Introduction to Noncommutative Geometry of Commutative Algebras and Applications in Physics

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Abstract

This is a selfcontained introduction to noncommutative geometry, concentrating on noncommutative differential calculi on commutative algebras and geometric structures built on them. Many examples of relevance for physics are discussed, including models on discrete spaces and applications of the formalism in the context of soliton equations.

RECENT DEVELOPMENTS IN GRAVITATION AND MATHEMATICAL PHYSICS
Proceedings of the Second Mexican School on Gravitation and Mathematical Physics
eds. A. Garcia, C. Lämmerzahl, A. Macias, T. Matos, and D. Nuñez
Science Network Publishing 1997
ISBN 3-9805735-0-8
on-line version: http://kaluza.physik.uni-konstanz.de/2MS

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1 Introduction

> The mathematical space has an extremely simple structure in the small. To whatever small dimensions one proceeds, there is always the same – apart from a "similarity transformation". This seems to be too simple to accomodate a map of the real events.
>
> *E. Schrödinger*[1]

An adequate mathematical description of nature has to involve 'geometric' structures beyond smooth manifolds. In particular, one has to admit fractal sets which appear in many natural systems and especially as strange attractors in nonlinear dynamical systems. It is also widely believed that quantum theory should revise our present conception of space and time. Proposals have been made towards a fundamental discrete and even noncommutative structure of space-time (see [2] for a short review with a large collection of references).

Manifolds, which are usually taken to model spaces and space-times in physical theories, are in particular topological spaces. The latter are in correspondence with commutative associative algebras. Given an abstract commutative algebra, it determines a topological space on which the elements of the algebra are realized as functions. On the other hand, the algebra of functions on a topological space defines a commutative and associative algebra. Instead of dealing with topological spaces, we may study commutative associative algebras. This point of view suggests a drastic generalization, namely to dispense with the requirement of commutativity of the algebra. Does it make sense to regard a (certain) noncommutative algebra as a kind of 'space' in which physical phenomena can be described? Indeed, an important example is provided by the canonical quantization procedure. Given a phase space, the classical observables are functions on it, i.e., functions of coordinates q^i and conjugate momenta p_j. Quantization replaces them by operators (on a Hilbert space) satisfying the Heisenberg commutation relations $[q^i, p_j] = i\,\hbar\,\delta^i_j$. The (suitably restricted) algebra generated by operators satisfying these relations is the Heisenberg algebra. It may serve as an example of a 'noncommutative space' (see [3, 4], in particular). A 'noncommutative torus' appears as a 'noncommutative Brillouin zone' in a treatment of the quantum Hall effect [5]. Further examples are provided by 'quantum groups' which are noncommutative analogues of (the algebra of functions on) a classical group.

Whereas in classical differential geometry the introduction of geometric structures essentially begins with the notion of a vector field, this appears not to be the adequate concept to start with in generalizing geometric concepts to an associative algebra $\mathcal{A}$.[2] Instead, it is more convenient to first introduce the concept of a differential form. The algebra $\mathcal{A}$ is extended to a 'differential algebra' on which further geometric structures are then built. In the case of the algebra of smooth functions on a (smooth) manifold there is a natural choice for such a differential algebra, namely the algebra of (ordinary) differential forms. For a general algebra there is no distinguished differential algebra and one has to understand what the significance is of the different choices. Even in the case of the algebra of smooth functions on a manifold, there is no longer a good argument to single out the ordinary calculus of differential forms. Indeed, the exploration of other differential calculi opened a door to a whole new world of geometry and applications in physics, as well as relations with other fields of mathematics. Indeed, differential calculi on commutative (associative) algebras and their noncommutative geometries shall occupy a considerable part of this report.[3]

The formalism of noncommutative geometry is an extremely radical abstraction of ordinary differential geometry. It includes the latter as a special case, but allows a huge variety of structures which may or

[1] This is my translation of the original German "Der mathematische Raum hat eine überaus einfache Struktur im Kleinen. Zu wie kleinen Dimensionen man auch übergeht, es liegt immer wieder dasselbe vor – abgesehen von einer "Ähnlichkeitstransformation". Das scheint zu einfach zu sein, um eine Landkarte des wirklichen Geschehens darin unterzubringen." [1].

[2] See [6], however. On (the algebra of functions on) a finite set there is no derivation except the trivial one (which maps all functions to zero).

[3] A 'noncommutative' geometry on a commutative algebra is characterized by noncommutativity of functions and (generalized) differentials.

may not be of any real use in mathematics or physics. An immediate prospect is that in this framework we have the possibility to 'deform' ordinary differential geometry and models built on it while keeping basic concepts and (simple) recipes on which the models are based.

Section 2 introduces to differential calculus on associative algebras and, in particular, commutative algebras. Section 3 is devoted to gauge theory on algebras, i.e., connections in noncommutative geometry. Section 4 presents some applications of the formalism in the context of integrable models and soliton equations. Some concluding remarks are collected in section 5.

This report is meant as an elementary introduction to some modern developments. We are not at all trying to cover the existing literature in the already wide field of noncommutative geometry. For various aspects quite remote from our selection, we refer to [7, 8], for example. Rather, we present various examples of applications in physics and cross relations with other areas of mathematics which are of relevance for physics. Our presentation should provide the reader with an easy access to understanding basic ideas behind and the usefulness of the new mathematical concepts.

An earlier report on various aspects of differential calculus on commutative algebras appeared in [9]. Though some of the material presented there is basically taken over to the present report with some improvements, a considerable amount of new results has been obtained since then. To a large extent this concerns my own work jointly with Aristophanes Dimakis.

2 Differential calculi on associative algebras

Let $\mathcal{A}$ be an associative algebra[4] with unit $\mathbb{1}$. A *differential calculus* on $\mathcal{A}$ is a $\mathbb{Z}$-graded associative algebra (over $\mathbb{R}$, respectively $\mathbb{C}$)[5]

$$\Omega(\mathcal{A}) = \bigoplus_{r \geq 0} \Omega^r(\mathcal{A}) \tag{2.1}$$

where the spaces $\Omega^r(\mathcal{A})$ are $\mathcal{A}$-bimodules[6] and $\Omega^0(\mathcal{A}) = \mathcal{A}$. There is a ($\mathbb{R}$- respectively $\mathbb{C}$-) linear map

$$\mathrm{d} \; : \quad \Omega^r(\mathcal{A}) \to \Omega^{r+1}(\mathcal{A}) \tag{2.2}$$

with the following properties,

$$\mathrm{d}^2 \;=\; 0 \tag{2.3}$$

$$\mathrm{d}(w\, w') \;=\; (\mathrm{d}w)\, w' + (-1)^r\, w\, \mathrm{d}w' \tag{2.4}$$

where $w \in \Omega^r(\mathcal{A})$ and $w' \in \Omega(\mathcal{A})$. The last relation is usually referred to as the (generalized) *Leibniz rule*. Assuming that $\mathbb{1}$ extends to a unit element in Ω, the Leibniz rule and the identity $\mathbb{1}\mathbb{1} = \mathbb{1}$ imply

$$\mathrm{d}\mathbb{1} \;=\; 0\,. \tag{2.5}$$

We shall furthermore assume that d generates the spaces $\Omega^r(\mathcal{A})$ for $r > 0$ in the sense that $\Omega^r(\mathcal{A}) = \mathcal{A}\, \mathrm{d}\Omega^{r-1}(\mathcal{A})\, \mathcal{A}$. Using the Leibniz rule, every element of $\Omega^r(\mathcal{A})$ can be written as a linear combination[7] of monomials $a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_r$. The action of d is then determined by

$$\mathrm{d}(a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_r) = \mathrm{d}a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_r\,. \tag{2.6}$$

Examples.
1. Setting $\Omega^r(\mathcal{A}) = \{0\}$ for $r > 0$ and $\mathrm{d} \equiv 0$ provides us with a trivial example, the smallest possible

[4] More precisely, we consider only algebras over $\mathbb{R}$ or $\mathbb{C}$. Finite linear combinations of elements of $\mathcal{A}$ with coefficients in $\mathbb{R}$ or $\mathbb{C}$, respectively, and also finite products are again elements of $\mathcal{A}$. The multiplication is assumed to be associative.

[5] Though in many interesting cases one has $\Omega^r(\mathcal{A}) = \{0\}$ when r is larger than some $r_0 \geq 0$, one also encounters examples where $\Omega(\mathcal{A})$ is actually an infinite sum. In such a case we should define $\Omega(\mathcal{A})$ as the space of finite sums.

[6] Hence its elements, called *r-forms*, can be multiplied from left and right by elements of $\mathcal{A}$.

[7] A technical problem arises if infinite sums of r-forms are admitted.

differential calculus.
2. There is also a largest differential calculus on $\mathcal{A}$. Let

$$\tilde{\Omega}^1(\mathcal{A}) = \{\sum_i a_i \otimes b_i \mid \sum_i a_i b_i = 0,\ a_i, b_i \in \mathcal{A}\} \tag{2.7}$$

(where only finite sums are admitted).[8] Then $a \mapsto \mathbb{1} \otimes a - a \otimes \mathbb{1}$ defines a linear map $\tilde{\mathrm{d}} : \mathcal{A} \to \tilde{\Omega}^1(\mathcal{A})$ satisfying (2.4) for $w, w' \in \mathcal{A}$. The space of r-forms is defined as

$$\tilde{\Omega}^r(\mathcal{A}) := \underbrace{\tilde{\Omega}^1(\mathcal{A}) \otimes_{\mathcal{A}} \tilde{\Omega}^1(\mathcal{A}) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \tilde{\Omega}^1(\mathcal{A})}_{r\text{-times}}, \qquad r > 0 \tag{2.8}$$

where the tensor product is now over $\mathcal{A}$ (which means that elements of $\mathcal{A}$ can be commuted from one side to the other). In order to extend the operator $\tilde{\mathrm{d}}$ to $\tilde{\Omega}(\mathcal{A})$, we next define its action on 1-forms. Since every 1-form can be written as a linear combination of terms of the form $w = a\,(\mathbb{1} \otimes b - b \otimes \mathbb{1})$ with $a, b \in \mathcal{A}$, it is sufficient to specify the action of $\tilde{\mathrm{d}}$ on such terms:

$$\begin{aligned} \tilde{\mathrm{d}}w &= (\mathbb{1} \otimes a - a \otimes \mathbb{1}) \otimes_{\mathcal{A}} (\mathbb{1} \otimes b - b \otimes \mathbb{1}) \\ &= \mathbb{1} \otimes a \otimes b - \mathbb{1} \otimes ab \otimes \mathbb{1} - a \otimes \mathbb{1} \otimes b + a \otimes b \otimes \mathbb{1}\,. \end{aligned} \tag{2.9}$$

Via the Leibniz rule (with respect to the product $\otimes_{\mathcal{A}}$) the linear operator $\tilde{\mathrm{d}}$ then extends to the whole differential algebra. The differential calculus $(\tilde{\Omega}(\mathcal{A}), \tilde{\mathrm{d}})$ is *universal* in the sense that every other differential calculus can be derived from it (as a quotient $\tilde{\Omega}(\mathcal{A})/\tilde{\mathcal{I}}$ where $\tilde{\mathcal{I}}$ is a two-sided ideal in $\tilde{\Omega}(\mathcal{A})$ closed under $\tilde{\mathrm{d}}$, i.e., $\tilde{\mathrm{d}}\tilde{\mathcal{I}} \subset \tilde{\mathcal{I}}$). In the mathematical literature it is usually called *universal differential envelope*. ♡

Between the smallest (trivial) and the biggest (universal) differential calculus there is a large variety of further differential calculi on an algebra $\mathcal{A}$. In contrast to the preceding two examples, they depend on the algebra under consideration. Some of them are distinguished via specific properties of the algebra or structures which can be defined on it.

Examples.
3. In case of a group, it is natural to require that the left and right action of the group on itself extends to a differential algebra (on the algebra of functions) on the group. More generally, in case of Hopf algebras (including quantum groups) there is a coaction which one might wish to extend to act on a differential calculus on the Hopf algebra. This leads to the notions of left-, right- and bi-covariant differential calculi [10].
4. If $\mathcal{A}$ is a C^*-algebra, there is a representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ in terms of bounded operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{D}$ be an operator on $\mathcal{H}$ such that $[\mathcal{D}, \rho(\mathcal{A})] \subset \mathcal{B}(\mathcal{H})$. Then ρ induces a representation $\rho : \tilde{\Omega}(\mathcal{A}) \to \mathcal{B}(\mathcal{H})$ via

$$a_0\, \tilde{\mathrm{d}}a_1 \cdots \tilde{\mathrm{d}}a_r \mapsto \rho(a_0)\,[\mathcal{D}, \rho(a_1)] \cdots [\mathcal{D}, \rho(a_r)]\,. \tag{2.10}$$

Though this construction takes care of the Leibniz rule, the property $\tilde{\mathrm{d}}^2 = 0$ would severely constrain the possible choices of an operator $\mathcal{D}$. Rather, one defines

$$\Omega(\mathcal{A})_{\mathcal{D}} := \tilde{\Omega}(\mathcal{A})/\{\ker\rho + \tilde{\mathrm{d}}\ker\rho\} \tag{2.11}$$

which carries the structure of a differential calculus. Here it is the choice of the operator $\mathcal{D}$ which determines the differential calculus. If $\mathcal{M}$ is a smooth manifold with a Riemannian metric, choosing $\mathcal{A} = C^\infty(\mathcal{M})$ and for $\mathcal{D}$ the Dirac operator, the differential algebra turns out to be isomorphic with the algebra of (ordinary) differential forms on $\mathcal{M}$. Behind all this is the idea of a 'quantum mechanical approach' towards geometry. Suppose that in some way we obtain from measurements a set of 'geometrical' data which form the spectrum of some selfadjoint operator $\mathcal{D}$. Notions of classical differential geometry, or generalizations thereof, may then be derived from this operator. The construction outlined above is due to A. Connes (see [11], in particular).

[8] The tensor product is over $\mathbb{R}$, respectively $\mathbb{C}$.

5. Let $\mathcal{A}$ be the (commutative) algebra of smooth functions on a differentiable manifold $\mathcal{M}$. The algebra of smooth differential forms together with the exterior derivative d is a differential calculus on $\mathcal{A}$. From other differential calculi on $\mathcal{A}$ it is distinguished by the property that functions and differentials commute, i.e., $[f, \mathrm{d}h] = 0$ for all $f, h \in \mathcal{A}$. Using (2.4), this implies anticommutativity of 1-forms, a property which is intimately related to our classical conception how to measure volumes. Originally differentials were introduced in a rather naive way with the interpretation of being 'infinitesimally small'. Modern mathematics put them on a rigorous footing as objects acting on vector fields. See also [12, 13].
6. Superspace differential forms [14, 3]. These play an important role in the context of supersymmetric field theories. ♡

Further examples are treated in more detail in the following subsections.

Remark. In general, we have to face the problem of 'differentiability', i.e., the applicability of d to the algebra $\mathcal{A}$. In case of commutative algebras we are familiar with corresponding notions, whereas the treatment of noncommutative algebras requires a more sophisticated machinery like that of Connes sketched above. But the problem of applicability of d extends to $\Omega(\mathcal{A})$, of course. In the following we shall treat differentials as algebraic objects and mostly leave aside the question how to introduce them in a perhaps more satisfactory way or how to 'realize' them.

2.1 Differential calculi on a finite set

Let $\mathcal{M}$ be a finite set and $\mathcal{A}$ the algebra of all $\mathbb{C}$-valued functions on it. $\mathcal{A}$ is generated by $\{e_i\}$ where $e_i(j) = \delta_{ij}$ for $i, j \in \mathcal{M}$. These functions satisfy the two identities

$$e_i\, e_j = \delta_{ij}\, e_j\,, \qquad \sum_i e_i = \mathbb{1}\,. \tag{2.12}$$

The following result shows that there are no 'vector fields' in the sense of derivations on a finite set.

Lemma. There are no derivations $\partial : \mathcal{A} \to \mathcal{A}$ besides $\partial \equiv 0$.
Proof: If ∂ is a derivation on $\mathcal{A}$, so that it satisfies the product rule of differentiation, then from the first equation of (2.12) one finds

$$(\partial e_i)\, e_i + e_i\, \partial e_i = \partial e_i$$

which implies

$$e_i\, (\partial e_i)\, e_i = 0\,.$$

Writing $\partial e_i = \sum_k f_i(k)\, e_k$ with constants $f_i(k)$, we get

$$0 = e_i\, (\sum_k f_i(k)\, e_k)\, e_i = f_i(i)\, e_i$$

and thus $f_i(i) = 0$ for all $i \in \mathcal{M}$. Furthermore,

$$\partial e_i = \sum_k f_i(k)\, e_k\, e_i + e_i \sum_k f_i(k)\, e_k = 2\, f_i(i)\, e_i = 0$$

and therefore $\partial \equiv 0$. ♡

As a consequence of the identities (2.12) and the Leibniz rule, the differentials $\mathrm{d}e_i$ of a differential calculus on $\mathcal{A}$ are subject to the following relations,

$$\mathrm{d}e_i\; e_j = -e_i\, \mathrm{d}e_j + \delta_{ij}\, \mathrm{d}e_j\,, \qquad \sum_i \mathrm{d}e_i = 0\,. \tag{2.13}$$

Without additional constraints, we are dealing with the universal differential calculus $(\tilde{\Omega}(\mathcal{A}), \tilde{\mathrm{d}})$. Introducing the 1-forms

$$e_{ij} = e_i\, \tilde{\mathrm{d}} e_j \qquad (i \neq j) \tag{2.14}$$

one finds the following.

Lemma. $\{e_{ij}\}_{i\neq j}$ is a basis over $\mathbb{C}$ of the space $\tilde\Omega^1$ of universal 1-forms.

Proof: First we show completeness. By definition, a 1-form is a linear combination of terms of the form $f\,(\mathrm{d}g)\,h$ with $f,g,h\in\mathcal{A}$. Now

$$\begin{aligned} f\,(\mathrm{d}g)\,h &= \sum_{i,j,k} f(i)\,e_i\,g(j)\,\mathrm{d}e_j\,h(k)\,e_k \\ &= \sum_{i,j,k} f(i)\,g(j)\,h(k)\,e_i\,(-e_j\,\mathrm{d}e_k+\delta_{jk}\,\mathrm{d}e_k) \\ &= \sum_{i,j} f(i)\,[g(j)-g(i)]\,h(j)\,e_i\,\mathrm{d}e_j \\ &= \sum_{i\neq j} f(i)\,[g(j)-g(i)]\,h(j)\,e_i\,\mathrm{d}e_j \\ &= \sum_{i,j} f(i)\,[g(j)-g(i)]\,h(j)\,e_{ij}\ . \end{aligned}$$

To demonstrate linear independence of the e_{ij} let us assume that

$$\sum_{i\neq j} c_{ij}\,e_{ij}=0$$

with constants c_{ij}. Using $e_i\,e_{jk}=\delta_{ij}\,e_{jk}$ and

$$e_{ij}\,e_k=e_i\,\tilde{\mathrm{d}}e_j\,e_k=e_i\,[-e_j\,\tilde{\mathrm{d}}e_k+\delta_{jk}\,\tilde{\mathrm{d}}e_k]=\delta_{jk}\,e_{ij}$$

for $i\neq j$, we have

$$0=e_k\,(\sum_{i\neq j} c_{ij}\,e_{ij})\,e_l=c_{kl}\,e_{kl}\ .$$

Since $e_{kl}\neq 0$ in the universal calculus, this implies $c_{kl}=0$. ♡

Lemma. All first order differential calculi are obtained from the universal one by setting some of the e_{ij} to zero.

Proof: A general theorem (see [15], for example) tells us that all first order differential calculi are obtained as the quotient of $\tilde\Omega^1$ by a two-sided ideal ($\mathcal{A}$-sub-bimodule) in $\tilde\Omega^1$. So we have to determine all two-sided ideals in $\tilde\Omega^1$. First we note that, as a consequence of the relations $e_k\,e_{ij}=\delta_{ki}\,e_{ij}$ and $e_{ij}\,e_k=\delta_{jk}\,e_k$, $\{c\,e_{ij}\mid c\in\mathbb{C}\}$ is such an ideal and moreover a primitive one, it has no proper subideals. Since $\{e_{ij}\}_{i\neq j}$ is a basis of $\tilde\Omega^1$ over $\mathbb{C}$, it follows that every two-sided ideal is a union of such primitive ones. ♡

Let us associate with each nonvanishing e_{ij} of some differential calculus (Ω,d) an arrow from the point i to the point j :

$$e_{ij}\neq 0 \quad\Longleftrightarrow\quad i\bullet\longrightarrow\bullet j\ . \tag{2.15}$$

The universal (first order) differential calculus then corresponds to the complete digraph where all vertices are connected with each other by a pair of antiparallel arrows. Other first order differential calculi are obtained by deleting some of the arrows.

Hence, the choice of a (first order) differential calculus on a finite set means choosing a connection structure on it. This is not just an abstract correspondence since the formula for the differential of a function $f\in\mathcal{A}$ precisely displays this structure:[9]

$$\mathrm{d}f=\sum_{i,j}[f(j)-f(i)]\,e_{ij}\ . \tag{2.16}$$

[9]More precisely, the summation runs over all i,j with $i\neq j$. Note that e_{ii} has not been defined. We may, however, set $e_{ii}:=0$.

Returning to the universal calculus, concatenation of the 1-forms e_{ij} leads to the $(r-1)$-forms

$$e_{i_1\dots i_r} := e_{i_1 i_2}\, e_{i_2 i_3} \cdots e_{i_{r-1} i_r} \ . \tag{2.17}$$

The significance of these forms should be evident from the following.

Lemma. Let $r > 1$. Then the forms $e_{i_1\dots i_r}$ constitute a basis of $\tilde{\Omega}^{r-1}$ over $\mathbb{C}$. ♡

We omit the proof. The special forms (2.17) satisfy the simple relations

$$e_{i_1\dots i_r}\, e_{j_1\dots j_s} = \delta_{i_r j_1}\, e_{i_1\dots i_{r-1} j_1 \dots j_s} \ . \tag{2.18}$$

In particular, this implies

$$f\, e_{i_1\dots i_r} = f(i_1)\, e_{i_1\dots i_r} \, , \qquad e_{i_1\dots i_r}\, f = e_{i_1\dots i_r}\, f(i_r) \ . \tag{2.19}$$

for $f \in \mathcal{A}$. Furthermore, we have

$$\tilde{\mathrm{d}} e_i = \sum_j (e_{ji} - e_{ij}) \tag{2.20}$$

$$\tilde{\mathrm{d}} e_{ij} = \sum_k (e_{kij} - e_{ikj} + e_{ijk}) \tag{2.21}$$

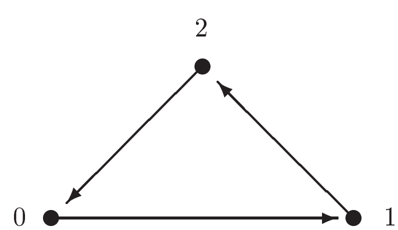
$$\vdots$$

where the dots stand for corresponding formulas for the differentials of the higher basis forms. The first equation is a special case of (2.16). In a 'reduced' differential calculus (Ω, d) where not all of the e_{ij} are present, the possibilities to build (nonvanishing) higher forms $e_{i_1\dots i_r}$ are restricted and the above formulas for $\mathrm{d} e_{i_1\dots e_r}$ impose further constraints on them. An example is treated in the following subsection.

A discrete set together with a differential calculus on it has been named a *discrete differential manifold* in [16] and some general properties have been studied there.

2.1.1 Differential calculi and topology

Let us consider the differential calculus on a 3-point set associated with the following graph:



The nonvanishing basis 1-forms are thus e_{01}, e_{12}, e_{20}. The only basis 2-forms we can construct from these are e_{012}, e_{120} and e_{201}. But $e_{10} = 0$ implies

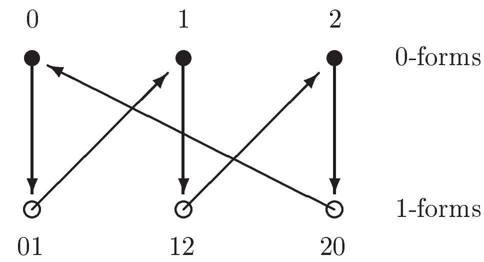
$$0 = \mathrm{d} e_{10} = \sum_{k=1}^{3} (e_{k10} - e_{1k0} + e_{10k}) = -e_{120} \tag{2.22}$$

and similarly $e_{012} = e_{201} = 0$ as a consequence of $e_{02} = e_{21} = 0$. Hence there are no 2-forms and we can assign the *dimension* 1 to the 3-point set with the differential calculus specified above.[10] The action of d can be visualized as follows. In our example, we have

$$\mathrm{d} e_0 = e_{20} - e_{01} \, , \quad \mathrm{d} e_1 = e_{01} - e_{12} \, , \quad \mathrm{d} e_2 = e_{12} - e_{20} \ . \tag{2.23}$$

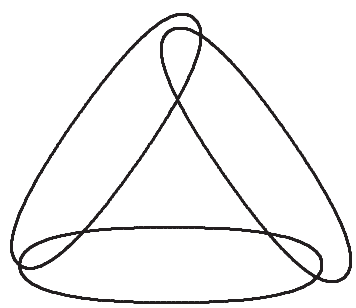
[10] This notion of dimension is a local one. In general, it varies from subgraph to subgraph.

This determines the following diagram,[11]

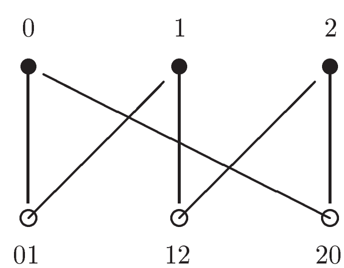


If in the expression for $\mathrm{d}e_i$ a 1-form e_{jk} appears on the rhs, an arrow is drawn between the vertices representing the 0-form e_i and the 1-form e_{jk}. The sign of a term on the rhs of (2.23) determines its orientation. This scheme extends to higher order forms via (2.21).

An interpretation. An n-dimensional manifold can be covered with a finite set $\mathcal{M} = \{\mathcal{U}_\alpha\}$ of sets $\mathcal{U}_\alpha$ which are homeomorphic to open balls in $\mathbb{R}^n$. The global topology of the space is then encoded in the way in which these sets intersect. The overlap relations can be expressed using a *Hasse diagram*. How this works is explained with an example:



The corresponding Hasse diagram is



The points of the upper row represent the three big sets $\mathcal{U}_\alpha$, $\alpha = 1, 2, 3$, depicted in the last figure. Those of the lower row represent the intersection sets. If a line connects two points in different (neighbouring) rows, this means that the set which the lower point represents is a subset of the one for which the upper point stands. A vertex together with all lower lying vertices which are connected to it forms an open set. In the present case, $\{01\}, \{12\}, \{20\}, \{0, 01, 20\}, \{1, 01, 12\}, \{2, 12, 20\}$ are the open sets (besides the empty and the whole set). Ignoring the additional structure encoded in the orientation of the arrows in the previous diagram relating 0-forms and 1-forms, what we have there is just the Hasse diagram. We

[11] In our example, this diagram does not contain more information than the triangle digraph we started with. This is so since here we did not impose restrictions on the universal differential calculus on the level of r-forms with $r > 1$ (besides those induced by the restrictions on the 1-form level).

conclude that a differential calculus on a finite set determines a topology on the set. For an exploration of these relations see [17, 16]. In particular, the formalism gives an answer to the question how to discretize a field theory on a manifold in such a way that global topological properties are preserved. It also appears to be of interest in the context of attempts to formulate a theory of unprecise space and space-time measurements avoiding the notion of a (sharp) point (see [18, 19], for example).

2.1.2 The Ritz-Rydberg principle

Let ν_{ij} be the frequency of emitted (or absorbed) light in an electronic transition from level i to level j in an atom. Then there is an addition law,

$$\nu_{ij} + \nu_{jk} = \nu_{ik} \tag{2.24}$$

which is known from the period before the birth of quantum mechanics as the *Ritz-Rydberg combination principle*. Let us define

$$\nu := \sum_{i,j} \nu_{ij}\, e_{ij} \ . \tag{2.25}$$

Then

$$\begin{aligned} \mathrm{d}\nu &= \sum_{i,j,k} \nu_{ij}\,(e_{kij} - e_{ikj} + e_{ijk}) \\ &= \sum_{i,j,k} (\nu_{jk} - \nu_{ik} + \nu_{ij})\, e_{ijk} \end{aligned} \tag{2.26}$$

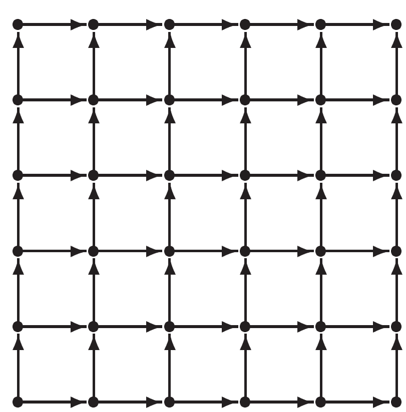
in terms of which the above combination principle becomes $\mathrm{d}\nu = 0$, i.e., ν is closed. If the first cohomology group of the differential calculus is trivial (so that every closed 1-form is exact), then we have $\nu = \mathrm{d}(H/h)$ with a function $H = \sum_n E_n\, e_n$. This is the energy if h is taken to be Planck's constant. The cohomology condition is satisfied in particular for the universal differential calculus. See also [7, 17].

2.1.3 A lattice differential calculus

Let $\mathcal{M} = \mathbb{Z}^n$. For $a, b \in \mathcal{M}$ we define a differential calculus by

$$e_{ab} \neq 0 \quad \Longleftrightarrow \quad b = a + \hat{\mu} \qquad \text{where} \quad \hat{\mu} = \left(\delta^\nu_\mu\right) \ . \tag{2.27}$$

This corresponds to an oriented lattice graph, a finite part of which is drawn below.



In this example we are actually dealing with an *infinite* set and thus infinite sums in some formulas which requires special care. Introducing the lattice coordinate functions

$$x^\mu := \sum_a a^\mu \, e_a \tag{2.28}$$

one obtains[12]

$$[\mathrm{d}x^\mu, x^\nu] = \delta^{\mu\nu} \, \mathrm{d}x^\mu \tag{2.29}$$

using (2.18). $\{\mathrm{d}x^\mu\}$ is a left $\mathcal{A}$-module basis of Ω^1. We will return to this example in section 2.2.1.

2.1.4 Representations of first order differential calculi on finite sets

We have seen that first order differential calculi on a set of N elements are in bijective correspondence with digraphs with N vertices (and at most a pair of antiparallel arrows between any two vertices). Let us define an $N \times N$-matrix $\mathcal{D}$ in the following way: $\mathcal{D}_{ij} = 1$ if there is an arrow from i to j and $\mathcal{D}_{ij} = 0$ otherwise. In graph theory this matrix is known as the *adjacency matrix* of the digraph. It encodes the complete structure of the digraph. We should then expect that the (first order) differential calulus determined by a digraph can be expressed in terms of the adjacency matrix. How to construct a derivation $\mathrm{d} : \mathcal{A} \to \Omega^1(\mathcal{A})$ with this matrix? The easiest way to obtain a derivation is via a commutator,

$$\mathrm{d}f := [\mathcal{D}, f] \tag{2.30}$$

which, of course, only makes sense if the elements of $\mathcal{A}$ can be represented as $N \times N$-matrices. The simplest way of achieving this is via

$$f \quad \mapsto \quad \begin{pmatrix} f(1) & & 0 \\ & \ddots & \\ 0 & & f(N) \end{pmatrix} . \tag{2.31}$$

Representing e_{ij} as the $N \times N$-matrix E_{ij} with a 1 in the ith row and jth column and zeros elsewhere, the above expression for $\mathrm{d}f$ in terms of $\mathcal{D}$ is precisely our formula (2.16). Note that the adjacency matrix represents $\sum_{i,j} e_{ij}$.

Proceeding beyond 1-forms, the above representation will not respect the $\mathbb{Z}_2$-grading of a differential algebra $\Omega(\mathcal{A})$. Therefore, one considers instead a 'doubled' representation[13]

$$e_i \mapsto \begin{pmatrix} E_{ii} & 0 \\ 0 & E_{ii} \end{pmatrix} , \qquad e_{ij} \mapsto \begin{pmatrix} 0 & E_{ij}^\dagger \\ E_{ij} & 0 \end{pmatrix} . \tag{2.32}$$

The grading can be expressed in terms of a grading operator which in our case is given by

$$\gamma := \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} . \tag{2.33}$$

It is selfadjoint and satisfies

$$\gamma^2 = \mathbf{1} , \qquad \gamma \, \hat{\mathcal{D}} = -\hat{\mathcal{D}} \, \gamma \qquad \gamma \, \hat{f} = \hat{f} \, \gamma \tag{2.34}$$

with

$$\hat{\mathcal{D}} := \begin{pmatrix} 0 & \mathcal{D}^\dagger \\ \mathcal{D} & 0 \end{pmatrix} \qquad \hat{f} := \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \tag{2.35}$$

[12] Let us give a proof for the one-dimensional case ($n = 1$). Using $\mathrm{d}x = \sum_a a \, \mathrm{d}e_a = \sum_{a,b} a \, (e_{ba} - e_{ab}) = \sum_a a \, (e_{a-1,a} - e_{a,a+1}) = \sum_a e_{a,a+1}$ we find $[\mathrm{d}x, x] = \sum_{a,b} b \, [e_{a,a+1}, e_b] = \sum_a e_{a,a+1} = \mathrm{d}x$.

[13] See also [20] for further generalizations.

where f has to be represented as in (2.31).

The above representations of (first order) differential calculi make contact with Connes' formalism (cf also example 4 in the beginning of section 2). It should be noticed that $\mathcal{D}$ in the present context is *not*, in general, a *selfadjoint* operator (on the Hilbert space $\mathbb{C}^N$). The 'doubling' in (2.32) leads to a selfadjoint operator on the Hilbert space $\mathcal{H} = \mathbb{C}^{2N}$, however. $(\mathcal{A}, \mathcal{H}, \hat{\mathcal{D}})$ is an example of an *even spectral triple*, a basic structure in Connes' approach to noncommutative geometry. According to Connes [21] (see also [11] for a refinement), a *spectral triple* $(\mathcal{A}, \mathcal{H}, \hat{\mathcal{D}})$ consists of an involutive algebra $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}$ together with a selfadjoint operator $\hat{\mathcal{D}}$ satisfying some technical conditions. It is called *even* when there is a grading operator γ as in our example.

2.1.5 Connes' distance function for differential calculi on finite sets

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. A *state* on $\mathcal{A}$ is a linear map $\phi : \mathcal{A} \to \mathbb{C}$ which is positiv, i.e., $\phi(a^* a) \geq 0$ for all $a \in \mathcal{A}$, and normalized, i.e., $\phi(\mathbb{1}) = 1$. According to Connes, the distance between two states ϕ and ϕ' is given by

$$d(\phi, \phi') := \sup\{|\phi(a) - \phi'(a)| \; ; \; a \in \mathcal{A}, \, \|[\mathcal{D}, a]\| \leq 1\} \,. \tag{2.36}$$

Given a set $\mathcal{M}$, each point $p \in \mathcal{M}$ defines a state ϕ_p via $\phi_p(f) := f(p)$ for all functions f on $\mathcal{M}$. The above formula then becomes

$$d(p, p') := \sup\{|f(p) - f(p')| \; ; \; f \in \mathcal{A}, \, \|[\mathcal{D}, f]\| \leq 1\} \,. \tag{2.37}$$

In the following, we give some simple examples (see also [22]).

Example 1. The universal first order differential calculus on a set of two elements p, q is described by a graph consisting of two points which are connected by a pair of antiparallel arrows. Its adjacency matrix is

$$\mathcal{D} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.38}$$

so that

$$[\mathcal{D}, f] = \begin{pmatrix} 0 & f(p) - f(q) \\ f(q) - f(p) & 0 \end{pmatrix} \,. \tag{2.39}$$

Then

$$\|[\mathcal{D}, f]\|^2 = \sup_{\|\psi\|=1} \|[\mathcal{D}, f]\, \psi\|^2 = \sup_{\|\psi\|=1} |f(p) - f(q)|^2 \, (|\psi_1|^2 + |\psi_2|^2) = |f(p) - f(q)|^2 \tag{2.40}$$

for $\psi \in \mathbb{C}^2$. It follows that Connes' distance function defined with the adjacency matrix gives $d(p, q) = 1$ in this case. See also [21].

Example 2. Let us consider the first order differential calculus on a set of N elements determined by the following graph.

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \;\cdots\; \bullet \longrightarrow \bullet$$

The corresponding adjacency matrix is

$$\mathcal{D}_N = \begin{pmatrix} 0 & 1 & 0 & \cdots 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \,. \tag{2.41}$$

With a complex function f we associate a real function h via

$$h_1 := 0\,, \quad h_{i+1} := h_i + |f_{i+1} - f_i| \qquad i = 1, \ldots, N-1\;. \tag{2.42}$$

where $f_i := f(i)$, numbering the consecutive lattice sites by $1, \ldots, N$. Then $|h_{i+1} - h_i| = |f_{i+1} - f_i|$ and

$$\|[\hat{\mathcal{D}}_N, \hat{f}]\,\psi\| = \|[\hat{\mathcal{D}}_N, \hat{h}]\,\psi\| \tag{2.43}$$

for all $\psi \in \mathbb{C}^{2N}$. As a consequence, in calculating the supremum over all functions f in the definition of Connes' distance function, it is now sufficient to consider only *real* functions (see also [23] for a related example). Since $\hat{\mathcal{D}}_N$ is hermitean, the matrix $i\,[\hat{\mathcal{D}}_N, \hat{f}]$ is then also hermitean and its norm is equal to the maximal absolute value of its eigenvalues. For $N = 2$ one finds the characteristic polynomial

$$\det\Big(i\,[\hat{\mathcal{D}}_2, \hat{f}] - \lambda\,\mathbf{1}\Big) = \lambda^2\,(\lambda^2 - (f_2 - f_1)^2) \tag{2.44}$$

(with a real constant λ) and for arbitrary $N > 2$

$$\det\Big(i\,[\hat{\mathcal{D}}_N, \hat{f}] - \lambda\,\mathbf{1}\Big) = (\lambda^2 - (f_N - f_{N-1})^2)\,\det\Big([\hat{\mathcal{D}}_{N-1}, \hat{f}] - \lambda\,\mathbf{1}\Big)\;. \tag{2.45}$$

This implies

$$\|[\hat{\mathcal{D}}_N, \hat{f}]\| = \max\{|f_2 - f_1|, \ldots, |f_N - f_{N-1}|\} \tag{2.46}$$

from which we conclude that $d(i,j) = |i - j|$. ♡

2.2 A class of noncommutative differential calculi on a commutative algebra

Let $\mathcal{A}$ be the associative and commutative algebra over $\mathbb{R}$ (or $\mathbb{C}$) freely generated by elements x^μ, $\mu = 1, \ldots, n$. For example, the x^μ could be the canonical coordinates on $\mathbb{R}^n$ (or $\mathbb{C}^n$). The ordinary differential calculus on $\mathcal{A}$ has the property $[\mathrm{d}x^\mu, x^\nu] = 0$, i.e., differentials and functions commute. Relaxing this property, there is a class of *noncommutative differential calculi* such that[14]

$$[\mathrm{d}x^\mu, x^\nu] = C^{\mu\nu}{}_\kappa\,\mathrm{d}x^\kappa \tag{2.47}$$

with *structure functions* $C^{\mu\nu}{}_\kappa(x^\lambda)$ which have to satisfy some consistency condition. First, we have

$$\begin{aligned} [\mathrm{d}x^\mu, x^\nu] &= (\mathrm{d}x^\mu)\,x^\nu - x^\nu\,\mathrm{d}x^\mu \\ &= \mathrm{d}(\underbrace{x^\mu x^\nu - x^\nu x^\mu}_{=0}) - x^\mu\,\mathrm{d}x^\nu + (\mathrm{d}x^\nu)\,x^\mu = [\mathrm{d}x^\nu, x^\mu]\;. \end{aligned} \tag{2.48}$$

Assuming the differentials $\mathrm{d}x^\mu$, $\mu = 1, \ldots, n$, to be linearly independent[15], this implies

$$C^{\mu\nu}{}_\kappa = C^{\nu\mu}{}_\kappa\;. \tag{2.49}$$

Furthermore,

$$\begin{aligned} 0 &= ([\mathrm{d}x^\mu, x^\nu] - C^{\mu\nu}{}_\kappa\,\mathrm{d}x^\kappa)\;x^\lambda \\ &= [(\mathrm{d}x^\mu)\,x^\lambda, x^\nu] - C^{\mu\nu}{}_\kappa\,(\mathrm{d}x^\kappa)\,x^\lambda \\ &= [x^\lambda\,\mathrm{d}x^\mu + C^{\mu\lambda}{}_\rho\,\mathrm{d}x^\rho, x^\nu] - C^{\mu\nu}{}_\kappa\,(x^\lambda\,\mathrm{d}x^\kappa + C^{\kappa\lambda}{}_\rho\,\mathrm{d}x^\rho) \\ &= x^\lambda\,[\mathrm{d}x^\mu, x^\nu] + C^{\mu\lambda}{}_\rho\,[\mathrm{d}x^\rho, x^\nu] - x^\lambda\,C^{\mu\nu}{}_\kappa\,\mathrm{d}x^\kappa - C^{\mu\nu}{}_\kappa\,C^{\kappa\lambda}{}_\rho\,\mathrm{d}x^\rho \\ &= (C^{\mu\lambda}{}_\rho\,C^{\rho\nu}{}_\kappa - C^{\mu\nu}{}_\rho\,C^{\rho\lambda}{}_\kappa)\,\mathrm{d}x^\kappa \end{aligned} \tag{2.50}$$

which leads to

$$C^{\lambda\mu}{}_\rho\,C^{\nu\rho}{}_\kappa = C^{\nu\mu}{}_\rho\,C^{\lambda\rho}{}_\kappa \tag{2.51}$$

[14] On the rhs of this equation we are using the summation convention.
[15] More precisely, we assume here that the $\mathrm{d}x^\mu$ form a left $\mathcal{A}$-module basis of $\Omega^1(\mathcal{A})$.

or, in terms of the matrices C^μ with entries $(C^\mu)^\nu{}_\kappa := C^{\mu\nu}{}_\kappa$,

$$C^\mu \, C^\nu = C^\nu \, C^\mu \ . \tag{2.52}$$

For constant $C^{\mu\nu}{}_\kappa$ and $n \leq 3$, a classification of all solutions of the consistency conditions (2.49) and (2.51) has been given in [24] (see also [25] for some earlier results).

Remark. Defining a product on an n-dimensional vector space over $\mathbb{R}$ (or $\mathbb{C}$) with basis ξ^μ by

$$\xi^\mu \cdot \xi^\nu = C^{\mu\nu}{}_\kappa \, \xi^\kappa$$

with constants $C^{\mu\nu}{}_\kappa$, the two conditions (2.49) and (2.51) mean that we are dealing with a commutative and associative algebra. The classification of first order differential calculi on $\mathbb{R}^n$ (or $\mathbb{C}^n$) with constant structure functions thus corresponds to the classification of commutative associative algebras. ♡

2.2.1 The simplest example

Let $\mathcal{A}$ be the algebra of all functions on $\mathbb{R}$. It is generated by the canonical coordinate function x. The simplest deformation of the ordinary differential calculus on $\mathcal{A}$ is then determined by

$$[\mathrm{d}x, x] = \ell \, \mathrm{d}x \tag{2.53}$$

where ℓ is a constant which we will choose to be real and positive. This is a special case of the commutation structure (2.47) considered in the previous subsection and we encountered it already in section 2.1.3. Written in the form

$$\mathrm{d}x \, x = (x + \ell) \, \mathrm{d}x \ , \tag{2.54}$$

the above commutation relation extends to $\mathcal{A}$ as

$$\mathrm{d}x \, f(x) = f(x + \ell) \, \mathrm{d}x \ . \tag{2.55}$$

Furthermore,

$$\begin{aligned} \mathrm{d}f &=: (\partial_{+x} f) \, \mathrm{d}x = \frac{1}{\ell} \, (\partial_{+x} f) \, [\mathrm{d}x, x] \\ &= \frac{1}{\ell} \, [(\partial_{+x} f) \, \mathrm{d}x, x] = \frac{1}{\ell} \, [\mathrm{d}f, x] \\ &= \frac{1}{\ell} \, (\mathrm{d}(f \, x - x \, f) - [f, \mathrm{d}x]) = \frac{1}{\ell} (\mathrm{d}x \, f - f \, \mathrm{d}x) \\ &= \frac{1}{\ell} [f(x + \ell) - f(x)] \, \mathrm{d}x \end{aligned} \tag{2.56}$$

so that the *left partial derivative* defined via the first equality turns out to be the right discrete derivative, i.e.,

$$\partial_{+x} f = \frac{1}{\ell} [f(x + \ell) - f(x)] \ . \tag{2.57}$$

Introducing a *right partial derivative* via $\mathrm{d}f = \mathrm{d}x \, \partial_{-x} f$, an application of (2.55) shows that it is the left discrete derivative, i.e.,

$$\partial_{-x} f = \frac{1}{\ell} [f(x) - f(x - \ell)] \ . \tag{2.58}$$

An *indefinite integral* should have the property

$$\int \mathrm{d}f = f + \text{‘constant’} \tag{2.59}$$

where 'constants' are functions annihilated by d. These are just the functions with period ℓ (so that $f(x+\ell)=f(x)$). It turns out that every function can be integrated and an explicit formula can be found in [26].

Example. Using the Leibniz rule,

$$\int x\,\mathrm{d}x = \int \mathrm{d}x^2 - \int \underbrace{(\mathrm{d}x)\,x}_{=(x+\ell)\,\mathrm{d}x} = x^2 - \ell\,x - \int x\,\mathrm{d}x + \text{periodic function} \tag{2.60}$$

and thus

$$\int x\,\mathrm{d}x = \frac{1}{2}\,x\,(x-\ell) + \text{periodic function}\ . \tag{2.61}$$

In [25] a recursion formula is given for the integral of an arbitrary monomial in x. ♡

Since the indefinite integral is only determined up to the addition of an arbitrary function with period ℓ, it defines a *definite integral* only if the region of integration is an interval the length of which is a multiple of ℓ (or a union of such intervals). Then one obtains

$$\int_{x_0-m\ell}^{x_0+n\ell} f(x)\,\mathrm{d}x = \ell \sum_{k=-m}^{n-1} f(x_0+k\ell) \tag{2.62}$$

and in particular

$$\int_{x_0-\infty}^{x_0+\infty} f(x)\,\mathrm{d}x = \ell \sum_{k=-\infty}^{\infty} f(x_0+k\ell)\ . \tag{2.63}$$

The integral thus simply picks out the values of f on a *lattice* with spacings ℓ and forms the Riemann integral for the corresponding piecewise constant function on $\mathbb{R}$.

All this shows that the differential calculus with $\ell > 0$ is also well-defined on the algebra of functions on a lattice with spacings ℓ.

2.2.2 q-Calculus

Let us consider the new coordinate

$$y = q^{x/\ell} \tag{2.64}$$

with $q \in \mathbb{C} \setminus \{0\}$, not a root of unity. Using (2.56), we have

$$\mathrm{d}y = \frac{q-1}{\ell}\,y\,\mathrm{d}x \tag{2.65}$$

and with the help of (2.55) the commutation relation (2.53) is transformed into

$$\mathrm{d}y\,y = q\,y\,\mathrm{d}y\ . \tag{2.66}$$

The generalized partial derivatives with respect to the new coordinate y are defined by

$$\mathrm{d}f = \partial_{+y}f\ \mathrm{d}y = \mathrm{d}y\,\partial_{-y}f \tag{2.67}$$

and turn out to be the *q-derivatives*

$$\partial_{+y}f(y) = \frac{f(qy)-f(y)}{(q-1)\,y}\,, \qquad \partial_{-y}f(y) = \frac{f(y)-f(q^{-1}y)}{(1-q^{-1})\,y}\ . \tag{2.68}$$

These satisfy the 'quantum plane' relation

$$\partial_{-y}\,\partial_{+y} - q\,\partial_{+y}\,\partial_{-y} = 0 \tag{2.69}$$

and q-deformed canonical commutation relations with the coordinate function y (regarded as an operator), i.e.,

$$\partial_{-y}\,y - q^{-1}\,y\,\partial_{-y} = 1\,, \qquad \partial_{+y}\,y - q\,y\,\partial_{+y} = 1\,. \tag{2.70}$$

Moreover, choosing $x_0 = 0$ in (2.63), one finds

$$\begin{aligned}\int_0^\infty f(y)\,\mathrm{d}y &= \int_{0-\infty}^{0+\infty} f(y(x))\,\frac{q-1}{\ell}\,y(x)\,\mathrm{d}x \\ &= (q-1)\sum_{k=-\infty}^{\infty} f(y(k\ell))\,y(k\ell) \\ &= (q-1)\sum_{k=-\infty}^{\infty} f(q^k)\,q^k \end{aligned} \tag{2.71}$$

which is the *Jackson q-integral*. We have indeed established it as an 'integral', associated with a non-commutative differential calculus. The above equations show that there is a method to perform a change of variables under the q-integral, something mathematicians have actually been looking for (cf [27], p. 46). For further details we refer to [26]. q-Calculus has various applications (see [27], in particular). It experienced a revival in the context of integrable models and quantum groups (see [28], for example).

2.2.3 Lattice differential calculus

The 'lattice differential calculus' considered in subsection 2.2.1 generalizes to higher dimensions as follows,

$$[\mathrm{d}x^\mu, x^\nu] = \ell\,\delta^{\mu\nu}\,\mathrm{d}x^\nu\,, \tag{2.72}$$

a relation which we already encountered in subsection 2.1.3.[16] More generally,

$$\mathrm{d}x^\mu\,f(x) = f(x+\ell^\mu)\,\mathrm{d}x^\mu \tag{2.73}$$

with $(x+\ell^\mu)^\nu := x^\nu + \delta^{\mu\nu}\,\ell$ and $f \in \mathcal{A}$, where $\mathcal{A}$ is the space of all functions on $\mathbb{R}^n$.

Assuming that the $\mathrm{d}x^\mu$, $\mu = 1,\ldots,n$, constitute a basis of the space of 1-forms as a left and as a right $\mathcal{A}$-module, we can introduce left and right partial derivatives via

$$\mathrm{d}f = \partial_{+\mu} f\;\mathrm{d}x^\mu = \mathrm{d}x^\mu\,\partial_{-\mu} f\,. \tag{2.74}$$

Then we find

$$\partial_{+\mu} f = \frac{1}{\ell}\,[f(x+\ell^\mu) - f(x)]\,, \qquad \partial_{-\mu} f = \frac{1}{\ell}\,[f(x) - f(x-\ell^\mu)]\,. \tag{2.75}$$

Acting with d on (2.72) leads to

$$\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\mathrm{d}x^\nu\,\mathrm{d}x^\mu\,. \tag{2.76}$$

This property allows us to introduce a Hodge $\star$ operator in the familiar way,

$$\star\,(\mathrm{d}x^{\mu_1}\ldots\mathrm{d}x^{\mu_r}) := \frac{1}{(n-r)!}\sum \epsilon^{\mu_1\ldots\mu_r}{}_{\mu_{r+1}\ldots\mu_n}\,\mathrm{d}x^{\mu_{r+1}}\cdots\mathrm{d}x^{\mu_n} \tag{2.77}$$

[16] More generally, we may consider $[\mathrm{d}x^\mu, x^\nu] = \ell^\mu\,\delta^{\mu\nu}\,\mathrm{d}x^\nu$ with (possibly) different constants ℓ^μ, $\mu = 1,\ldots,n$. Note that there is no summation over repeated indices in these commutation relations.

where $\epsilon_{\mu_1\ldots\mu_n}$ is totally antisymmetric with $\epsilon_{1\ldots n} = 1$ and indices are raised with $\delta^{\mu\nu}$ or $\eta^{\mu\nu}$, in analogy with the cases of Euclidean and Minkowski geometry. The extension to arbitrary forms is done via[17]

$$\star(w\, f) = f\, \star w \tag{2.78}$$

for $f \in \mathcal{A}$ and $w \in \Omega^r$.

An indefinite integral is determined by the property

$$\int \mathrm{d}f = f + \text{‘constant’} \tag{2.79}$$

where a ‘constant’ is a function with period ℓ in each argument. It therefore only defines a definite integral over special sets, namely those which are a union of hypercubes with edges of length ℓ. Then it turns out that

$$\int_{x_0^1}^{x_0^1+\ell} \cdots \int_{x_0^n}^{x_0^n+\ell} f(x)\, \mathrm{d}x^1 \cdots \mathrm{d}x^n = \ell^n\, f(x_0) \tag{2.80}$$

(cf [25]). Choosing a decomposition of $\mathbb{R}^n$ into such hypercubes, we obtain

$$\int_{\mathbb{R}^n} f(x)\, \mathrm{d}^n x = \ell^n \sum_{k \in \mathbb{Z}^n} f(x_0 + k\ell)\ . \tag{2.81}$$

The integral depends on the choice of a point $x_0 \in \mathbb{R}^n$ which determines how the lattice $\ell\mathbb{Z}^n$ is embedded in $\mathbb{R}^n$. The above formulas show that we could have started equally well with the algebra of functions on the lattice $\ell\mathbb{Z}^n$ instead of the algebra of functions on $\mathbb{R}^n$. Thus (2.72) defines a differential calculus on a lattice.

Example. The Lagrangian for a free massive scalar field ϕ on Euclidean space $\mathbb{R}^n$ reads

$$\mathcal{L} = \mathrm{d}\phi \wedge \star\, \mathrm{d}\phi + m^2\, \phi \wedge \star\, \phi \tag{2.82}$$

in terms of ordinary differential forms. This expression also makes sense for the deformed differential calculus introduced above. Then

$$\begin{aligned}
\mathrm{d}\phi \star \mathrm{d}\phi &= \partial_{+\mu}\phi\, \mathrm{d}x^\mu \star (\partial_{+\nu}\phi\, \mathrm{d}x^\nu) \\
&= \partial_{+\mu}\phi\, \mathrm{d}x^\mu \star \mathrm{d}x^\nu\, (\partial_{+\nu}\phi)(x - \ell^\nu) \\
&= \partial_{+\mu}\phi\, \mathrm{d}x^\mu\, (\partial_{+\nu}\phi)(x - \ell^\nu) \star \mathrm{d}x^\nu \\
&= \partial_{+\mu}\phi\, (\partial_{+\nu}\phi)(x + \ell^\mu - \ell^\nu)\, \mathrm{d}x^\mu \star \mathrm{d}x^\nu \\
&= \partial_{+\mu}\phi\, (\partial_{+\nu}\phi)(x + \ell^\mu - \ell^\nu)\, \delta^{\mu\nu}\, \mathrm{d}x^1 \ldots \mathrm{d}x^n \\
&= \sum_\mu (\partial_{+\mu}\phi)^2\, \mathrm{d}x^1 \ldots \mathrm{d}x^n
\end{aligned} \tag{2.83}$$

and

$$\phi \star \phi = \phi(x)^2\, \star 1 = \phi(x)^2\, \mathrm{d}x^1 \ldots \mathrm{d}x^n \tag{2.84}$$

shows that $S = \int_{\mathbb{R}^n} \mathcal{L}$ is the usual lattice action for the free scalar field. ♡

2.2.4 A class of noncommutative differential calculi on smooth manifolds

The following is taken from [29, 30, 31] to which we refer for further information.

[17] An alternative (different) definition would be $\star(f\, w) = f\, \star w$. In the applications discussed below, however, the ‘twist’ in (2.78) appears to be essential.

Let $\mathcal{A}$ be the algebra of smooth functions on a smooth n-dimensional manifold $\mathcal{M}$, i.e., $\mathcal{A} = C^\infty(\mathcal{M})$. Let x^μ be local coordinate functions and $g^{\mu\nu}$ the corresponding components of a symmetric tensor field on $\mathcal{M}$. We define a differential calculus (Ω, d) via

$$[\mathrm{d}x^\mu, x^\nu] = g^{\mu\nu}\,\tau \tag{2.85}$$

where τ is a 1-form satisfying

$$[\tau, x^\mu] = 0\,, \quad \tau^2 = 0\,, \quad \mathrm{d}\tau = 0\,. \tag{2.86}$$

Furthermore, we assume that $\mathrm{d}x^\mu$ and τ constitute a basis of Ω^1 as a left $\mathcal{A}$-module. The differential of a function f can then be expressed as follows,

$$\mathrm{d}f = \hat{\partial}_\tau f\,\tau + \hat{\partial}_\mu f\,\mathrm{d}x^\mu \tag{2.87}$$

where $\hat{\partial}_\tau$ and $\hat{\partial}_\mu$ are generalized (left) partial derivatives.

Lemma.

$$\mathrm{d}f = \frac{1}{2}\,g^{\mu\nu}\partial_\mu\partial_\nu f\,\tau + \partial_\mu f\,\mathrm{d}x^\mu \tag{2.88}$$

Proof: Exploiting the Leibniz rule, we find

$$\begin{aligned} & \hat{\partial}_\tau(fh)\,\tau + \hat{\partial}_\mu(fh)\,\mathrm{d}x^\mu = \mathrm{d}(fh) = (\mathrm{d}f)\,h + f\,\mathrm{d}h \\ = \;& (\hat{\partial}_\tau f)\,\tau\,h + (\hat{\partial}_\mu f)\,\mathrm{d}x^\mu\,h + f\,(\hat{\partial}_\tau h)\,\tau + f\,(\hat{\partial}_\mu h)\,\mathrm{d}x^\mu \\ = \;& ((\hat{\partial}_\tau f)\,h + f\,\hat{\partial}_\tau h + g^{\mu\nu}\,\hat{\partial}_\mu f\,\hat{\partial}_\nu h)\,\tau + ((\hat{\partial}_\mu f)\,h + f\,\hat{\partial}_\mu h)\,\mathrm{d}x^\mu \end{aligned}$$

using

$$[\mathrm{d}x^\mu, h] = [\mathrm{d}h, x^\mu] = \hat{\partial}_\nu h\,[\mathrm{d}x^\nu, x^\mu]\,.$$

Hence

$$\begin{aligned} \hat{\partial}_\mu(fh) &= (\hat{\partial}_\mu f)\,h + f\,\hat{\partial}_\mu h \\ \hat{\partial}_\tau(fh) &= (\hat{\partial}_\tau f)\,h + f\,\hat{\partial}_\tau h + g^{\mu\nu}\,\hat{\partial}_\mu f\,\hat{\partial}_\nu h\,. \end{aligned}$$

According to the first equation, $\hat{\partial}_\mu$ is a derivation. But derivations on $C^\infty(\mathcal{M})$ are vector fields. Using

$$\hat{\partial}_\mu x^\nu = \delta^\nu_\mu$$

which follows from the above formula for $\mathrm{d}f$, we conclude that

$$\hat{\partial}_\mu = \frac{\partial}{\partial x^\mu} =: \partial_\mu\,.$$

Writing $\hat{\partial}_\tau$ as $\hat{\partial}_\tau = \frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\partial_\nu + \delta$ with an operator δ, the other equation which we obtained from the evaluation of the Leibniz rule is turned into $\delta(fh) = (\delta f)\,h + f\,\delta h$, so that δ is a derivation and thus a vector field. Using $\hat{\partial}_\tau x^\mu = 0$ one finds $\delta = 0$. ♡

Lemma. If $g^{\mu\nu}$ are the components of a symmetric tensor field and τ a scalar 1-form on $\mathcal{M}$, then the commutation relations (2.85) are coordinate-independent (and thus define a global structure on $\mathcal{M}$).

Proof:

$$\begin{aligned} [\mathrm{d}x^{\mu'}, x^{\nu'}] &= [\hat{\partial}_\tau x^\mu\,\tau + \partial_\lambda x^{\mu'}\,\mathrm{d}x^\lambda, x^{\nu'}] \\ &= \partial_\lambda x^{\mu'}\,[\mathrm{d}x^\lambda, x^{\nu'}] \\ &= \partial_\lambda x^{\mu'}\,[\mathrm{d}x^{\nu'}, x^\lambda] \\ &= \partial_\lambda x^{\mu'}\,\partial_\kappa x^{\nu'}\,[\mathrm{d}x^\kappa, x^\lambda] \\ &= \partial_\lambda x^{\mu'}\,\partial_\kappa x^{\nu'}\,g^{\kappa\lambda}\,\tau \\ &= g^{\mu'\nu'}\,\tau\,. \end{aligned}$$

Relation with bicovariant differential calculus on quantum groups.
The structure of a (Lie) group can be reformulated as an algebraic structure on the space of functions on the group. This leads to a commutative *Hopf algebra*. Such an algebra can then be deformed into a noncommutative Hopf algebra. The analogy with the canonical quantization procedure motivated the name *quantum group*. An example is $SL_q(2)$, $q \in \mathbb{C}$, which is a deformation of (the algebra of functions on) $SL(2,\mathbb{C})$. It is the algebra $\mathcal{A}$ generated by elements x^k, $k = 1, \ldots, 4$, which satisfy certain commutation relations and the 'q-determinant' constraint $x^1 x^4 - q\, x^2 x^3 = \mathbb{1}$ (with a unit element $\mathbb{1}$). The additional Hopf algebra structure can be used to narrow down the large number of possible differential calculi on such an algebra and leads to the concept of a *bicovariant* differential calculus [10]. Of particular interest appear to be those calculi for which the dimension of the space of 1-forms (as a left or right $\mathcal{A}$-module) coincides with the number of generators of $\mathcal{A}$. For $SL_q(2)$ this is four or three, depending whether one believes that the above quadratic constraint should eliminate one of the generators. It turns out that there are no three-dimensional bicovariant calculi. There are two four-dimensional bicovariant calculi which both do not yield the ordinary differential calculus on $SL(2,\mathbb{C})$ in the classical limit $q \to 1$. One of them has the form (2.85) after elimination of x^4 via the determinant constraint [32] (see also [30, 31]).

Relation with proper time theories and stochastic calculus on manifolds.
When $\tau = \gamma\, \mathrm{d}t$ where t is a parameter (extra coordinate), we may consider (smooth) functions $f(x^\mu, t)$ depending also on t. (2.87) then has to be replaced by[18]

$$\mathrm{d}f = (\partial_t + \frac{\gamma}{2}\, g^{\mu\nu}\, \partial_\mu \partial_\nu)\, f\ \mathrm{d}t + \partial_\mu f\ \mathrm{d}x^\mu\ . \tag{2.89}$$

1. Let $\gamma = -i\,\hbar$. The (generalized) partial derivative associated with t is then the operator $i\,\hbar\,\partial_t + (\hbar^2/2)\, g^{\mu\nu}\, \partial_\mu \partial_\nu$ which appears in the Schrödinger equation of quantum mechanics. If $g^{\mu\nu}$ is the Minkowski metric, this is the five-dimensional Schrödinger operator of *proper time* quantum theory (see [33] for a review). The noncommutative differential calculus may thus be viewed as a basic structure underlying such proper time theories. See [34, 29].

2. The formula (2.89), with a positive definite metric $\gamma\, g^{\mu\nu}$, is known in the theory of stochastic processes as the *Itô formula*. This suggests that the noncommutative differential calculus provides us with a convenient framework to deal with stochastic processes on manifolds [35]. Up to first order there is indeed a translation [30] to the (Itô) calculus of stochastic differentials. In contrast to the Itô calculus, our differential calculus admits an extension to higher order forms. It is not known, however, whether there is a stochastic interpretation of the higher order forms. See also [31]. For an application of the formalism to kinetic theory see [36].

There are differential calculi with (generalized) partial derivatives which are differential operators of n-th order with an arbitrary $n \in \mathbb{N}$ [24]. See also section 4.2 for an example.

3 Connections in noncommutative geometry

We start with a rather abstract definition of a *connection*[19] and then deduce familiar formulas from it (though still their contents is pretty much unfamiliar, in general).

Let (Ω, d) be a differential calculus on an associative algebra $\mathcal{A}$. A *connection* on a left $\mathcal{A}$-module Γ is a $\mathbb{C}$-linear map

$$\nabla\ :\ \Gamma \to \Omega^1 \otimes_{\mathcal{A}} \Gamma \tag{3.90}$$

such that

$$\nabla(f\,\gamma) = \mathrm{d}f \otimes_{\mathcal{A}} \gamma + f\, \nabla\gamma \tag{3.91}$$

[18] In the proof of the first Lemma, the operator δ now becomes ∂_t.
[19] In the context of classical differential geometry this is due to Koszul. It has been generalized to the framework of noncommutative geometry by Karoubi and Connes.

for all $f \in \mathcal{A}$ and $\gamma \in \Gamma$. This extends to a linear map

$$\nabla : \Omega \otimes_{\mathcal{A}} \Gamma \to \Omega \otimes_{\mathcal{A}} \Gamma \tag{3.92}$$

via

$$\nabla(\chi \, \Psi) = \mathrm{d}\chi \, \Psi + (-1)^r \, \chi \, \nabla\Psi \qquad (\chi \in \Omega^r, \; \Psi \in \Omega \otimes_{\mathcal{A}} \Gamma) \; . \tag{3.93}$$

The *curvature* of ∇ is the map ∇^2.

If Γ has a basis e^A, $A = 1, \ldots, n$, then

$$\gamma = \gamma_A \, e^A \qquad (\gamma_A \in \mathcal{A}) \tag{3.94}$$

$$\nabla\gamma = \mathrm{d}\gamma_A \otimes_{\mathcal{A}} e^A + \gamma_A \, \nabla e^A = \underbrace{(\mathrm{d}\gamma_A - \gamma_B \, A^B{}_A)}_{=: \, D\gamma_A} \otimes_{\mathcal{A}} e^A \; . \tag{3.95}$$

Here we have introduced *connection 1-forms* $A^A{}_B$ via

$$\nabla e^A =: -A^A{}_B \otimes_{\mathcal{A}} e^B \; . \tag{3.96}$$

Furthermore,

$$\begin{aligned} \nabla^2\gamma &= \mathrm{d}(\mathrm{d}\gamma_A - \gamma_B \, A^B{}_A) \otimes_{\mathcal{A}} e^A - (\mathrm{d}\gamma_A - \gamma_B \, A^B{}_A) \, \nabla e^A \\ &= -\gamma_A \, \underbrace{(\mathrm{d}A^A{}_B + A^A{}_C \, A^C{}_B)}_{=: \, F^A{}_B} \otimes_{\mathcal{A}} e^B \end{aligned} \tag{3.97}$$

where the *curvature* (or *field strength*) *2-form*

$$F = \mathrm{d}A + A \, A \tag{3.98}$$

appears. A *gauge transformation* is given by

$$e^A \mapsto e^{A'} = a^A{}_B \, e^B \tag{3.99}$$

with $a \in GL(n, \mathcal{A})$. This induces the following transformation laws,

$$A \mapsto A' = a \, A \, a^{-1} + a \, \mathrm{d}a^{-1} \tag{3.100}$$

$$F \mapsto F' = a \, F \, a^{-1} \; . \tag{3.101}$$

Let $\Omega(\mathcal{A})$ be a differential calculus on $\mathcal{A}$, ∇ a left $\mathcal{A}$-module connection on an $\mathcal{A}$-bimodule Γ and ∇' a connection on a left $\mathcal{A}$-module Γ'. Is is possible to build from these a left $\mathcal{A}$-module connection $\nabla_\otimes$ on the left $\mathcal{A}$-module $\Gamma \otimes_{\mathcal{A}} \Gamma'$? In classical differential geometry we only need to introduce a connection on 1-forms. It then induces a connection on arbitrary tensor fields. Is there a similar construction in the more general framework? Obviously the naive ansatz $\nabla_\otimes = \nabla \otimes \mathrm{id}' + \mathrm{id} \otimes \nabla'$ does not work since the last part maps into the wrong space. What is needed is a suitable 'twist map' $\Psi_\nabla : \Gamma \otimes_{\mathcal{A}} \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Gamma$ which may depend on the connection ∇. This suggests the ansatz

$$\nabla_\otimes = \nabla \otimes \mathrm{id}' + (\Psi_\nabla \otimes \mathrm{id}') \circ (\mathrm{id} \otimes \nabla') \; . \tag{3.102}$$

Then $\nabla_\otimes$ is a left $\mathcal{A}$-module connection iff Ψ_∇ is an $\mathcal{A}$-bimodule homomorphism with the property

$$\Psi_\nabla(\gamma \otimes_{\mathcal{A}} \mathrm{d}f) = \nabla(\gamma \, f) - (\nabla\gamma) \, f \; . \tag{3.103}$$

If such a map exists, then ∇ is called *extensible*. See [37] for an analysis of the extension problem for connections and for related references.

3.1 Connections on a finite set

Let (Ω, d) be a differential calculus on the algebra $\mathcal{A}$ of functions on a finite set (cf section 2.1). A connection 1-form can be expressed as

$$A = \sum_{i,j} A_{ij}\, e_{ij} \tag{3.104}$$

where A_{ij} are $m \times m$-matrices with entries in $\mathbb{C}$. We introduce

$$U_{ij} := \mathbf{1} + A_{ij} \quad \text{for } i \neq j\,, \qquad U_{ii} := \mathbf{1}\,. \tag{3.105}$$

Lemma. Under a gauge transformation with $a \in GL(m, \mathcal{A})$,

$$U_{ij} \;\mapsto\; a(i)\, U_{ij}\, a(j)^{-1}\,. \tag{3.106}$$

Proof: We only need to consider the components U_{ij} with $i \neq j$ which are collected in

$$U := \sum_{i,j} e_{ij} + A = \sum_{i,j} U_{ij}\, e_{ij}\,.$$

With $A' = a\, A\, a^{-1} + a\, \mathrm{d}a^{-1}$ we find

$$\begin{aligned} U' &= \sum_{i,j} e_{ij} + A' \\ &= \sum_{i,j} e_{ij} + a\, A\, a^{-1} + a \sum_{i,j} [a(j)^{-1} - a(i)^{-1}]\, e_{ij} \\ &= a\, A\, a^{-1} + a \sum_{i,j} e_{ij}\, a(j)^{-1} = a\, U\, a^{-1} \\ &= \sum_{i,j} a(i)\, U_{ij}\, a(j)^{-1}\, e_{ij} \end{aligned}$$

using (2.16) and (2.19). ♡

For the curvature (or field strength) of A we get

$$F = \mathrm{d}A + A\, A = \sum_{i,j,k} (U_{ij}\, U_{jk} - U_{ik})\, e_{ijk}\,. \tag{3.107}$$

U_{ij} should be regarded as a 'transport operator' (which maps a vector at point j to one at point i). Vanishing curvature, i.e., $F = 0$ then means (oriented) path independence of the transport.

Example. Let us consider the differential calculus determined by the following digraph on a set of four points.

$$\begin{array}{ccc} 3 \bullet & \longrightarrow & \bullet\ 2 \\ \uparrow & & \uparrow \\ 0 \bullet & \longrightarrow & \bullet\ 1 \end{array}$$

From the basis 1-forms e_{ij} we can only build the two-forms e_{012} and e_{032} by concatenation. The rules of differential calculus impose the relation $e_{032} = -e_{012}$ (cf section 2.1.1). Now (3.107) becomes $F = (U_{01}U_{12} - U_{03}U_{32})\, e_{012}$. Hence $F = 0$ means $U_{01}U_{12} = U_{03}U_{32}$. ♡

3.1.1 Involutions, 'symmetric' differential calculi, and the Higgs potential

On the algebra $\mathcal{A}$ of complex functions on $\mathcal{M}$ there is a natural involution given by complex conjugation: $(fh)^* = h^* f^*$. This extends to some differential calculi if we require that $(\mathrm{d}f)^* = -\mathrm{d}(f^*)$ where $f \in \mathcal{A}$. As a consequence, using the Leibniz rule we have

$$e_{ij}^* = e_{ji} \tag{3.108}$$

so that the digraph associated with the (first order) differential calculus has to be symmetric (in the sense that if two vertices are connected by an arrow, then also the reverse arrow must be present).[20] Hence, the involution only exists on differential calculi with symmetric graphs.

A connection 1-form A is *anti-Hermitean* if

$$A^\dagger = -A \qquad \text{where} \quad A^\dagger = \sum_{i,j} e_{ij}^* \, A_{ij}^\dagger = \sum_{i,j} A_{ij}^\dagger \, e_{ij}^* = \sum_{i,j} A_{ij}^\dagger \, e_{ji} \; . \tag{3.109}$$

The dagger extends the involution $*$ to matrices of forms. For matrices of 0-forms it means taking the Hermitean conjugate matrix. The above condition implies

$$U_{ij}^\dagger = U_{ji} \; . \tag{3.110}$$

Example. Let us consider $\mathcal{M} = \mathbb{Z}_2 = \{0,1\}$ with the universal differential calculus. The only basis 1-forms are e_{01} and e_{10}. Concatenation only yields the 2-forms e_{010} and e_{101}. The involution acts on them as follows,

$$e_{010}^* = e_{010} \, , \quad e_{101}^* = e_{101} \; . \tag{3.111}$$

For an anti-Hermitean connection, (3.107) reduces to

$$F = (U_{01}\, U_{10} - U_{00})\, e_{010} + (U_{10}\, U_{01} - U_{11})\, e_{101} = (\phi^\dagger \, \phi - \mathbf{1})\, e_{010} + (\phi\, \phi^\dagger - \mathbf{1})\, e_{101} \tag{3.112}$$

where we introduced $\phi := U_{10}$. Defining a Hermitean inner product on the space of 2-forms by

$$(e_{010}, e_{010}) = (e_{101}, e_{101}) = 1 \, , \quad (e_{010}, e_{101}) = 0 \tag{3.113}$$

and a Yang-Mills action as

$$S_{YM} := \mathrm{tr}\, (F, F) \, , \tag{3.114}$$

the latter is gauge invariant under gauge transformations with $a^\dagger = a^{-1}$. Inserting the above expression for F, one finds

$$S_{YM} = 2\, \mathrm{tr}\, (\phi^\dagger \, \phi - \mathbf{1})^2 \tag{3.115}$$

which physicists recognize as a *Higgs potential*, a substantial ingredient of the action for the standard model of elementary particles. Extending the space to $\mathbb{R}^4 \times \mathbb{Z}_2$, ϕ indeed becomes a field on $\mathbb{R}^4$ and in this way one recovers the Higgs field in elementary particle physics models (see [39]). Such a result is not too surprising since it has been known for quite a while that dimensional reduction of a pure gauge theory on a space $\mathbb{R}^4 \times S^2$ leads to Higgs fields [40]. One should think of $\mathbb{Z}_2$ as a discretization of the sphere S^2. A really new aspect is that on a discrete set one can accomodate different gauge groups. ♡

[20] See also [38] for some aspects of noncommutative geometry of symmetric digraphs (i.e., symmetric first order differential calculi).

3.2 Connections on a lattice

Let us now choose $\mathcal{A}$ as the algebra of all functions on $\mathbb{R}^n$ (or on an n-dimensional lattice with spacings ℓ) and the differential calculus introduced in section 2.2.3 with $\ell > 0$. A connection 1-form A can then be expressed as $A = A_\mu \, \mathrm{d}x^\mu$ with matrices A_μ (the entries of which are functions). We introduce left-coefficients of its field strength F via

$$F = F_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu \tag{3.116}$$

and define[21]

$$F^\dagger := F^\dagger_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu \tag{3.117}$$

where the dagger acts on matrices of functions by Hermitean conjugation. An analogue of the classical Yang-Mills action is

$$S_{YM} := \int \mathcal{L}_{YM} \quad \text{where} \quad \mathcal{L}_{YM} := \mathrm{tr}(F^\dagger \star F) \; . \tag{3.118}$$

Here the integral and the $\star$-operator are those of section 2.2.3. In the definition of the $\star$-operator we choose the Euclidean metric for what follows. Using

$$\begin{aligned} F^\dagger \star F &= F^\dagger_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu \star F_{\kappa\lambda} \, \mathrm{d}x^\kappa \, \mathrm{d}x^\lambda = F^\dagger_{\mu\nu}(x) \, F_{\kappa\lambda}(x + \ell^\mu + \ell^\nu - \ell^\kappa - \ell^\lambda) \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu \star (\mathrm{d}x^\kappa \, \mathrm{d}x^\lambda) \\ &= 2 \, F^\dagger_{\mu\nu}(x) \, F^{\mu\nu}(x) \, \mathrm{d}x^1 \cdots \mathrm{d}x^n \end{aligned} \tag{3.119}$$

one finds

$$\mathcal{L}_{YM} = 2 \, \mathrm{tr}[F_{\mu\nu}(x)^\dagger \, F^{\mu\nu}(x)] \, \mathrm{d}^n x \; . \tag{3.120}$$

Lemma. $\mathcal{L}_{YM}$ is invariant under unitary gauge transformations.
Proof: From

$$F' = a \, F \, a^{-1} = a(x) \, F_{\mu\nu}(x) \, a(x + \ell^\mu + \ell^\nu)^{-1} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu$$

we get

$$F^{\dagger'} = a^{-1}(x + \ell^\mu + \ell^\nu)^\dagger \, F_{\mu\nu}(x)^\dagger \, a(x)^\dagger \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu \; .$$

With $a^\dagger = a^{-1}$ this implies

$$F^\dagger_{\mu\nu}(x)' = a(x + \ell^\mu + \ell^\nu) \, F_{\mu\nu}(x)^\dagger \, a(x)^{-1} \; .$$

Hence

$$\begin{aligned} \mathrm{tr}[F_{\mu\nu}(x)^\dagger \, F^{\mu\nu}(x)]' &= \mathrm{tr}[a(x + \ell^\mu + \ell^\nu) \, F_{\mu\nu}(x)^\dagger \, a(x)^{-1} \, a(x) \, F^{\mu\nu}(x) \, a(x + \ell^\mu + \ell^\nu)^{-1}] \\ &= \mathrm{tr}[F_{\mu\nu}(x)^\dagger \, F^{\mu\nu}(x)] \end{aligned}$$

using the cyclicity of trace. ♡

Let us introduce

$$U := \frac{1}{\ell} \sum_\mu \mathrm{d}x^\mu + A = \frac{1}{\ell} \, U_\mu(x) \, \mathrm{d}x^\mu \; . \tag{3.121}$$

In the following, we prove some properties of U. We then express S_{YM} in terms of $U_\mu(x)$ and establish contact with a familiar formulation of lattice gauge theory.

[21] This definition differs from that in section 3.1.1. The latter cannot be used here because the lattice differential calculus is not symmetric (cf section 2.1.3).

Lemma.

$$U' = a\,U\,a^{-1}\ . \tag{3.122}$$

Proof:

$$\begin{aligned} U' &= \frac{1}{\ell}\sum_\mu \mathrm{d}x^\mu + a\,A\,a^{-1} - (\mathrm{d}a)\,a^{-1} \\ &= \frac{1}{\ell}\sum_\mu \mathrm{d}x^\mu + a\,A\,a^{-1} - \sum_\mu \frac{1}{\ell}[a(x+\ell^\mu) - a(x)]\,\mathrm{d}x^\mu\,a^{-1} \\ &= a\left(\frac{1}{\ell}\sum_\mu \mathrm{d}x^\mu + A\right)a^{-1} \end{aligned}$$

using (2.73). ♡

Lemma.

$$F := \mathrm{d}A + A^2 = U^2\ . \tag{3.123}$$

Proof:

$$\begin{aligned} U^2 &= \frac{1}{\ell^2}\sum_{\mu,\nu}\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \frac{1}{\ell}\sum_\mu(\mathrm{d}x^\mu\,A + A\,\mathrm{d}x^\mu) + A^2 \\ &= \frac{1}{\ell^2}\sum_{\mu,\nu}\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \frac{1}{\ell}\sum_\mu(\underbrace{\mathrm{d}x^\mu\,A_\nu(x)\,\mathrm{d}x^\nu}_{=A_\nu(x+\ell^\mu)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu} + A\,\mathrm{d}x^\mu) + A^2 \\ &= \frac{1}{\ell}\sum_{\mu,\nu}[A_\nu(x+\ell^\mu) - A_\nu(x)]\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + A^2 \\ &= \mathrm{d}A + A^2\ . \end{aligned}$$

♡

Now we have

$$\begin{aligned} F &= U^2 = \frac{1}{\ell^2}\,U_\mu(x)\,U_\nu(x+\ell^\mu)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \\ &= \frac{1}{2\ell^2}\,[U_\mu(x)\,U_\nu(x+\ell^\mu) - U_\nu(x)\,U_\mu(x+\ell^\nu)]\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \end{aligned} \tag{3.124}$$

and

$$\begin{aligned} \mathrm{tr}[F_{\mu\nu}(x)^\dagger\,F^{\mu\nu}(x)] &= \frac{1}{2\ell^4}\,\mathrm{tr}[U_\nu(x+\ell^\mu)^\dagger\,U_\mu(x)^\dagger\,U^\mu(x)\,U^\nu(x+\ell^\mu) \\ &\quad -U_\nu(x+\ell^\mu)^\dagger\,U_\mu(x)^\dagger\,U^\nu(x)\,U^\mu(x+\ell^\nu)]\ . \end{aligned} \tag{3.125}$$

Assuming

$$U_\mu^\dagger = U_\mu^{-1} \tag{3.126}$$

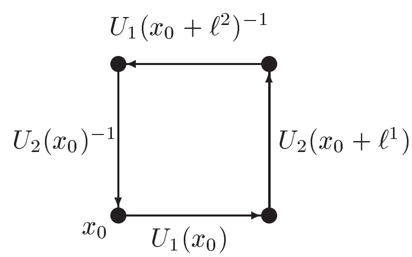
(which restricts the connection), this becomes

$$\begin{aligned} &= \mathrm{tr}[F_{\mu\nu}(x)^\dagger\,F^{\mu\nu}(x)] \\ &= \frac{1}{2\ell^4}\sum_{\mu,\nu}\mathrm{tr}\,[\mathbf{1} - U_\nu(x+\ell^\mu)^{-1}\,U_\mu(x)^{-1}\,U^\nu(x)\,U^\mu(x+\ell^\nu)] \end{aligned} \tag{3.127}$$

where $\mathbf{1}$ is the unit matrix in the gauge group. For the Yang-Mills action we now obtain

$$S_{YM} = \frac{1}{\ell^4} \int \mathrm{tr} \sum_{\mu,\nu} [\mathbf{1} - U^\nu(x)\, U^\mu(x+\ell^\nu)\, U_\nu(x+\ell^\mu)^{-1}\, U_\mu(x)^{-1}]\, \mathrm{d}^n x \tag{3.128}$$

using the cyclicity of the trace. The integral sums the values of the integrand on a hypercubic lattice (cf section 2.2.3). The result is then precisely the action of lattice gauge theory [41, 42]. Regarding $U_\mu(x)$ as a transport operator from the lattice site x to the neighbouring site $x+\ell^\mu$, the action involves a summation over all transports ('Wilson loops') around plaquettes of the hypercubic lattice (see the figure below).



The lattice version of gauge theory is crucial for numerical calculations in high energy physics, giving insight into nonperturbative features of the corresponding quantum theory. In particular, this concerns the problems of quark confinement and masses of elementary particles.

3.3 Linear connections in noncommutative geometry

A connection is called a *linear connection* when the module Γ is the space of 1-forms, i.e., $\Gamma = \Omega^1$. Hence it is a map

$$\nabla \,:\, \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \tag{3.129}$$

such that

$$\nabla(f\, w) = \mathrm{d}f \otimes_{\mathcal{A}} w + f\, \nabla w \qquad (f \in \mathcal{A},\ w \in \Omega^1)\ . \tag{3.130}$$

It extends to a map

$$\Omega \otimes_{\mathcal{A}} \Omega^1 \to \Omega \otimes_{\mathcal{A}} \Omega^1 \tag{3.131}$$

via

$$\nabla(\chi\, w) = \mathrm{d}\chi \otimes_{\mathcal{A}} w + (-1)^r\, \chi\, \nabla w \qquad (\chi \in \Omega^r)\ . \tag{3.132}$$

Besides the curvature ∇^2 we now have *torsion*

$$T := \mathrm{d} - \pi \circ \nabla \tag{3.133}$$

where

$$\pi \,:\, \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \to \Omega^2 \tag{3.134}$$

is the canonical projection.

If Ω^1 has a left $\mathcal{A}$-module basis θ^i, $i = 1, \ldots, n$, then

$$w = w_i\, \theta^i\,, \qquad \nabla w = D w_i \otimes_{\mathcal{A}} \theta^i \qquad \nabla^2 \theta^i = -\Omega^i{}_j \otimes_{\mathcal{A}} \theta^j \tag{3.135}$$

where

$$
\begin{aligned}
Dw_i &= \mathrm{d}w_i - w_j\,\omega^j{}_i = \mathrm{d}w_i - w_j\,\Gamma^j_{ik}\,\theta^k && (3.136)\\
\Omega^i{}_j &= \mathrm{d}\omega^i{}_j + \omega^i{}_k\,\omega^k{}_j = \frac{1}{2}\,R^i{}_{jkl}\,\theta^k\,\theta^l\ . && (3.137)
\end{aligned}
$$

The 1-forms $\omega^i{}_j$ and the 0-forms $\Gamma^k_{ij}, R^i{}_{jkl}$ are defined by these formulas. Under a change of basis (i.e., a gauge transformation)

$$
\theta^i \mapsto a^i{}_j\,\theta^j \qquad (a^i{}_j) \in GL(n,\mathcal{A}) \tag{3.138}
$$

we have

$$
\begin{aligned}
w_i &\mapsto w_j\,(a^{-1})^j{}_i && (3.139)\\
Dw_i &\mapsto Dw_j\,(a^{-1})^j{}_i && (3.140)\\
\omega^i{}_j &\mapsto a^i{}_k\,\omega^k{}_l\,(a^{-1})^l{}_j + a^i{}_k\,\mathrm{d}(a^{-1})^k{}_j && (3.141)\\
\Omega^i{}_j &\mapsto a^i{}_k\,\Omega^k{}_l\,(a^{-1})^l{}_j\ . && (3.142)
\end{aligned}
$$

But $R^i{}_{jkl}$ does *not* transform in such a simple way as long as elements of $\mathcal{A}$ do not commute with 1-forms. As a consequence, there is no direct analogue of the Ricci tensor (which appears in the field equations of general relativity). For the torsion we find

$$
T(\theta^i) = \mathrm{d}\theta^i - \pi \circ \nabla\theta^i = \mathrm{d}\theta^i + \omega^i{}_j\,\theta^j\ . \tag{3.143}
$$

The classical Cartan formulas for curvature and torsion are thus carried over without any change to the general framework. As a consequence, also the first and second Bianchi identities hold unchanged, i.e.,

$$
\begin{aligned}
DT(\theta) &:= \mathrm{d}T(\theta) + \omega\,T(\theta) = \Omega\,\theta && (3.144)\\
D\Omega &:= \mathrm{d}\Omega + \omega\,\Omega - \Omega\,\omega = 0 && (3.145)
\end{aligned}
$$

in matrix notation.

Remark. So far we only considered connections on a *left* $\mathcal{A}$-module. There are corresponding formulas for *right* $\mathcal{A}$-module connections. Since Ω^1 is a *bi*-module, there are indeed left and also right connections. These have to be distinguished. Though $\nabla : \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1$ looks very much 'symmetric', the two factors Ω^1 on the right hand side play very different roles. ♡

3.3.1 Linear connections on finite groups

Very simple examples of spaces which can be equipped with geometric structures are given by finite sets and in particular finite groups. The left and the right action of the group on itself can then be used to narrow down the possibilities of differential calculi, connections, and tensor fields by imposing a symmetry condition. We shall not go much into all that here (see [37] for more details).

Let G be a finite group. There are left-covariant and right-covariant differential calculi.[22] The universal differential calculus is both, left- and right-covariant, i.e., bicovariant. We introduce the left-invariant *Maurer-Cartan forms*

$$
\theta^g := e_{\underline{h}\cdot g,\underline{h}} := \sum_{h\in G} e_{h\cdot g,h} \qquad (g \in G) \tag{3.146}
$$

where $e_{g,h} := e_g\,\tilde{\mathrm{d}}e_h$ for $g \neq h$ and $e_{g,g} = 0$, cf. section 2.1. Each other left-covariant differential calculus is obtained by setting some of the θ^g to zero. Note that $\theta^e = 0$ where e is the unit element in G. The nonvanishing θ^g then constitute a basis over $\mathbb{C}$ of the space of 1-forms. One also obtains an analogue of the Maurer-Cartan equation of classical Lie group differential geometry,

$$
\mathrm{d}\theta^g = -C^h_{\underline{g},\underline{g}'}\,\theta^{\underline{g}'}\,\theta^{\underline{g}} \tag{3.147}
$$

[22]More generally, such concepts apply to Hopf algebras including quantum groups, see [10].

with

$$C^h_{g,g'} := -\delta^h_g - \delta^h_{g'} + \delta^h_{g\cdot g'} \ . \tag{3.148}$$

There are also right-invariant Maurer-Cartan forms

$$\omega^g := e_{g\cdot \underline{h},\underline{h}} \tag{3.149}$$

in the universal differential calculus and all other right-covariant differential calculi on G are obtained by setting some of them to zero.

For a bicovariant differential calculus on G there is a unique bimodule homomorphism $\sigma : \Omega^1 \otimes_\mathcal{A} \Omega^1 \to \Omega^1 \otimes_\mathcal{A} \Omega^1$ such that

$$\sigma(\theta \otimes_\mathcal{A} \omega) = \omega \otimes_\mathcal{A} \theta \tag{3.150}$$

for all left-invariant 1-forms θ and right-invariant 1-forms ω (which are linear combinations of Maurer-Cartan forms with coefficients in $\mathbb{C}$). The map σ satisfies the braid equation.

An example of an extensible (left $\mathcal{A}$-module) linear connection on G with a bicovariant differential calculus is given by

$$\nabla^{(\sigma)} w = \rho \otimes_\mathcal{A} w - \sigma(w \otimes_\mathcal{A} \rho) \,, \qquad \rho := \sum_{g\in G} \theta^g \ . \tag{3.151}$$

It satisfies $\nabla^{(\sigma)}\theta^g = 0$ (i.e., the left-invariant Maurer-Cartan forms are covariantly constant) and, as a consequence, has vanishing curvature. This connection has torsion, however, and it is bi-invariant. It is a perfect analogue of the classical connection which determines the '+-parallelism' on a Lie group [43].

An example of a torsion-free connection is given by

$$\nabla \theta^h = -C^h_{\underline{g},\underline{g'}} \, \theta^{\underline{g'}} \otimes_\mathcal{A} \theta^{\underline{g}} \ . \tag{3.152}$$

It is left-/right-invariant for a left-/right-covariant differential calculus on G. This connection is not extensible, in general (cf [37]).

In this subsection we have touched upon a new approach towards geometry of finite groups. So far this is 'just' mathematics and applications in a physical context are still missing. If discrete spaces and concepts of discrete space-time are addressed, one should expect the above concepts to be of similar use as their counterparts in ordinary differential geometry.

4 Applications in the context of integrable models and soliton equations

For two-dimensional σ-models there is a construction of an infinite sequence of conserved currents [44] which can be formulated neatly in terms of ordinary differential forms. This then suggests to generalize the notion of a σ-model to noncommutative differential calculi such that the construction of conservation laws still works. In this way one obtains a simple though very much non-trivial application of the formalism developed in the previous sections. Our presentation is based on [45, 46].

In the second part of this section we reveal some interesting relations between certain noncommutative differential calculi on $\mathbb{R}^2$ and $\mathbb{R}^3$, and the KdV and KP equation, respectively. This material is taken from [47].

4.1 Generalized integrable σ-models

Let $\mathcal{A}$ be an associative and commutative algebra with unit $\mathbb{1}$ and (Ω, d) a differential calculus on it. Furthermore, let $\star : \Omega^1 \to \Omega^1$ be an invertible linear map such that

$$\star(w\, f) = f \star w \tag{4.153}$$

and

$$w \star w' = w' \star w \ . \tag{4.154}$$

In addition, we require that

$$\mathrm{d}w = 0 \quad \Leftrightarrow \quad w = \star \star \mathrm{d}\chi \tag{4.155}$$

with some function χ. Furthermore, let $a \in GL(n, \mathcal{A})$ and $A := a^{-1}\, \mathrm{d}a$. Then

$$F := \mathrm{d}A + AA \equiv 0 \tag{4.156}$$

since $\mathrm{d}a^{-1} = -a^{-1}\,(\mathrm{d}a)\, a^{-1}$. The above definitions are made in such a way that the field equation of a *generalized σ-model*

$$\mathrm{d} \star A = 0 \tag{4.157}$$

and the construction of an infinite set of conservation laws in two dimensions [44] generalize to a much more general framework.

Lemma. If $\mathrm{d} \star A = 0$, then

$$\mathrm{d} \star D\chi = D \star \mathrm{d}\chi$$

for an $n \times n$ matrix χ with entries in $\mathcal{A}$, where $D\chi := \mathrm{d}\chi + A\,\chi$.
Proof: Using the two relations (4.153) and (4.154) we find

$$\mathrm{d} \star (A^i{}_j\, \chi^j{}_k) = \mathrm{d}(\chi^j{}_k \star A^i{}_j) = (\mathrm{d}\chi^j{}_k) \star A^i{}_j + \chi^j{}_k\, \mathrm{d} \star A^i{}_j = A^i{}_j \star \mathrm{d}\chi^j{}_k$$

and thus

$$\mathrm{d} \star D\chi = \mathrm{d} \star \mathrm{d}\chi + \mathrm{d}(\star A\chi) = \mathrm{d} \star \mathrm{d}\chi + A \star \mathrm{d}\chi = D \star \mathrm{d}\chi \ .$$

♡

Let

$$\chi^{(0)} := \begin{pmatrix} \mathbb{1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbb{1} \end{pmatrix} \tag{4.158}$$

Then

$$J^{(1)} := D\chi^{(0)} = A \tag{4.159}$$

so that

$$\mathrm{d} \star J^{(1)} = 0 \tag{4.160}$$

as a consequence of the field equation. Thus, using (4.155),

$$J^{(1)} = \star\, \mathrm{d}\chi^{(1)} \tag{4.161}$$

with a matrix $\chi^{(1)}$. Now, let $J^{(m)}$ be a conserved current, i.e.,

$$J^{(m)} = \star\, \mathrm{d}\chi^{(m)} \; . \tag{4.162}$$

Then

$$J^{(m+1)} := D\chi^{(m)} \qquad (m \geq 0) \tag{4.163}$$

is also conserved since

$$\begin{aligned} \mathrm{d} \star J^{(m+1)} &= \mathrm{d} \star D\chi^{(m)} = D \star \mathrm{d}\chi^{(m)} = DJ^{(m)} = D^2 \chi^{(m-1)} \\ &= F\,\chi^{(m-1)} = 0 \qquad (m \geq 1) \end{aligned} \tag{4.164}$$

using the Lemma and (4.156). Starting with $J^{(1)}$, we obtain an infinite set of conserved currents. In case of the ordinary differential calculus on a two-dimensional Riemannian manifold, this construction reduces to that described in [44].

Let us (formally) define

$$\chi := \sum_{m \geq 0} \lambda^m \, \chi^{(m)} \tag{4.165}$$

with a constant $\lambda \neq 0$. Then (4.162) and (4.163) lead to

$$\star\, \mathrm{d}\chi = \lambda \, D\chi \; . \tag{4.166}$$

As a consequence of this equation we have

$$0 = \mathrm{d} \star D\chi^i{}_j = D \star \mathrm{d}\chi^i{}_j + \chi^k{}_j \, \mathrm{d} \star A^i{}_k \tag{4.167}$$

(cf the proof of the Lemma) and

$$D \star \mathrm{d}\chi = \lambda \, D^2 \chi = \lambda \, F \, \chi \; . \tag{4.168}$$

With $A = a^{-1}\, \mathrm{d}a$, the integrability condition of the linear equation (4.166) is the field equation (4.157).

We have extended the definition of (a class of) generalized σ-models to a rather general framework of noncommutative geometry, though still with the restriction to a commutative algebra $\mathcal{A}$. But already for *commutative* algebras with *non*commutative differential calculi (where functions and differentials do not commute, in general) a huge set of possibilities for integrable models appears. We refer to [45, 46] for further details and results.

4.1.1 A simple example: recovering the Toda lattice

Let $\mathcal{A}$ be the (commutative) algebra of functions on $\mathbb{R} \times \ell\mathbb{Z}$ which are smooth in the first argument. Here $\ell\mathbb{Z}$ stands for the one-dimensional lattice with spacings $\ell > 0$. A special differential calulus on $\mathcal{A}$ is then determined by the following commutation relations,

$$[\mathrm{d}t, t] = 0 \, , \quad [\mathrm{d}x, x] = \ell \, \mathrm{d}x \, , \quad [\mathrm{d}t, x] = [\mathrm{d}x, t] = 0 \tag{4.169}$$

where t and x are the canonical coordinate functions on $\mathbb{R}$ and $\ell\mathbb{Z}$, respectively. As a consequence, we have

$$\mathrm{d}t \, f(t,x) = f(t,x) \, \mathrm{d}t \, , \quad \mathrm{d}x \, f(t,x) = f(t, x+\ell) \, \mathrm{d}x \tag{4.170}$$

and

$$\mathrm{d}f = \dot{f} \, \mathrm{d}t + \frac{1}{\ell} \{ f(t, x+\ell) - f(t,x) \} \, \mathrm{d}x \tag{4.171}$$

where $\dot f := \partial f/\partial t$. Furthermore, acting with d on (4.169), we obtain

$$\mathrm{d}t\,\mathrm{d}x = -\mathrm{d}x\,\mathrm{d}t\,, \quad \mathrm{d}t\,\mathrm{d}t = 0 = \mathrm{d}x\,\mathrm{d}x\ , \tag{4.172}$$

but this familiar anticommutativity of the differentials does not extend to general 1-forms.

We define a generalized Hodge operator by

$$\star\,\mathrm{d}t = -\mathrm{d}x\,, \qquad \star\,\mathrm{d}x = -\mathrm{d}t \tag{4.173}$$

which copies the familiar rules for the Hodge operator associated with the two-dimensional Minkowski metric. The action of $\star$ extends to Ω^1 via (4.153). It is now easily verified that (4.154) and (4.155) are indeed satisfied. Therefore, the construction of conservation laws does work in the case under consideration. Let us look at the simplest generalized σ-model where a is just a function (i.e., a 1×1-matrix). Let us write

$$a = e^{-q(t,x)} \tag{4.174}$$

with a function q. Then

$$A = e^q\,\mathrm{d}e^{-q} = -\dot q_k\,\mathrm{d}t + \frac{1}{\ell}(e^{q_k - q_{k+1}} - 1)\,\mathrm{d}x \tag{4.175}$$

where $q_k(t) := q(t, k\ell)$, $k \in \mathbb{Z}$.[23] Now

$$\star\,A = \dot q_k\,\mathrm{d}x - \frac{1}{\ell}(e^{q_{k-1} - q_k} - 1)\,\mathrm{d}t \tag{4.176}$$

and

$$0 = \mathrm{d}\star A = \left(\ddot q_k + \frac{1}{\ell^2}(e^{q_k - q_{k+1}} - e^{q_{k-1} - q_k})\right)\mathrm{d}t\,\mathrm{d}x \tag{4.177}$$

which is the *nonlinear Toda lattice* equation [48]. In this way a new and simple understanding of its complete integrability has been achieved. We have revealed a 'geometry' behind the Toda lattice equation. Generalizations of the Toda lattice are obtained by replacing the function a with a $GL(n,\mathcal{A})$-matrix [45].

Remark. Consider a linear chain of equal particles with mass m connected by springs. If q_n is the displacement of the nth particle from its equilibrium position, then the equations of motion are

$$m\,\ddot q_n = V'(q_{n+1} - q_n) - V'(q_n - q_{n-1}) \tag{4.178}$$

where $-V'(r) = -\mathrm{d}V/\mathrm{d}r$ is the force of the spring when stretched by the amount r. In case of the nonlinear Toda lattice model, the potential is taken to be

$$V(r) := \frac{a}{b}\,e^{-b\,r} + a\,r \tag{4.179}$$

with constants $a > 0$ and $b > 0$. For small r this is a harmonic oscillator potential. The equations of motion now read

$$m\,\ddot q_n = a\,[e^{-b\,(q_n - q_{n-1})} - e^{-b\,(q_{n+1} - q_n)}] \tag{4.180}$$

By a rescaling of the q_n we can achieve $b = 1$ and recover (4.177). With

$$r_n := q_{n+1} - q_n \tag{4.181}$$

the equations of motion take the form

$$m\,\ddot r_n = a\,[2\,e^{-b\,r_n} - e^{-b\,r_{n+1}} - e^{-b\,r_{n-1}}]\ . \tag{4.182}$$

[23] More precisely, we should regard k as the canonical coordinate function on $\mathbb{Z}$.

A special solution is given by

$$e^{-b\,r_n} - 1 = \frac{m}{ab}\,\omega^2\,[\cosh(\kappa\,n \pm \omega\,t)]^{-2} \tag{4.183}$$

with $\omega = \sqrt{ab/m}\,\sinh\kappa$. This is a pulselike wave, a 'soliton'. There are in fact multi-soliton generalizations of this solution. Several solitons can move on the lattice, interact and afterwards emerge with the same shape they had before the scattering. The Toda lattice plays a role in the modelling of the propagation of sound waves through a crystal lattice and several other physical problems. ♡

4.2 Soliton equations and the zero curvature condition

Famous soliton equations are known to admit zero curvature formulations (see in particular [49]). For example, the KdV and sine-Gordon equation are obtained from the equation

$$F := \mathrm{d}A + A \wedge A = 0 \tag{4.184}$$

where A is a special $SL(2,\mathbb{C})$ connection. It is essential here to deal with a noncommutative matrix algebra since otherwise the nonlinear term $A \wedge A$ would drop out. The situation is very different for a noncommutative differential calculus where $AA \neq 0$ in general already for a single 1-form A, i.e., a $GL(1,\mathbb{R})$-connection.

4.2.1 A noncommutative differential calculus and the KdV equation

Given some partial differential operators, we may look for a differential calculus in which they appear as generalized partial derivatives. If such operators emerge from a certain mathematical or physical problem, there is some hope (but no guarantee) to gain an improved understanding of the underlying mathematical structure in this way.

As an example, let us consider $\mathcal{A} = C^\infty(\mathbb{R}^2)$ and the two differential operators

$$\hat{\partial}_t := \frac{\partial}{\partial t} + ab\,\frac{\partial^3}{\partial x^3}\,, \qquad \Delta := b\,\frac{\partial^2}{\partial x^2} \tag{4.185}$$

where a, b are constants and t, x coordinates on $\mathbb{R}^2$. The two differential operators appear (for a special choice of the constants) in the Zakharov-Shabat scheme as 'undressed operators' from which the Korteweg-deVries[24] (KdV) equation is recovered (see [50], section 6.2, for example). In order to incorporate these operators as generalized partial derivatives of a differential calculus, we make the following ansatz for the (generalized) differential of a function $f \in \mathcal{A}$,

$$\mathrm{d}f = (f_t + ab\,f_{xx})\,\mathrm{d}t + f_x\,\mathrm{d}x + b\,f_{xx}\,\xi \tag{4.186}$$

using the abbreviation $f_t := \partial f/\partial t$ and a 1-form ξ. So far we only require that $\{\mathrm{d}t, \mathrm{d}x, \xi\}$ is a left $\mathcal{A}$-module basis of the space of 1-forms Ω^1 (so that every 1-form can be expressed as a linear combination of $\mathrm{d}t, \mathrm{d}x, \xi$ with coefficients in $\mathcal{A}$ to the left of these special 1-forms, as in the above expression for $\mathrm{d}f$). Now we have to find the commutation relations which the basis 1-forms satisfy with a function $f \in \mathcal{A}$. These are obtained using the Leibniz rule. One finds

$$\begin{aligned} \mathrm{d}t\,f &= f\,\mathrm{d}t & (4.187)\\ \xi\,f &= f\,\xi + 3a\,f_{xx} & (4.188)\\ \mathrm{d}x\,f &= f\,\mathrm{d}x + 2b\,f_x\,\xi + 3ab\,f_{xx}\,\mathrm{d}t\,. & (4.189) \end{aligned}$$

[24]The KdV equation describes the propagation of shallow water waves. In 1844 John Scott Russel reported on an observation of a heap of water in the Edinburgh-Glasgow channel which kept its shape and velocity over several miles. He also performed laboratory experiments, generating such water 'solitons' by dropping a weight at one end of a water channel. In 1895 Korteweg and deVries derived a nonlinear equation (which now carries their names) describing the propagation of shallow water waves and showed the existence of solutions with the proposed behaviour. In 1965 Zabusky and Kruskal discovered that two such solitons emerge unchanged from a collision. Zakharov and Faddeev in 1971 used the inverse scattering theory (invented by Gardner, Greene, Kruskal and Miura) to prove complete integrability of the KdV equation. The latter appears in a variety of different physical systems like ion-acoustic and magnetohydrodynamic waves in a plasma, anharmonic lattices, longitudinal dispersive waves in elastic rods, pressure waves in liquid gas bubble mixtures, rotating flow down a tube, and thermally excited phonon packets in low temperature nonlinear crystals.

In terms of the coordinates t, x the only nonvanishing commutators are thus

$$[\mathrm{d}x, x] = 2b\,\xi\,, \qquad [\xi, x] = 3a\,\mathrm{d}t\,. \tag{4.190}$$

Applying d to these equations, using the Leibniz rule, $\mathrm{d}^2 = 0$ and the commutation relations again, leads to

$$\mathrm{d}t\,\mathrm{d}t = \mathrm{d}t\,\mathrm{d}x + \mathrm{d}x\,\mathrm{d}t = \xi\,\xi = \xi\,\mathrm{d}t + \mathrm{d}t\,\xi = \xi\,\mathrm{d}x + \mathrm{d}x\,\xi = 0 \tag{4.191}$$

and the less familiar relation

$$\mathrm{d}\xi = -\frac{1}{b}\,\mathrm{d}x\,\mathrm{d}x\,. \tag{4.192}$$

The most general $GL(1,\mathbb{R})$ connection 1-form is

$$A = w\,\mathrm{d}t + v\,\mathrm{d}x + u\,\xi \tag{4.193}$$

where $u, v, w \in \mathcal{A}$. Evaluation of the zero curvature condition

$$F = \mathrm{d}A + A\,A = 0 \tag{4.194}$$

leads to the following set of equations,

$$-\frac{1}{b}\,u + v_x + v^2 = 0 \tag{4.195}$$

$$v_t + ab\,v_{xxx} + 3ab\,v\,v_{xx} + 3a\,u\,v_x - w_x = 0 \tag{4.196}$$

$$u_t + ab\,u_{xxx} + 3a\,u\,u_x - b\,w_{xx} + b\,v\,(3a\,u_x - 2\,w)_x = 0 \tag{4.197}$$

where the first equation reminds us of the Miura transformation (see [50], for example). The third equation obviously decouples from the others if we choose[25]

$$w_x = \frac{3}{2}\,a\,u_{xx}\,. \tag{4.198}$$

(4.197) then becomes

$$u_t + 3a\,u\,u_x - \frac{1}{2}\,ab\,u_{xxx} = 0 \tag{4.199}$$

which for $a = -2$, $b = 1$ is the KdV equation (see [50], for example). With the help of (4.195), the equation (4.196) is turned into

$$v_t - \frac{1}{2}ab\,v_{xxx} + 3ab\,v^2\,v_x = 0 \tag{4.200}$$

which is known as a 'modified KdV equation' [50].[26] It is surprising that both, the KdV and the mKdV equation appear jointly in our mathematical scheme.

In the above differential calculus it is consistent to impose the additional condition that the 1-form ξ is closed, i.e. $\mathrm{d}\xi = 0$. The above formulas remain valid, except that now $\mathrm{d}x\,\mathrm{d}x = 0$. The zero curvature condition is then slightly less restrictive. It still leads to (4.196) and (4.197), but (4.195) is replaced by the weaker equation $\frac{1}{b}\,u = v_x + v^2 + \lambda$ with a function $\lambda(t)$. For constant λ we rediscover what is sometimes referred to as the 'Miura-Gardner transformation'.

[25] Taking (4.195) into account, one finds a more general solution of the decoupling problem, namely $w_x = \frac{3}{2}\,a\,u_{xx} + c\,v_x$ with a constant c. This takes care of the freedom of Galilean transformations of the KdV equation. See [47] for details.

[26] The modified KdV equation describes acoustic waves in certain anharmonic lattices as well as Alfén waves in a collisionless plasma, for example.

4.2.2 From KdV to KP

The differential calculus associated in the previous subsection with the KdV equation involved a 1-form ξ which could not be expressed as the differential of some coordinate, at least as long as we do not extend the algebra $C^\infty(\mathbb{R}^2)$. It is indeed tempting, however, to replace it by $\mathcal{A} = C^\infty(\mathbb{R}^3)$ and ξ by $\mathrm{d}y$ where y is the third coordinate. Using the Leibniz rule and the commutativity of the algebra $\mathcal{A}$, we find

$$3a\,\mathrm{d}t = [\xi, x] = [\mathrm{d}y, x] = [\mathrm{d}x, y] \tag{4.201}$$

which determines the minimal extension of the differential calculus considered in the last subsection. Then

$$\mathrm{d}t\, f = f\,\mathrm{d}t\,, \quad \mathrm{d}y\, f = f\,\mathrm{d}y + 3a\, f_x\,\mathrm{d}t\,, \quad \mathrm{d}x\, f = f\,\mathrm{d}x + 2b\, f_x\,\mathrm{d}y + 3a\,(f_y + b\, f_{xx})\,\mathrm{d}t \tag{4.202}$$

and

$$\mathrm{d}f = (f_t + 3a\, f_{xy} + ab\, f_{xxx})\,\mathrm{d}t + (f_y + b\, f_{xx})\,\mathrm{d}y + f_x\,\mathrm{d}x\,. \tag{4.203}$$

The basis 1-forms $\mathrm{d}t, \mathrm{d}x, \mathrm{d}y$ satisfy the ordinary Grassmann commutation relations.

The zero curvature condition $F = 0$ for a 1-form $A = w\,\mathrm{d}t + u\,\mathrm{d}y + v\,\mathrm{d}x$ leads to the set of equations

$$u_x = v_y + b\,(v_x + v^2)_x \tag{4.204}$$
$$w_x = v_t + 3a\, v_{xy} + ab\, v_{xxx} + 3a\, uv_x + 3a\, v(v_y + b\, v_{xx}) \tag{4.205}$$
$$w_y + b\, w_{xx} = u_t + 3a\, u_{xy} + ab\, u_{xxx} + 3a\, uu_x - v\,(2b\, w_x - 3a\,(u_y + b\, u_{xx}))\,. \tag{4.206}$$

The next step parallels that of the KdV case treated in the previous subsection. v is obviously eliminated from the last equation by setting

$$w_x = \frac{3a}{2b}\, u_y + \frac{3a}{2}\, u_{xx}\,. \tag{4.207}$$

Taking (4.204) into account, (4.206) then reduces to

$$w_y = u_t + \frac{3a}{2}\, u_{xy} - \frac{ab}{2}\, u_{xxx} + 3a\, uu_x\,. \tag{4.208}$$

Now there is an integrability condition. Comparing the results obtained by differentiating (4.207) with respect to y and (4.208) with respect to x, we obtain

$$(u_t - \frac{ab}{2}\, u_{xxx} + 3a\, uu_x)_x - \frac{3a}{2b}\, u_{yy} = 0 \tag{4.209}$$

which is the Kadomtsev-Petviashvili (KP) equation (for the choices of the constants a, b mentioned earlier, see [50] for example).[27]

Let us now turn to the equation for v which resulted from the zero curvature condition. Taking (4.207) into account, we have

$$\frac{3a}{2b}\, u_y = v_t + \frac{3a}{2}\, v_{xy} - \frac{ab}{2}\, v_{xxx} - 3ab\, v_x^2 + 3a\, vv_y + 3a\, uv_x\,. \tag{4.210}$$

Expressing v as

$$v = q_x \tag{4.211}$$

with a function q, (4.204) becomes

$$u_x = q_{xy} + b\,(q_{xx} + q_x^2)_x \tag{4.212}$$

[27] Relaxing the restriction to strictly one-dimensional waves in the derivation of the KdV equation, one is led to the KP equation [51]. See also [52]. Quite surprisingly, the KP equation appears in various mathematical problems.

and thus

$$u = q_y + b\,(q_{xx} + q_x^2) + f \tag{4.213}$$

where f is a function which does not depend on x, i.e. $f(t,y)$. Now we can eliminate u from (4.210) and obtain

$$(q_t - c\,q_x - \frac{ab}{2}\,q_{xxx} + ab\,q_x^3)_x + 3a\,(q_y + f)\,q_{xx} - \frac{3a}{2b}(q_{yy} + f_y) = 0\;. \tag{4.214}$$

Expressing f as $f = h_y$ with a function $h(t,y)$, a field redefinition $q \mapsto q - h$ eliminates f from the last equation and we get

$$(q_t - c\,q_x - \frac{ab}{2}\,q_{xxx} + ab\,q_x^3)_x + 3a\,q_y\,q_{xx} - \frac{3a}{2b}q_{yy} = 0\;. \tag{4.215}$$

This equation may be called a 'modified KP equation' (mKP). Given a solution q of the mKP equation, then u determined by (4.213) is a solution of the KP equation.

Although it is quite striking that the KdV and the KP equation are so nicely related to a certain differential calculus, it is not yet clear what is really behind all that. In particular, it still has to be seen in which way one could profit from this observation.

5 Final remarks

In this report we have collected many examples which, according to our opinion, contribute to an understanding of what noncommutative geometry is all about. More precisely, we have concentrated on examples where the basic algebra $\mathcal{A}$ is commutative, i.e., we dealt with noncommutative geometry of topological spaces (like discrete sets and manifolds). It should be clear from this report that already on this level, noncommutative geometry leads into a huge new world of interesting structures and possibilities for applications in mathematics and physics. Only a tiny portion of it has been explored so far.

In principle, all those classical models and theories which can be (nicely) formulated in terms of (ordinary) differential forms can be deformed to noncommutative differential calculi. But there is no guarantee, of course, that this procedure leads to something really interesting. From an exercise to a nontrivial result is still a long way. It is good to have a kind of guiding principle. Some of my own work with A. Dimakis originally aimed at a formulation of discrete gravity (an alternative to Regge calculus) in the framework of noncommutative geometry. This problem is still not satisfactorily solved, but on the way I think we made some nice observations. The guiding principle behind our construction of generalized σ-models (including the Toda lattice) was of a technical nature. The idea was to deform ordinary σ-models in such a way that the known construction of an infinite set of conserved currents still works.

Again, we have to stress that our collection of material and references only displays a very small portion of the field of noncommutative geometry. The material which we presented in these lectures centers around own work and was not intended to cover much about different approaches and results of other authors. Actually, applications of noncommutative geometry to *commutative* algebras have not really been taken much into consideration by other authors, except for the example of the two (and three) point set which plays a crucial role in the particle physics models of Connes and Lott [39] (see also [53] for a 'gravity' approach).

Acknowledgment. I have to thank the organizers of the conference and in particular Claus Lämmerzahl and Alfredo Macias for an enjoyable time in Mexico.

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