17. *Let* X *be a category and* $M \to X_0$, $N \to X_0$ *arrows in* S. *Then the following square is a strict pullback*

$$\begin{array}{ccc} X[M \times_{X_0} N] & \longrightarrow & X[N] \\ \big\downarrow & & \big\downarrow \\ X[M] & \longrightarrow & X \end{array}$$

when the various base changes exist.

Proof. Consider the following cube



The bottom and sides are pullbacks, either by definition, or using (4), and so the top is a pullback. □

The following technical lemma will be useful later. Even though Obj does not extend to a 2-functor, it captures some of the interaction between the fibrational nature of Obj and the 2-category nature of $\mathbf{Cat}'(S)$.

Lemma 2.18. *Let* $f, g\colon X \to Y$ *be functors and* $a\colon f \Rightarrow g$ *a natural isomorphism. There is an isomorphism*

$$X_0^2 \times_{f^2, Y_0^2} Y_1 \simeq X_0^2 \times_{g^2, Y_0^2} Y_1$$

commuting with the projections to X_0^2.

Proof. Supressing the canonical isomorphisms $X_0^2 \times_{Y_0^2} Y_1 \simeq X_0 \times_{Y_0} Y_1 \times_{Y_0} X_0$, the required isomorphism is

$$X_0 \times_{f,Y_0} Y_1 \times_{Y_0,f} X_0 \xrightarrow{(\mathrm{id},-a)\times \mathrm{id} \times (a,\mathrm{id})} X_0 \times_{g,Y_0} Y_1 \times_{Y_0} Y_1 \times_{Y_0} Y_1 \times_{Y_0,g} X_0$$
$$\xrightarrow{\mathrm{id}\times m \times \mathrm{id}} X_0 \times_{g,Y_0} Y_1 \times_{Y_0,g} X_0.$$

which is the identity map when restricted to the X_0 factors, from which the claim follows. □

Corollary 2.19. *If* $X = disc(M)$*, the categories* $Y[M \xrightarrow{f} Y_0]$ *and* $Y[M \xrightarrow{g} Y_0]$ *are isomorphic.*

PUT STUFF HERE ABOUT EXAMPLES OF CAT'(S)

3 SITES AND COVERS

The idea of *localness* is inherent in many constructions in algebraic topology and algebraic geometry. For an abstract category the concept of 'local' is encoded by a Grothendieck pretopology. Localness is needed to be able to talk about local sections of a map in a category – a concept that will replace surjectivity when moving from **Set** to more general categories. This section gathers definitions and notations for later use.

Definition 3.1. A *Grothendieck pretopology* (or simply *pretopology*) on a category S is a collection J of families

$$\{(U_i \to A)_{i\in I}\}_{A\in \mathrm{Obj}(S)}$$

of morphisms for each object $A \in S$ satisfying the following properties

1. $(\mathrm{id}\colon A \to A)$ is in J for every object A.
2. Given a map $B \to A$, for every $(U_i \to A)_{i\in I}$ in J the pullbacks $B \times_A A_i$ exist and $(B \times_A A_i \to B)_{i\in I}$ is in J.

3. For every $(U_i \to A)_{i\in I}$ in J and for a collection $(V^i_k \to U_i)_{k\in K_i}$ from J for each $i \in I$, the family of composites

$$(V^i_k \to A)_{k\in K_i, i\in I}$$

are in J.

Families in J are called *covering families*. A category S equipped with a pretopology J is called a *site*, denoted (S, J).

Example 3.2. The basic example is the lattice of open sets of a topological space, seen as a category in the usual way, where a covering family of an open $U \subset X$ is an open cover of U by opens in X. This is to be contrasted with the pretopology $\mathcal{O}$ on **Top**, where the covering families of a space are just open covers of the whole space.

Example 3.3. On **Grp** the class of surjective homomorphisms form a pretopology.

Example 3.4. On **Top** the class of numerable open covers (i.e. those that admit a subordinate partition of unity [Dol63]) form a pretopology. Much of traditional bundle theory is carried out using this site, for example, the Milnor classifying space classifies bundles which are locally trivial over numerable covers [Mil56, Dol63, tD66].

Definition 3.5. Let (S, J) be a site. The pretopology J is called a *singleton pretopology* if every covering family consists of a single arrow $(U \to A)$. In this case a covering family is called a *cover*.

Example 3.6. In **Top**, the classes of covering maps, local section admitting maps, surjective étale maps and open surjections are all examples of singleton pretopologies. The results of [Pro96] pertaining to topological groupoids were carried out using the site of open surjections.

Example 3.7. The class Subm of surjective submersions in **Diff**, the category of smooth manifolds, is a singleton pretopology.

There are many different and useful pretopologies on the category **Sch** of schemes, such as the Zariski, étale, *fpqc* and Nisnevich pretopologies (see [] for details of these - need reference/s!)

Definition 3.8. A covering family $(U_i \to A)_{i\in I}$ is called *effective* if A is the colimit of the following diagram: the objects are the U_i and the pullbacks $U_i \times_A U_j$, and the arrows are the projections

$$U_i \leftarrow U_i \times_A U_j \to U_j.$$

If the covering family consists of a single arrow $(U \to A)$, this is the same as saying $U \to A$ is a regular epimorphism.

Definition 3.9. A site is called *subcanonical* if every covering family is effective.

Example 3.10. On **Top**, the usual pretopology of opens, the pretopology of numerable covers and that of open surjections are subcanonical.

Example 3.11. In a regular category, the regular epimorphisms form a subcanonical singleton pretopology.

In fact, the (pullback stable) regular epimorphisms in any category form the largest subcanonical singleton pretopology, so it has its own name.

Of course, the nomenclature was decided the other way around; 'subcanonical' meaning 'contained in the canonical pretopology'.

Definition 3.12. The *canonical singleton pretopology* R is the largest class of regular epimorphisms which are pullback stable. It contains all the subcanonical singleton pretopologies.

Remark 3.13. If $U \to A$ is an effective cover, a functor $\check{C}(U) \to \mathrm{disc}(B)$ gives a unique arrow $A \to B$. This follows immediately from the fact A is the colimit of $\check{C}(U)$.

Definition 3.14. A *finitary* (resp. *infinitary*) *extensive* category is a category with finite (resp. small) coproducts such that the following condition holds: let I be a a finite set (resp. any set), then, given a collection of commuting diagrams

$$\begin{array}{ccc} X_i & \longrightarrow & Z \\ \downarrow & & \downarrow \\ A_i & \longrightarrow & \coprod_{i\in I} A_i\,, \end{array}$$

one for each $i \in I$, the squares are all pullbacks if and only if the collection $\{X_i \to Z\}_I$ forms a coproduct diagram.

In such a category there is a strict initial object (i.e. given a map $A \to 0$ with 0 initial, we have $A \simeq 0$).

Example 3.15. **Top** is infinitary extensive.

Example 3.16. $\mathbf{Ring}^{op}$ is finitary extensive.

In **Top** we can take an open cover $\{U_i\}_I$ of a space X and replace it with the single map $\coprod_I U_i \to X$, and work just as before using this new sort of cover, using the fact **Top** is extensive. The sort of sites that mimic this behaviour are called *superextensive*.

Definition 3.17. (Bartels-Shulman) A *superextensive site* is an extensive category S equipped with a pretopology J containing the families

$$(U_i \to \coprod_I U_i)_{i\in I}$$

and such that all covering families are bounded; this means that for a finitely extensive site, the families are finite, and for an infinitary site, the families are small. The pretopology in this instance will also be called superextensive.

Example 3.18. Given an extensive category S, the *extensive pretopology* has as covering families the bounded collections $(U_i \to \coprod_I U_i)_{i\in I}$. The pretopology on any superextensive site contains the extensive pretopology.

Example 3.19. The category **Top** with its usual pretopology of open covers is a superextensive site.

Example 3.20. A topos with the regular pretopology is finitary superextensive, and a Grothendieck topos with the regular pretopology is infinitary superextensive.

Given a superextensive site, one can form the class $\amalg J$ of arrows $\coprod_I U_i \to A$.

Proposition 3.21. *The class* $\amalg J$ *is a singleton pretopology, and is subcanonical if and only if* J *is.*

Proof. Since identity arrows are covers for J they are covers for $\amalg J$. The pullback of a $\amalg J$-cover $\coprod_I U_i \to A$ along $B \to A$ is a $\amalg J$-cover as coproducts and pullbacks commute by definition of an extensive category. Now for the third condition we use the fact that in an extensive category a map

$$f\colon B \to \coprod_I A_i$$

implies that $B \simeq \coprod_I B_i$ and $f = \coprod_i f_i$. Given $\amalg J$-covers $\coprod_I U_i \to A$ and $\coprod_J V_j \to (\coprod_I U_i)$, we see that $\coprod_J V_j \simeq \coprod_I W_i$. By the previous point, the pullback

$$\coprod_I U_k \times_{\coprod_I U_{i'}} W_i$$

is a ∐J-cover of U_i, and hence $(U_k \times_{\coprod_I U_i} W_i \to U_k)_{i \in I}$ is a J-covering family for each $k \in I$. Thus

$$(U_k \times_{\coprod_I U_i} W_i \to A)_{i,k \in I}$$

is a J-covering family, and so

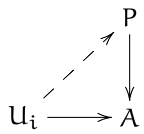
$$\coprod_J V_j \simeq \coprod_{k \in I} \left(\coprod_I U_k \times_{\coprod_I U_i} W_i \right) \to A$$

is a ∐J-cover.

The map $\coprod_I U_i \to A$ is the coequaliser of $\coprod_{I \times I} U_i \times_A U_j \rightrightarrows \coprod_I U_i$ if and only if A is the colimit of the diagram in definition 3.8. Hence $(\coprod_I U_i \to A)$ is effective if and only if $(U_i \to A)_{i \in I}$ is effective □

Notice that the original superextensive pretopology J is generated by the union of ∐J and the extensive pretopology.

Definition 3.22. Let (S, J) be a site. An arrow $P \to A$ in S is called a J-*epimorphism* (or simply J-*epi*) if there is a covering family $(U_i \to A)_{i \in I}$ and a lift



for every $i \in I$. The class of J-epimorphisms will be denoted (J-epi).

This definition is equivalent to the definition in III.7.5 in [MM92]. The dotted maps in the above definition are called local sections, after the case of the usual open cover pretopology on **Top**. If the pretopology is left unnamed, we will refer to *local epimorphisms*.

One reason we are interested in superextensive sites is the following

Lemma 3.23. *If* (S, J) *is a superextensive site, the class of* J*-epimorphisms is precisely the class of* ∐J*-epimorphisms.*

If S has all pullbacks then the class of J-epimorphisms form a pretopology. In fact they form a pretopology with an additional property – it is *saturated*. The following is adapted from [BW84].

Note that what we are calling a Grothendieck pretopology *is referred to as a Grothendieck* topology *in [BW84].*

Definition 3.24. A singleton pretopology J is *saturated* if whenever the composite $V \to U \to A$ is in J, then $U \to A$ is in J.

In fact only a slightly weaker condition on S is necessary for (J-epi) to be a pretopology.

Example 3.25. Let (S, J) be a site. If pullbacks of J-epimorphisms exist then the collection (J-epi) of J-epimorphisms is a saturated pretopology.

There is a definition of 'saturated' for arbitrary pretopologies, but we will use only this one. Another way to pass from an arbitrary pretopology to a singleton one in a canonical way is this:

Definition 3.26. The *XX* of a pretopology J on an arbitrary category S is the largest class $J_{sing} \subset$ (J-epi) of those J-epimorphisms which are pullback stable.

It is clear that $(J_{sing})_{sing} = J_{sing}$, and that when pullbacks exist, (J-epi) $= J_{sing}$.

Example 3.27. The singleton XX of the class of open covers $\mathcal{O}$ in **Diff** is $\mathcal{S}ubm$, the class of surjective submersions. Notice that all surjective submersions admit local sections (essentially by the implicit function theorem), whereas not all maps in ($\mathcal{O}$-epi) are submersions, so that $\mathcal{O}_{sing} \neq$ ($\mathcal{O}$-epi).

If J is a singleton pretopology, it is clear that $J \subset J_{sing}$. In fact J_{sing} contains all the covering families of J with only one element when J is any pretopology.

From lemma 3.23 we have

Corollary 3.28. *In a superextensive site* (S, J), *we have* $J_{sing} = (\amalg J)_{sing}$.

One class of extensive categories which are of particular interest is those that also have finite/small limits. These are called *lextensive*. For example, **Top** is infinitary lextensive, as is a Grothendieck topos. In contrast, a general topos is finitary lextensive. In a lextensive category

$$J_{sing} = (\amalg J)_{sing} = (J\text{-epi}).$$

Sometimes a pretopology J contains a smaller pretopology that still has enough covers to compute the same J-epis.

Definition 3.29. If J and K are two singleton pretopologies with $J \subset K$, such that $K \subset J_{sing}$, then J is said to be *cofinal* in K, denoted $J \leqslant K$.

Clearly $J \leqslant J_{sing}$ for any singleton pretopology J.

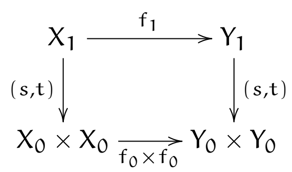
Lemma 3.30. *If* $J \leqslant K$, *then* $J_{sing} = K_{sing}$.

4 WEAK EQUIVALENCES

Equivalences in **Cat** – assuming the axiom of choice – are precisely the fully faithful, essentially surjective functors. For internal categories, however, this is not the case. In addition, we need to make use of a pretopology to make the 'surjective' part of essentially surjective meaningful. To start with we shall just assume that our ambient category is equipped with a class E of morphisms. The following definition first made its appearence in [BP79] for S finitely complete and regular, and E the class of regular epimorphisms, in the context of stacks and indexed categories.

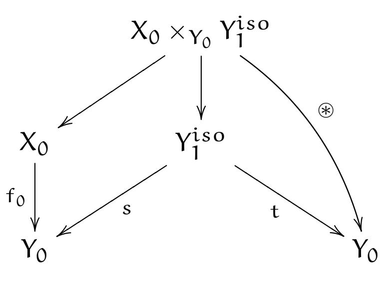
Definition 4.1. [BP79, EKvdL05] Let S be a category with a specified class E of morphisms. An internal functor $f : X \to Y$ in S is called

1. *fully faithful* if



is a pullback diagram

2. *essentially* E*-surjective* if the arrow labelled ⊛ is in E



3. an E*-equivalence* if it is fully faithful and essentially E-surjective.

The class of E-equivalences will be denoted W_E.

If (S, J) is a site, then we are interested in the class $E = J_{sing}$. The class of J_{sing}-equivalences will be denoted W_J and they will, following [EKvdL05], be referred to as J-equivalences. If mention of J is suppressed, they will be called *weak equivalences*. This usage differs from *loc. cit.* where the class of (J-epi)-equivalences are referred to as J-equivalences. In a finitely complete category there is no difference, but this definition allows later proofs to hold for non-finitely complete categories.

Example 4.2. The canonical functor $X[M] \to X$ is always fully faithful, by definition.

Example 4.3. If $X \to Y$ is an internal equivalence, then it is a J-equivalence for all pretopologies J such that split epimorphisms are contained in J_{sing} [EKvdL05]. In fact, if T denotes the trivial pretopology (only isomorphisms are covers) on a finitely complete category, the T-equivalences are precisely the internal equivalences.

Remark 4.4. This example does not include Lie groupoids as $\mathcal{O}_{sing} = \mathrm{Subm}$ does not contain the split epimorphisms. Internal equivalences *are* $\mathcal{O}$-equivalences, but for another reason. In fact we have chosen to take J_{sing}-equivalences as standard for non-finitely complete categories as this reflects the usage in the Lie groupoid literature.

Lemma 4.5. *If $f : X \to Y$ is a functor such that f_0 is in J_{sing}, then f is essentially J_{sing}-surjective.*

Corollary 4.6. *If (S, J) is a site, X a category in S and $(U \to X_0)$ is a covering family (e.g. J is a singleton pretopology), the functor $X[U] \to X$ is a J-equivalence.*

Proof. The object component of the canonical functor $X[U] \to X$ is $U \to X_0$ and since it is in J it is in J_{sing}. Hence as $X[U] \to X$ is fully faithful it is a J-equivalence. □

We now consider some easy results on the behavious of weak equivalences under pullbacks, both strict and weak. First, fully faithful functors are stable under strict pullback.

Lemma 4.7. *If $f \colon X \to Y$ is fully faithful, and $g \colon Z \to Y$ is any functor, pr_1 in*

$$\begin{array}{ccc} Z \times_Y X & \longrightarrow & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \underset{g}{\longrightarrow} & Y \end{array}$$

is fully faithful whenever the strict pullback exists.

Proof. The following chain of isomorphisms establishes the claim

$$\begin{aligned} (Z_0 \times_{Y_0} X_0)^2 \times_{Z_0^2} Z_1 &\simeq X_0^2 \times_{Y_0^2} Z_1 \\ &\simeq (X_0^2 \times_{Y_0^2} Y_1) \times_{Y_1} Z_1 \\ &\simeq X_1 \times_{Y_1} Z_1, \end{aligned}$$

the last following from the fact f is fully faithful. □

The following terminology is adapted from [EKvdL05], although strictly speaking this map is only a fibration when model structure from *loc. cit.* exists.

Definition 4.8. An internal functor $f \colon X \to Y$ is called a *trivial* E*-fibration* if it is fully faithful and $f_0 \in E$.

Lemma 4.9. *If a functor $f \colon X \to Y$ is an E-equivalence,*

$$X \times_Y Y^{\mathbf{I}} \xrightarrow{t \circ \mathrm{pr}_2} Y$$

is a trivial E-fibration.

Proof. The object component of $\mathbf{t} \circ \mathrm{pr}_2$ is $t \circ \mathrm{pr}_2$, which is in E by definition as f is essentially E-surjective. Consider now the pullback

$$\begin{CD}
(X_0 \times_{Y_0} Y_1^{\mathrm{iso}})^2 \times_{Y_0^2} Y_1 @>>> Y_1 \\
@VVV @VVV \\
(X_0 \times_{Y_0} Y_1^{\mathrm{iso}})^2 @>>> Y_0 \times Y_0
\end{CD}$$

Remark 2.12 tells is that the pullback is isomorphic to $X_0^2 \times_{Y_0^2} Y_1^{\mathbf{I}}$ in the pullback

$$\begin{CD}
X_0^2 \times_{Y_0^2} Y_1^{\mathbf{I}} @>{\mathrm{pr}_2}>> Y_1^{\mathbf{I}} \\
@. @VV{\mathrm{pr}_1}V \\
@VVV Y_1 \\
@. @VVV \\
X_0^2 @>>> Y_0 \times Y_0
\end{CD}$$

but if f is fully faithful,

$$\begin{aligned}
X_0^2 \times_{Y_0^2} Y_1^{\mathbf{I}} &\simeq X_0^2 \times_{Y_0^2} Y_1 \times_{Y_1} Y_1^{\mathbf{I}} \\
&\simeq X_1 \times_{Y_1} Y_1^{\mathbf{I}},
\end{aligned}$$

hence $\mathbf{t} \circ \mathrm{pr}_2$ is fully faithful. □

The internal category $X \times_Y Y^{\mathbf{I}}$ is called the mapping path space construction in [EKvdL05] and if the model structure therein exists, the above follows from cofibration-acyclic fibration factorisation.

Corollary 4.10. *If the weak pullback of an* E*-equivalence exists, it is again an* E*-equivalence.*

5 ANAFUNCTORS

Now assume that J is a subcanonical singleton pretopology on the ambient category S. In this section we assume that $\mathbf{Cat}'(S)$ admits base change along arrows in the given pretopology J. This is a slight generalisation of what is considered in [Bar06], where only $\mathbf{Cat}'(S) = \mathbf{Cat}(S)$ is considered.

Definition 5.1. [Mak96, Bar06] An *anafunctor* in (S, J) from a category X to a category Y consists of a cover $(U \to X_0)$ and an internal functor

$$f \colon X[U] \to Y.$$

Since $X[U]$ is an object of $\mathbf{Cat}'(S)$, an anafunctor is a span in $\mathbf{Cat}'(S)$, and will be denoted

$$(U, f) \colon X \rightarrow\!\!\!\!\!\!+\; Y.$$

Example 5.2. For an internal functor $f \colon X \to Y$ in S, define the anafunctor $(X_0, f) \colon X \rightarrow\!\!\!\!\!\!+\; Y$ as the following span

$$X \xleftarrow{=} X[X_0] \xrightarrow{f} Y.$$

We will blur the distinction between these two descriptions. If $f = \mathrm{id} \colon X \to X$, then (X_0, id) will be denoted simply by id_X.

Example 5.3. If $U \to A$ is a cover in (S, J) and $\mathbf{B}G$ is a groupoid with one object in S (i.e. a group), an anafunctor $(U, g) \colon \mathrm{disc}(A) \rightarrow\!\!\!\!\!\!+\; \mathbf{B}G$ is the same thing as a Čech cocycle.

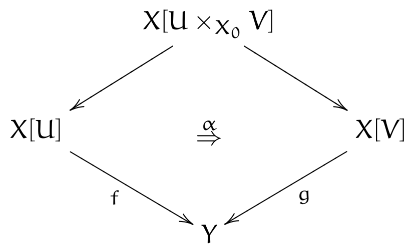
Definition 5.4. [Mak96, Bar06] Let (S, J) be a site and let

$$(U, f), (V, g) \colon X \twoheadrightarrow Y$$

be anafunctors in S. A *transformation*

$$\alpha \colon (U, f) \Rightarrow (V, g)$$

from (U, f) to (V, g) is an internal natural transformation



If $\alpha \colon U \times_{X_0} V \to Y_1$ factors through Y_1^{iso}, then α is called an *isotransformation*. In that case we say (U, f) *is isomorphic to* (V, g). Clearly all transformations between anafunctors between internal groupoids are isotransformations.

Example 5.5. Given functors $f, g \colon X \to Y$ between categories in S, and a natural transformation $a \colon f \Rightarrow g$, there is a transformation $a \colon (X_0, f) \Rightarrow (X_0, g)$ of anafunctors, given by the component $X_0 \times_{X_0} X_0 = X_0 \xrightarrow{a} Y_1$.

Example 5.6. If $(U, g), (V, h) \colon \mathrm{disc}(A) \twoheadrightarrow \mathbf{B}G$ are two Čech cocycles, a transformation between them is a coboundary on the cover $U \times_A V \to A$.

Example 5.7. Let $(U, f) \colon X \twoheadrightarrow Y$ be an anafunctor in S. There is an isotransformation $1_{(U,f)} \colon (U, f) \Rightarrow (U, f)$ called the *identity transformation*, given by the natural transformation with component

$$U \times_{X_0} U \simeq (U \times U) \times_{X_0^2} X_0 \xrightarrow{id_U^2 \times e} X[U]_1 \xrightarrow{f_1} Y_1 \qquad (5)$$

Example 5.8. [Mak96] Given anafunctors $(U, f) \colon X \to Y$ and $(V, f \circ k) \colon X \to Y$ where $k \colon V \simeq U$ is an isomorphism over X_0, a *renaming transformation*

$$(U, f) \Rightarrow (V, f \circ k)$$

is an isotransformation with component

$$1_{(U,f)} \circ (k \times \mathrm{id}) : V \times_{X_0} U \to U \times_{X_0} U \to Y_1.$$

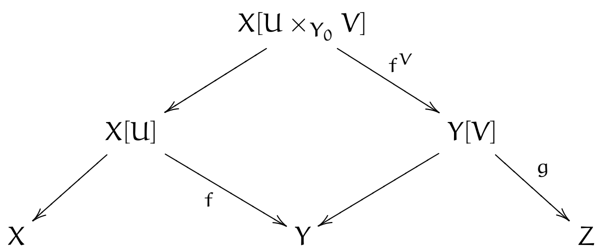
The isomorphism k will be referred to as a *renaming isomorphism*. More generally, we could let $k \colon V \to U$ be any refinement, and this prescription also gives an isotransformation $(U, f) \Rightarrow (V, f \circ k)$.

See example 5.11 below for another useful example of an isotransformation.

We define (following [Bar06]) the composition of anafunctors as follows. Let

$$(U, f) \colon X \twoheadrightarrow Y \quad \text{and} \quad (V, g) \colon Y \twoheadrightarrow Z$$

be anafunctors in the site (S, J). Their composite $(V, g) \circ (U, f)$ is the composite span defined in the usual way. It is again a span in $\mathbf{Cat}'(S)$.



The square is a pullback by lemma 2.17, and the resulting span is an anafunctor because $V \to Y_0$, and hence $U \times_{Y_0} V \to X_0$, is a cover, and using (4). We will sometimes denote the composite by $(U \times_{Y_0} V, g \circ f^V)$.

Remark 5.9. If one does not impose the existence of pullbacks on S (as in say **Diff**, see comment 2.20), this composite span still exists, because $V \to Y_0$ is a cover.

Consider the special case when $V = Y_0$, and hence (Y_0, g) is just an ordinary functor. Then there is a renaming transformation (the identity transformation!) $(Y_0, g) \circ (U, f) \Rightarrow (U, g \circ f)$, using the equality $U \times_{Y_0} Y_0 = U$. If we let $g = \mathrm{id}_Y$, then we see that (Y_0, id_Y) is a strict unit on the left for anafunctor composition. Similarly, considering $(V, g) \circ (Y_0, \mathrm{id})$, we see that (Y_0, id_Y) is a two-sided strict unit for anafunctor composition. In fact, we have also proved

Lemma 5.10. *Given two functors* $f\colon X \to Y$, $g\colon Y \to Z$ *in* S, *their composition as anafunctors is equal to their composition as functors:*

$$(Y_0, g) \circ (X_0, f) = (X_0, g \circ f).$$

Example 5.11. As a concrete and relevant example of a renaming transformation we can consider the triple composition of anafunctors

$$\begin{aligned}(U, f)\colon X &\rightarrow\!\!\!\!\!\!\!+\!\!\!\!\!\!\rightarrow Y,\\ (V, g)\colon Y &\rightarrow\!\!\!\!\!\!\!+\!\!\!\!\!\!\rightarrow Z,\\ (W, h)\colon Z &\rightarrow\!\!\!\!\!\!\!+\!\!\!\!\!\!\rightarrow A.\end{aligned}$$

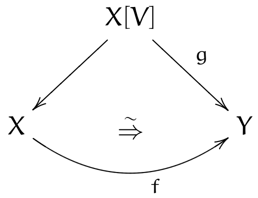
The two possibilities of composing these are

$$\Big((U \times_{Y_0} V) \times_{Z_0} W, h \circ (g f^V)^W\Big), \quad \Big(U \times_{Y_0} (V \times_{Z_0} W), h \circ g^W \circ f^{V \times_{Z_0} W}\Big)$$

The unique isomorphism $(U \times_{Y_0} V) \times_{Z_0} W \simeq U \times_{Y_0} (V \times_{Z_0} W)$ commuting with the various projections is then the required renaming isomorphism. The isotransformation arising from this renaming transformation is called the *associator*.

A simple but useful criterion for describing isotransformations where one of the anafunctors involved is a functor is as follows.

Lemma 5.12. *An anafunctor* $(V, g)\colon X \rightarrow\!\!\!\!\!\!\!+\!\!\!\!\!\!\rightarrow Y$ *is isomorphic to a functor* $f\colon X \to Y$ *if and only if there is a natural isomorphism*



In a site (S, J) where the axiom of choice holds – every J-epimorphism has a section – one can prove that every J-equivalence between internal categories is in fact an internal equivalence of categories. It is precisely the lack of splittings that prevents this theorem from holding in general sites. The best one can do in a general site is described in the the following two lemmas.

In other words, existence of local sections is enough to guarantee a global section.

Lemma 5.13. *Let* $f\colon X \to Y$ *be a* J*-equivalence, and choose a cover* $U \to Y_0$ *and a local section* $s\colon U \to X_0 \times_{Y_0} Y_1^{iso}$. *Then there is a functor* $Y[U] \to X$ *with object component* $s' := \mathrm{pr}_1 \circ s\colon U \to X_0$.

Proof. The object component is given, we just need the arrow component. Denote the local section by $(s', \iota) : U \to X_0 \times_{Y_0} Y_1^{iso}$. Consider the composite

$$\begin{aligned} Y[U]_1 \simeq U \times_{Y_0} Y_1 \times_{Y_0} U &\xrightarrow{(s',\iota)\times \mathrm{id} \times (-\iota, s')} (X_0 \times_{Y_0} Y_1^{iso}) \times_{Y_0} Y_1 \times_{Y_0} (Y_1^{iso} \times_{Y_0} X_0) \\ &\hookrightarrow X_0 \times_{Y_0} Y_3 \times_{Y_0} X_0 \xrightarrow{\mathrm{id} \times m \times \mathrm{id}} X_0 \times_{Y_0} Y_1 \times_{Y_0} X_0 \simeq X_1 \end{aligned}$$

where the last isomorphism arises from f being full faithful. It is clear that this commutes with source and target, because these are projection on the first and last factor at each step. To see that it respects identities and composition, just use the fact that the ι component will cancel with the $-\iota$ component. □

Hence we have an anafunctor $Y \twoheadrightarrow X$, and the next proposition tells us this is a pseudoinverse to f (in a sense to be made precise in proposition 5.19 below).

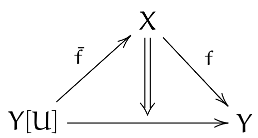
Lemma 5.14. *Let* $f\colon X \to Y$ *be a* J*-equivalence in* S*. There is an anafunctor*

$$(U, \bar{f})\colon Y \twoheadrightarrow X$$

and isotransformations

$$\iota\colon (X_0, f) \circ (U, \bar{f}) \Rightarrow id_Y$$
$$\epsilon\colon (U, \bar{f}) \circ (X_0, f) \Rightarrow id_X$$

Proof. We have the anafunctor $(U, \bar{f})$ from lemma 5.13. Since the anafunctors id_X, id_Y are actually functors, we can use lemma 5.12. Using the special case of anafunctor composition when the second is a functor, this tells us that ι will be given by a natural isomorphism



This has component $\iota\colon U \to Y_1^{iso}$, using the notation from the proof of the previous lemma. Notice that the composite $f_1 \circ \bar{f}_1$ is just

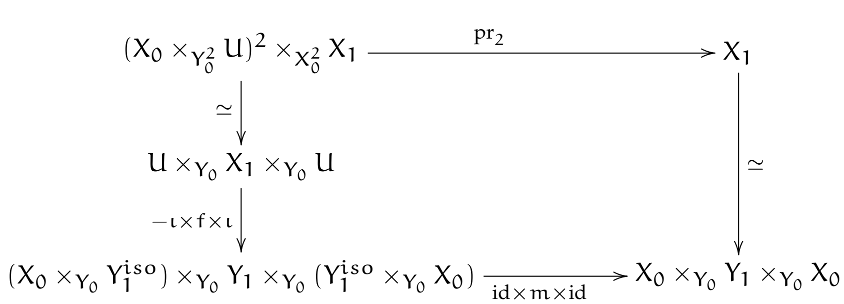
$$Y[U]_1 \simeq U \times_{Y_0} Y_1 \times_{Y_0} U \xrightarrow{\iota \times id \times -\iota} Y_1^{iso} \times_{Y_0} Y_1 \times_{Y_0} Y_1^{iso} \hookrightarrow Y_3 \xrightarrow{m} Y_1.$$

Since the arrow component of $Y[U] \to Y$ is $U \times_{Y_0} Y_1 \times_{Y_0} U \xrightarrow{pr_2} Y_1$, ι is indeed a natural isomorphism using the diagram (2).

The other isotransformation is between $(X_0 \times_{Y_0} U, \bar{f} \circ pr_2)$ and (X_0, id_X), and is given by the arrow

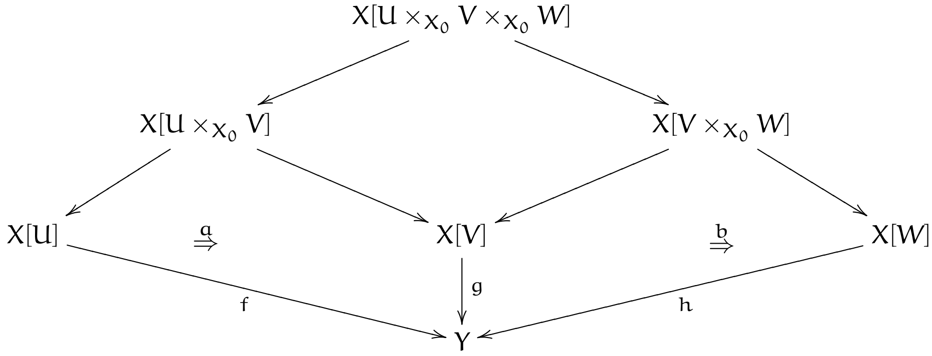
$$\epsilon\colon X_0 \times_{X_0} X_0 \times_{Y_0} U \simeq X_0 \times_{Y_0} U \xrightarrow{id \times (s', a)} X_0 \times_{Y_0} (X_0 \times_{Y_0} Y_1) \simeq X_0^2 \times_{Y_0^2} Y_1 \simeq X_1$$

This has the correct source and target, as the object component of $\bar{f}$ is s', and the source is given by projection on the first factor of $X_0 \times_{Y_0} U$. This diagram



commutes, and using (2) we see that ϵ is natural. □

Just as there is composition of natural transformations between internal functors, there is a composition of transformations between internal anafunctors [Bar06]. This is where the effectiveness of our covers will be used in order to construct a map locally over some cover. Consider the following diagram



from which we can form a natural transformation between the leftmost and the rightmost composites as functors in S. This will have as its component the arrow

$$\widetilde{ba}\colon U\times_{X_0}V\times_{X_0}W \xrightarrow{\mathrm{id}\times\Delta\times\mathrm{id}} U\times_{X_0}V\times_{X_0}V\times_{X_0}W \xrightarrow{a\times b} Y_1\times_{Y_0}Y_1 \xrightarrow{m} Y_1$$

in S. Notice that the Čech groupoid of the cover

$$U\times_{X_0}V\times_{X_0}W \to U\times_{X_0}W \tag{6}$$

is

$$U\times_{X_0}V\times_{X_0}V\times_{X_0}W \rightrightarrows U\times_{X_0}V\times_{X_0}W,$$

using the two projections $V\times_{X_0}V \to V$. Denote this pair of parallel arrows by $s,t\colon UV^2W \rightrightarrows UVW$ for brevity. In [Baro6], section 2.2.3, we find the commuting diagram

$$\begin{array}{ccc} UV^2W & \xrightarrow{t} & UVW \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \widetilde{ba}} \\ UVW & \xrightarrow[\widetilde{ba}]{} & Y_1 \end{array} \tag{7}$$

and so we have a functor $\check{C}(U\times_{X_0}V\times_{X_0}W)\to \mathrm{disc}(Y_1)$ (this Čech groupoid is associated to the cover $UVW\to UW$). Our pretopology J is assumed to be subcanonical, and using remark 3.13 this gives us a unique arrow $ba\colon U\times_{X_0}W\to Y_1$, the composite of a and b.

Remark 5.15. In the special case that $U\times_{X_0}V\times_{X_0}W\to U\times_{X_0}W$ is an isomorphism (or is even just split), the composite transformation has

$$U\times_{X_0}W\to U\times_{X_0}V\times_{X_0}W \xrightarrow{\widetilde{ba}} Y_1$$

as its component arrow. In particular, this is the case if one of a or b is a renaming transformation.

Example 5.16. Let $(U,f)\colon X \to\!\!\!\!\!+ Y$ be an anafunctor and $U'' \xrightarrow{j'} U' \xrightarrow{j} U$ successive refinements of $U\to X_0$ (e.g isomorphisms). Let $(U',f_{U'})$ and $(U'',f_{U''})$ denote the composites of f with $X[U']\to X[U]$ and $X[U'']\to X[U]$ respectively. The arrow

$$U\times_{X_0}U'' \xrightarrow{\mathrm{id}_U\times j\circ j'} U\times_{X_0}U\to Y_1$$

is the component for the composition of the isotransformations $(U,f)\Rightarrow (U',f_{U'}),\Rightarrow (U'',f_{U''})$ described in example 5.8. Thus we can see that the composite of renaming transformations associated to isomorphisms ϕ_1,ϕ_2 is simply the renaming transformation associated to their composite $\phi_1\circ\phi_2$.

Example 5.17. If $a\colon f \Rightarrow g,\ b\colon g \Rightarrow h$ are natural transformations between functors $f, g, h\colon X \to Y$ in S, their composite as transformations between anafunctors

$$(X_0, f), (X_0, g), (X_0, h)\colon X \twoheadrightarrow Y.$$

is just their composite as natural transformations. This uses the equality

$$X_0 \times_{X_0} X_0 \times_{X_0} X_0 = X_0 \times_{X_0} X_0 = X_0,$$

which is due to our choice in remark 2.16 of canonical pullbacks

The first half of the following theorem is proposition 12 in [Bar06], and the second half follows because all the constructions of categories involved in dealing with anafunctors outlined above are still objects of $\mathbf{Cat}'(S)$.

Theorem 5.18. *[Bar06] For a site (S, J) where J is a subcanonical singleton pretopology, internal categories, anafunctors and transformations form a bicategory $\mathbf{Cat}_{ana}(S, J)$. If we restrict attention to a sub-2-category $\mathbf{Cat}'(S)$ which admits base change for arrows in J, we have an analogous full subbicategory $\mathbf{Cat}'_{ana}(S, J)$.*

There is a strict 2-functor $\mathbf{Cat}'_{\text{ana}}(S, J) \to \mathbf{Cat}_{\text{ana}}(S, J)$ which is the identity on 0-cells and induces isomorphisms on hom-categories. The following is the main result of this section, and allows us to relate anafunctors to the localisations considered in the next section.

Proposition 5.19. *There is a strict 2-functor*

$$\alpha_J\colon \mathbf{Cat}'(S) \to \mathbf{Cat}'_{ana}(S, J)$$

sending J-equivalences to equivalences, and commuting with the respective inclusions into $\mathbf{Cat}(S)$ and $\mathbf{Cat}_{ana}(S, J)$.

Proof. We define α_J to be the identity on objects, and as described in examples 5.2, 5.5 on 1-cells and 2-cells (i.e. functors and transformations). We need first to show that this gives a functor $\mathbf{Cat}'(S)(X, Y) \to \mathbf{Cat}'_{\text{ana}}(S, J)(X, Y)$. This is precisely the content of example 5.17. Since the identity 1-cell on a category X in $\mathbf{Cat}'_{\text{ana}}(S, J)$ is the image of the identity functor on S in $\mathbf{Cat}'(S)$, α_J respects identity 1-cells. Also, lemma 5.10 tells us that α_J respects composition. That α_J sends J-equivalences to equivalences is the content of lemma 5.14. □

6 LOCALISING BICATEGORIES AT A CLASS OF 1-CELLS

Ultimately we are interesting in inverting all weak equivalences in $\mathbf{Cat}'(S)$ and so need to discuss what it means to add the formal pseudoinverses to a class of 1-cells in a 2-category – a process known as *localisation*. This was done in [Pro96] for the more general case of a class of 1-cells in a bicategory, where the resulting bicategory is constructed and its universal properties (analogous to those of a quotient) examined. The application in *loc. cit.* is to showing the equivalence of various bicategories of stacks to localisations of 2-categories of smooth, topological and algebraic groupoids. The results of this article can be seen as one-half of a generalisation of these results to more general sites.

Definition 6.1. [Pro96] Let B be a bicategory and $W \subset B_1$ a class of 1-cells. A *localisation of* B *with respect to* W is a bicategory $B[W^{-1}]$ and a weak 2-functor

$$U\colon B \to B[W^{-1}]$$

such that U sends elements of W to equivalences, and is universal with this property i.e. composition with U gives an equivalence of bicategories

$$U^*\colon \mathrm{Hom}(B[W^{-1}], D) \to \mathrm{Hom}_W(B, D),$$

where Hom_W denotes the sub-bicategory of weak 2-functors that send elements of W to equivalences (call these W*-inverting*, abusing notation slightly).

The universal property means that W-inverting weak 2-functors $F\colon B \to D$ factor, up to a transformation, through $B[W^{-1}]$, inducing an essentially unique weak 2-functor $\widetilde{F}\colon B[W^{-1}] \to D$.

Definition 6.2. [Pro96] Let B be a bicategory B with a class W of 1-cells. W is said to *admit a right calculus of fractions* if it satisfies the following conditions

2CF1. W contains all equivalences

2CF2. a) W is closed under composition
b) If $a \in W$ and a iso-2-cell $a \overset{\sim}{\Rightarrow} b$ then $b \in W$

2CF3. For all $w\colon A' \to A$, $f\colon C \to A$ with $w \in W$ there exists a 2-commutative square

$$\begin{array}{ccc} P & \xrightarrow{\ g\ } & A' \\ {\scriptstyle v}\downarrow & \overset{\simeq}{\swarrow} & \downarrow{\scriptstyle w} \\ C & \xrightarrow[\ f\ ]{} & A \end{array}$$

with $v \in W$.

2CF4. If $\alpha\colon w \circ f \Rightarrow w \circ g$ is a 2-cell and $w \in W$ there is a 1-cell $v \in W$ and a 2-cell $\beta\colon f \circ v \Rightarrow g \circ v$ such that $\alpha \circ v = w \circ \beta$. Moreover: when α is an iso-2-cell, we require β to be an isomorphism too; when v' and β' form another such pair, there exist 1-cells u, u' such that $v \circ u$ and $v' \circ u'$ are in W, and an iso-2-cell $\epsilon\colon v \circ u \Rightarrow v' \circ u'$ such that the following diagram commutes:

$$\begin{array}{ccc} f \circ v \circ u & \xRightarrow{\ \beta \circ u\ } & g \circ v \circ u \\ {\scriptstyle f \circ \epsilon}\Downarrow\simeq & & \simeq\Downarrow{\scriptstyle g \circ \epsilon} \\ f \circ v' \circ u' & \xRightarrow[\ \beta' \circ u'\ ]{} & g \circ v' \circ u' \end{array} \tag{8}$$

Remark 6.3. In particularly nice cases (as in the next section), the first half of 2CF4 holds due to left-cancellability of elements of W, giving us the canonical choice $v = I$.

Theorem 6.4. *[Pro96] A bicategory* B *with a class* W *that admits a calculus of right fractions has a localisation with respect to* W.

From now on we shall refer to a calculus of right fractions as simply a calculus of fractions, and the resulting localisation as a bicategory of fractions. Since $B[W^{-1}]$ is defined only up to equivalence, it is of great interest to know when a bicategory D in which elements of W are converted to equivalences is itself equivalent to $B[W^{-1}]$. In particular, one would be interested in finding such an equivalent bicategory with a simpler description than that which appears in [Pro96]. Thanks are due to Matthieu Dupont for pointing out (in personal communication) that the statement of proposition 6.5 actually only holds in one direction, as stated below, not in both, as in [Pro96].

Proposition 6.5. *[Pro96] A weak 2-functor* $F : B \to D$ *which sends elements of* W *to equivalences induces an equivalence of bicategories*

$$\widetilde{F}\colon B[W^{-1}] \xrightarrow{\sim} D$$

if the following conditions hold

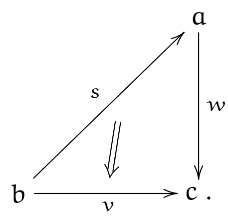
EF1. F *is essentially surjective,*

EF2. For every 1-cell $f \in D_1$ *there are 1-cells* $w \in W$ *and* $g \in B_1$ *such that* $Fg \stackrel{\sim}{\Rightarrow} f \circ Fw$*,*

EF3. F *is locally fully faithful.*

The following is useful in showing a weak 2-functor sends weak equivalences to equivalences, because this condition only needs to be checked on a class that is in some sense cofinal in the weak equivalences.

Theorem 6.6. *In the bicategory* B *and let* $V \subset W$ *be two classes of 1-cells such that for all* $w \in W$*, there exists* $v \in V$ *and* $s \in W$ *such that there is an invertible 2-cell*

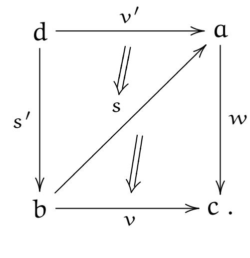


Then a weak 2-functor $F\colon B \to D$ *that sends elements of* V *to equivalences also sends elements of* W *to equivalences.*

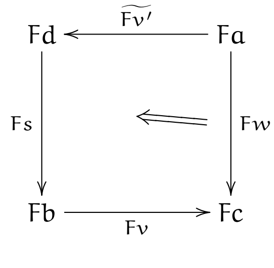
Proof. In the following the coherence cells will be implicit. First we show that Fw has a pseudosection in D for any $w \in W$. Let v, s be as above. Let $\widetilde{Fv}$ be a pseudoinverse of Fv, and let $j = Fs \circ \widetilde{Fv}$. Then there is the following invertible 2-cell

$$Fw \circ j \Rightarrow F(w \circ s) \circ \widetilde{Fv} \Rightarrow Fv \circ \widetilde{Fv} \Rightarrow I.$$

We now show that j is in fact a pseudoinverse for Fw. Since $s \in W$, there is a $v' \in V$ and $s' \in W$ and a 2-cell giving the following diagram



Apply the functor F, and denote pseudoinverses of Fv, Fv' by $\widetilde{Fv}, \widetilde{Fv'}$. Using the 2-cell $I \Rightarrow Fv' \circ \widetilde{Fv'}$ we get the following 2-cell



Then there is this composite invertible 2-cell

$$j \circ Fw \Rightarrow (Fs \circ \widetilde{Fv}) \circ (Fv \circ (Fs \circ \widetilde{Fv'})) \Rightarrow (Fs \circ Fs') \circ \widetilde{Fv'} \Rightarrow Fv' \circ \widetilde{Fv'} \Rightarrow I,$$

making Fw is an equivalence. Hence F sends all elements of W to equivalences. □

7 ANAFUNCTORS ARE A LOCALISATION

In this section we prove the result that $\mathbf{Cat}'(S)$ admits a calculus of fractions for the E-equivalences, and the bicategory of anafunctors is a localisation. Note that E is not required to be subcanonical, but rather that it satisfies a weak saturation condition.

Definition 7.1. Let E be a class of arrows in the ambient category S. E is called a *class of admissible maps* if it is a singleton pretopology containing the split epimorphisms which satisfies the following condition:

(S) If $e \colon A \to B$ is a split epimorphism, and $A \xrightarrow{e} B \xrightarrow{p} C$ is in E, then $p \in E$.

Example 7.2. If E is a saturated singleton pretopology, it is a class of admissible maps. In particular, E could be $J_{sing} = $ (J-epi) for a non-singleton pretopology J on a finitely complete category.

Example 7.3. The singleton pretopology Subm of surjective submersions on **Diff** is subcanonical and satisfies condition (S), but does not contain the split epimorphisms, so is not admissible.

Recall that $\mathbf{Cat}'(S)$ is assumed to be such that all pullbacks of all source and target maps exist.

Theorem 7.4. *Let S be a category with a class E of admissible maps. Assume the 2-category $\mathbf{Cat}'(S)$ admits base change along maps in E and has weak pullbacks. Then $\mathbf{Cat}'(S)$ admits a right calculus of fractions for the class W_E of E-equivalences.*

Proof. We show the conditions of definition 6.2 hold.

2CF1. Since E contains all the split epis, an internal equivalence is essentially E-surjective (c.f. example 4.3). Let $f \colon X \to Y$ be an internal equivalence, and $g \colon Y \to X$ a pseudoinverse. By definition there are natural isomorphisms $a \colon g \circ f \Rightarrow id_X$ and $b \colon f \circ g \Rightarrow id_Y$. To show that f is fully faithful, we first show that the map

$$q \colon X_1 \to X_0^2 \times_{f,Y_0^2} Y_1$$

is a split monomorphism over X_0^2. This diagram commutes

$$\begin{array}{ccc}
X_1 & \longrightarrow & X_0^2 \times_{f,Y_0^2} Y_1 \\
\Vert & & \downarrow \\
X_1 & \xleftarrow{\ \simeq\ } & X_0^2 \times_{gf,X_0^2} X_1\,,
\end{array}$$

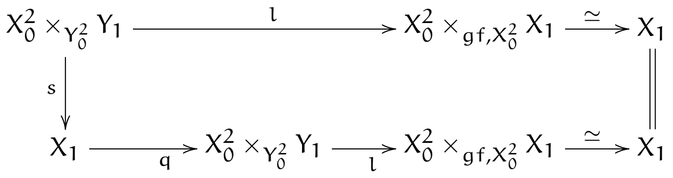
by the naturality of a, the marked isomorphism coming from lemma 2.18, giving the desired splitting (call it s). The splitting commutes with projection to X_0^2 because the isomorphism does. The same argument implies that

$$Y_1 \to Y_0^2 \times_{X_0^2} X_1$$

is a split monomorphism over Y_0^2, and this implies the composite arrow

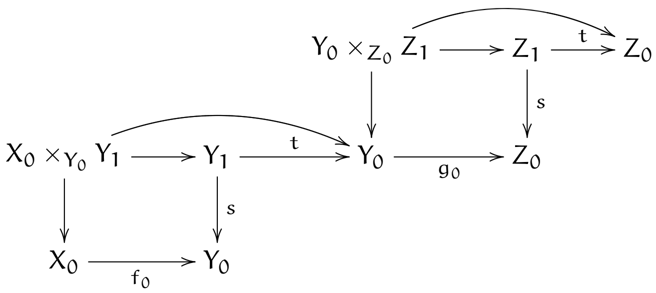
$$l \colon X_0^2 \times_{Y_0^2} Y_1 \to X_0^2 \times_{Y_0^2} Y_0^2 \times_{X_0^2} X_1 \simeq X_0^2 \times_{gf,X_0^2} X_1$$

is a split monomorphism. This diagram commutes



using naturality again, and so $q \circ s = \mathrm{id}$, using the fact l is a monomorphism. Thus q is an isomorphism, and f is fully faithful.

2CF2 a). That the composition of fully faithful functors is again fully faithful is trivial. To show that the composition of essentially E-surjective functors $f\colon X \to Y$, $g\colon Y \to Z$ is again so, consider the following diagram



where the curved arrows are in E by assumption. The lower such arrow pulls back to an arrow $X_0 \times_{Y_0} Y_1 \times_{Z_0} Z_1 \to Y_0 \times_{Z_0} Z_1$ (again in E). Hence the composite

$$X_0 \times_{Y_0} Y_1 \times_{Z_0} Z_1 \to Y_0 \times_{Z_0} Z_1 \xrightarrow{t \circ pr_2} Z_0$$

is in E, and is equal to the composite

$$X_0 \times_{Y_0} Y_1 \times_{Z_0} Z_1 \xrightarrow{\mathrm{id}\times g\times \mathrm{id}} X_0 \times_{Z_0} Z_1 \times_{Z_0} Z_1 \xrightarrow{\mathrm{id}\times m} X_0 \times_{Z_0} Z_1 \xrightarrow{t \circ pr_2} Z_0.$$

The map

$$X_0 \times_{Z_0} Z_1 \simeq X_0 \times_{Y_0} Y_0 \times_{Z_0} Z_1 \xrightarrow{\mathrm{id}\times e\times \mathrm{id}} X_0 \times_{Y_0} Y_1 \times_{Z_0} Z_1$$

is a section of

$$X_0 \times_{Y_0} Y_1 \times_{Z_0} Z_1 \xrightarrow{\mathrm{id}\times g\times \mathrm{id}} X_0 \times_{Z_0} Z_1 \times_{Z_0} Z_1 \xrightarrow{\mathrm{id}\times m} X_0 \times_{Z_0} Z_1.$$

Now condition (S) tells us that $X_0 \times_{Z_0} Z_1 \xrightarrow{t \circ pr_2} Z_0$ is in E, hence $g \circ f$ is essentially E-surjective.

2CF2 b). We will show this in two parts: fully faithful functors are closed under isomorphism, and essentially E-surjective functors are closed under isomorphism. Let $w, f\colon X \to Y$ be functors and $a\colon w \Rightarrow f$ be a natural isomorphism. First, let w be essentially E-surjective. That is,

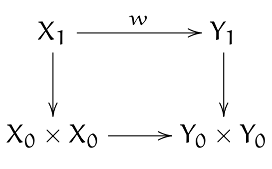
$$X_0 \times_{w,Y_0,s} Y_1 \xrightarrow{t \circ pr_2} Y_0 \qquad (9)$$

is in E. Now note that the map

$$X_0 \times_{f,Y_0,s} Y_1 \xrightarrow{(\mathrm{id},-a)\times \mathrm{id}} X_0 \times \times_{w,Y_0,s} Y_1 \times_{t,Y_0,s} Y_1 \xrightarrow{\mathrm{id}\times m} X_0 \times_{w,Y_0,s} Y_1 \qquad (10)$$

is an isomorphism, and so the composite of (10) and (9) is in E. Thus f is essentially E-surjective.

Now let w be fully faithful. Thus



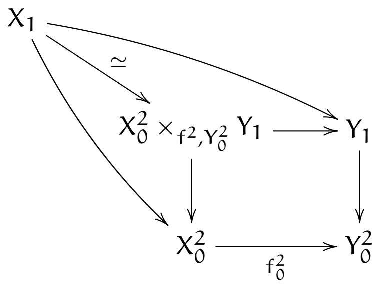
is a pullback square. Using lemma 2.18 there is an isomorphism

$$X_1 \simeq X_0 \times_{w,Y_0,s} Y_1 \times_{t,Y_0,w} X_0 \simeq X_0 \times_{f,Y_0,s} Y_1 \times_{t,Y_0,f} X_0.$$

The composite of this with projection on X_0^2 is $(s,t)\colon X_1 \to X_0^2$, and the composite with

$$\mathrm{pr}_2\colon X_0 \times_{f,Y_0,s} Y_1 \times_{t,Y_0,f} X_0 \to Y_1$$

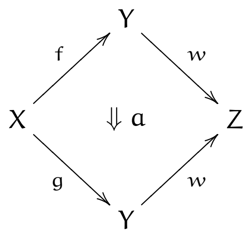
is just f_1 by the diagram 2, and so this diagram commutes



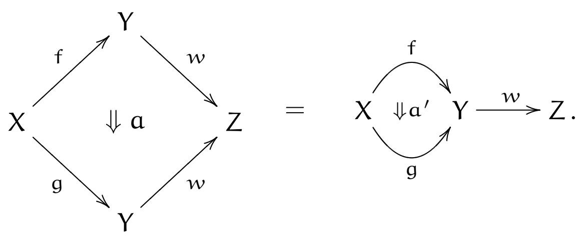
i.e. f is fully faithful.

2CF3. The existence of a 2-commuting square is easy: take the weak pullback (definition 2.14) which exists by definition. 2CF3 follows from 4.10.

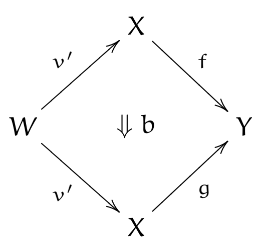
2CF4. Section 4.1 in [Pro96] shows that given a natural transformation



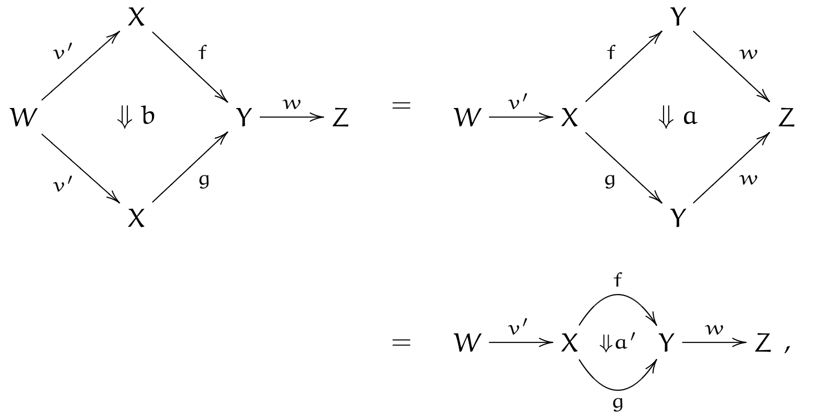
where w is fully faithful (e.g. w is in W_E), there is a unique $a'\colon f \Rightarrow g$ such that



This is the first half of 2CF4, where $v = \mathrm{id}_X$. If $v'\colon W \to X \in W_E$ such that there is a transformation

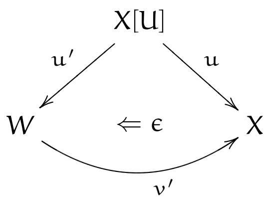


satisfying

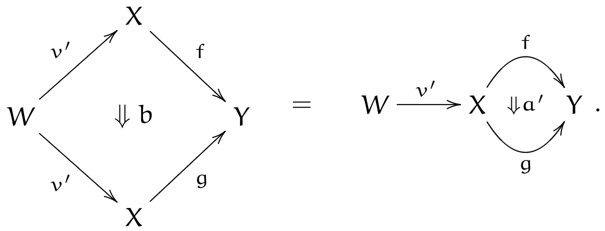


(11)

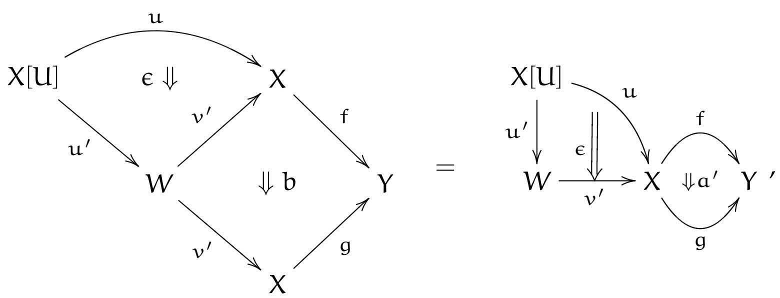
we can choose a J-cover $U \to X_0$, a functor $u' \colon X[U] \to W$ and a natural isomorphism



where, since $J \subset E$, $u \in W_E$, and since $v' \circ u' \overset{\simeq}{\Rightarrow} u$, we have $v' \circ u' \in W_E$ by 2CF2 a) above as required by 2CF4. The uniqueness result from Pronk's argument, together with equation (11) to give us



We paste this with ϵ,



which is precisely the diagram (8) with $v = \mathrm{id}_X$. Hence 2CF4 holds. □

Remark 7.5. If we replace the assumption that E contains the split epimorphisms by the slightly weaker assumption that all internal equivalences in **Cat**′(S) are E-equivalences, then theorem 7.4 still holds as split epimorphisms are only used to prove 2CF1. By a result of [MM05] that internal equivalences of Lie groupoids are Subm-equivalences, we recover the result that theorem 7.4 holds for Lie groupoids and the class of Subm-equivalences, as well as for various sub-2-categories, such as proper étale Lie groupoids aka orbifolds.

Definition 7.6. Given a singleton pretopology J and a class E of admissible maps, we say E is *admissible for* J if $J \leqslant E$.

Example 7.7. J_{sing} is a class of admissible maps for J if J_{sing} contains the split epimorphisms. A saturated singleton pretopology is a class of admissible maps for itself.

If E is a class of admissible maps for J, E-equivalences are J-equivalences and so $W_E \subset W_J$. This means that the 2-functor α_J in proposition 5.19 sends E-equivalences to equivalences. We use this fact and proposition 6.5 to show the following.

Theorem 7.8. *Let (S, J) be a site with a subcanonical singleton pretopology J and let E be a class of admissible maps for J. Then there is an equivalence of bicategories*

$$\textbf{\textit{Cat}}'_{ana}(S, J) \simeq \textbf{\textit{Cat}}'(S)[W_E^{-1}]$$

Proof. Let us show the conditions in proposition 6.5 hold.

EF1. α_J is the identity on 0-cells, and hence surjective on objects.

EF2. This is equivalent to showing that for any anafunctor $(U, f) \colon X \twoheadrightarrow Y$ there are functors w, g such that w is in W_E and

$$(U, f) \overset{\sim}{\Rightarrow} \alpha_J(g) \circ \alpha_J(w)^{-1}$$

where $\alpha_J(w)^{-1}$ is some pseudoinverse for $\alpha_J(w)$.

Let w be the functor $X[U] \to X$ (this has object component in $J \subset E$, hence is an E-equivalence) and let $g = f \colon X[U] \to Y$. First, note that

$$X \longleftarrow X[U] \xrightarrow{=} X[U]$$

is a pseudoinverse for

$$\alpha_J(w) \quad = \quad X[U] \xleftarrow{=} X[U][U] \longrightarrow X \quad .$$

Then the composition $\alpha_J(f) \circ \alpha_J(w)^{-1}$ is

$$X \longleftarrow X[U \times_U U \times_U U] \longrightarrow Y$$

which is isomorphic to (U, f) by the renaming transformation arising from the isomorphism $U \times_U U \times_U U \simeq U$.

EF3. If $a \colon (X_0, f) \Rightarrow (X_0, g)$ is a transformation of anafunctors for functors $f, g \colon X \to Y$, it is given by a natural transformation with component

$$X_0 \times_{X_0} X_0 \to Y_1.$$

But we have declared $X_0 \times_{X_0} X_0 = X_0$. Hence we get a unique natural transformation $a \colon f \Rightarrow g$ such that a is the image of a' under α_J. □

We now give a series of results following from this theorem, using basic properties of pretopologies from section 3.

Corollary 7.9. *When J and K are two subcanonical singleton pretopologies on S such that $J_{sing} = K_{sing}$, there is an equivalence of bicategories*

$$\textbf{\textit{Cat}}'_{ana}(S, J) \simeq \textbf{\textit{Cat}}'_{ana}(S, K).$$

Using corollary 7.9 we see that using a cofinal pretopology gives an equivalent bicategory of anafunctors.

If E is any class of admissible maps for subcanonical J, the bicategory of fractions inverting W_E is equivalent to that of J-anafunctors. Hence

Corollary 7.10. *Let* E *be a class of admissible maps for the subcanonical pretopology* J*. There is an equivalence of bicategories*

$$\mathbf{Cat}'(S)[W_E^{-1}] \simeq \mathbf{Cat}'(S)[W_J^{-1}]$$

where of course $W_J = W_{J_{sing}}$.

Finally, if (S, J) is a superextensive site (like **Top** with its usual pretopology of open covers), we have the following result which is useful when J is not a singleton pretopology.

Corollary 7.11. *Let* (S, J) *be a superextensive site where* J *is a subcanonical pretopology. Then*

$$\mathbf{Cat}'(S)[W_{J_{sing}}^{-1}] \simeq \mathbf{Cat}'_{ana}(S, \amalg J).$$

Proof. This essentially follows from the corollary to lemma 3.23. □

Obviously this can be combined with previous results, for example if $K \leqslant \amalg J$, for J a non-singleton pretopology, K-anafunctors localise $\mathbf{Cat}'(S)$ at the class of J-equivalences.

8 SIZE CONSIDERATIONS

The 2-category $\mathbf{Cat}'(S)$ is locally small, similar to the case of the 2-category of small categories (and in fact the latter is cartesian closed). However the construction of $B[W^{-1}]$ given by Pronk, even for a locally small bicategory B is *a priori* not necessarily locally small (or even locally essentially small). Recall that the axiom of choice for a site (S, J) is that for all J-epimorphisms $p\colon P \to A$ there exists a section of p. This is too strong an assumption in practice. In many algebraic situations one has projective covers, for instance in **Grp** (every group has an epimorphism from a free group). We can rephrase this by saying the full subcategory of **Grp**$/G$ consisting of the epimorphisms has a weakly initial object. More generally one could ask only that the category of all singleton covers of an object (see definition 8.3 below) has a *set* of weakly initial objects. This is the content of the axiom WISC below. We first give some more precise definitions.

Definition 8.1. A category C has a *weakly initial set* $\mathcal{I}$ of objects if for every object A of C there is an arrows $O \to A$ from some object $O \in \mathcal{I}$.

Every small category has a weakly initial set, namely its set of objects.

Example 8.2. The category $\mathtt{Field}$ of fields has a weakly initial set, consisting of the prime fields $\{\mathbb{Q}, \mathbb{F}_p | p \text{ prime}\}$. To contrast, the category of sets with surjections for arrows doesn't have a weakly initial set of objects.

Definition 8.3. Let (S, J) be a site. For any object A, the *category of covers of* A, denoted J/A has as objects the covering families $(U_i \to A)_{i \in I}$ and as morphisms $(U_i \to A)_{i \in I} \to (V_j \to A)_{j \in J}$tuples consisting of a function $r\colon I \to J$ and arrows $U_i \to V_{r(i)}$ in S/A.

When J is a singleton pretopology this is simply a full subcategory of S/A. We now define the axiom WISC, due to Mike Shulman, which in a sense limits how much Choice fails to hold. Let (S, J) be a site.

WISC (Weakly Initial Set of Covers). *For every object* A *of* S*, the category* J/A *has a weakly initial set of objects.*

When S is **Set** with surjections as covers, this is implied by the axiom COSHEP (category of sets has enough projectives). Without the condition that this is a *set* of objects (as opposed to a class or large set) then this would be true of all sites.

Example 8.4. Any regular category with enough projectives with the regular pretopology satisfies WISC.

Example 8.5. Assuming Choice in the metalogic – that is, in **Set** – $(\mathbf{Top}, \mathcal{O})$ and $(\mathbf{Diff}, \mathcal{O})$ satisfy WISC.

Choice may be more than is necessary here; it would be interesting to see if WISC in $(\mathbf{Set}, \mathtt{surjections})$ is enough to prove WISC in these cases, analogous to how enough injectives in a topos proves enough injectives for abelian group objects therein.

Lemma 8.6. *If* (S, J) *satisfies WISC, then so does* (S, J_{sing}).

Lemma 8.7. *If* (S, J) *is a superextensive site,* (S, J) *satisfies WISC if and only if* $(S, \amalg J)$ *does.*

Proposition 8.8. *Let* (S, J) *be a site with a subcanonical singleton pretopology* J*, satisfying WISC. Then* $\mathbf{Cat}'_{ana}(S, J)$ *is locally essentially small.*

Proof. Let $I(A)$ be a weakly initial set for J/A. Consider the locally full sub-2-category of $\mathbf{Cat}'_{ana}(S, J)$ with the same objects, and arrows those anafunctors $(U, f) : X \rightarrow\!\!\!\!\!\!+\; Y$ such that $U \to X_0$ is in $I(X_0)$. Every anafunctor is then isomorphic, by the generalisation of example 5.8, to one in this sub-2-category. □

Corollary 8.9. *Any localisation* $\mathbf{Cat}'(S)[W^{-1}_{J_{sing}}]$*, when it exists, is locally essentially small for* (S, J) *satisfying WISC.*

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