Categorical Homotopy Type Theory

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The emergence of Homotopy Type Theory

Gestation:

- **Russell**: *Mathematical logic based on the theory of types* (1908)
- **Church**: *A formulation of the simple theory of types* (1940)
- **Lawvere**: *Equality in hyperdoctrines and comprehension schema as an adjoint functor* (1968)
The emergence of Homotopy Type Theory

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▶ Russell: *Mathematical logic based on the theory of types* (1908)
▶ Church: *A formulation of the simple theory of types* (1940)
▶ Lawvere: *Equality in hyperdoctrines and comprehension schema as an adjoint functor* (1968)

Birth:

▶ Voevodsky: *Notes on type systems* (2006∼2009)
Suggested readings

*Recent work in homotopy type theory*
Slides of a talk by Steve Awodey at the AMS meeting January 2014

*Notes on homotopy λ-calculus*
Vladimir Voevodsky

*Homotopy Type Theory*
A book by the participants to the Univalent Foundation Program held at the IAS in 2012-13
Axiomatic Homotopy Theory

Henry Whitehead (1950):

The ultimate aim of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that analytic is equivalent to pure projective geometry.
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Examples of axiomatic systems

- Triangulated categories (Verdier 1963);
- Homotopical algebra (Quillen 1967);
- Homotopy theories (Heller 1988);
- Theory of derivators (Grothendieck 1987);
- Homotopy type theory;
- Elementary higher topos?
Some features of Hott

Hott replaces

- sets by spaces,
- isomorphisms by equivalences,
- proofs of equality $x = y$ by paths $x \rightsquigarrow y$,
- the relation $x = y$ by the homotopy relation $x \sim y$,
- equivalences $X \simeq Y$ by paths $X \rightsquigarrow Y$.

The formal system of Hott is decidable in a precise way.
Potential applications

▶ to constructive mathematics,
▶ to proof verification and proof assistant,
▶ to homotopy theory.

A wish list:

▶ to higher topos theory,
▶ higher category theory,
▶ derived algebraic geometry.
Category theory as a bridge

- Category theory
  - Type theory
  - Homotopy theory
  - Hott
Overview of the talk

\[
\begin{array}{c}
\text{tribe} \\
\pi \ - \ \text{tribe} \\
\text{Martin-Löf tribe} \\
\text{Voevodsky tribe} \\
\text{Elementary higher topos?} \\
h \ - \ \text{tribe}
\end{array}
\]
Quadrable objects and maps

An object $X$ of a category $C$ is **quadrable** if the cartesian product $A \times X$ exists for every object $A \in C$.

A map $p : X \to B$ is **quadrable** if the object $(X, p)$ of the category $C/B$ is quadrable. This means that the pullback square

$$
\begin{array}{ccc}
A \times_B X & \xrightarrow{p_2} & X \\
\downarrow{p_1} & & \downarrow{p} \\
A & \xrightarrow{f} & B
\end{array}
$$

exists for every map $f : A \to B$.

The projection $p_1$ is called the **base change** of $p : X \to B$ along $f : A \to B$. 
Let $\mathcal{C}$ be a category with terminal object $\star$.

**Definition**

A **tribe structure** on $\mathcal{C}$ is a class of maps $\mathcal{F} \subseteq \mathcal{C}$ satisfying the following conditions:

- $\mathcal{F}$ contains the isomorphisms and is closed under composition;
- every map in $\mathcal{F}$ is quadrable and $\mathcal{F}$ is closed under base changes;
- the map $X \rightarrow \star$ belongs to $\mathcal{F}$ for every object $X \in \mathcal{C}$.

A **tribe** is a category $\mathcal{C}$ with terminal object equipped with a tribe structure $\mathcal{F}$. A map in $\mathcal{F}$ is called a **fibration**.
Examples of tribes

- A category with finite products, if the fibrations are the projections;
- The category of small groupoids \( \text{Grpd} \) if the fibrations are the iso-fibrations;
- The category of Kan complexes \( \text{Kan} \) if the fibrations are the Kan fibrations;
- The category of fibrant objects of a Quillen model category.
Types and terms

An object $E$ of a tribe $C$ is called a **type**. Notation:

$$\vdash E : Type$$

A map $t : \star \to E$ in $C$ is called a **term** of type $E$. Notation:

$$\vdash t : E$$
Fibrations and families

The fiber $E(a)$ of a fibration $p : E \to A$ at a point $a : A$ is defined by the pullback square

$$
\begin{array}{ccc}
E(a) & \longrightarrow & E \\
\downarrow & & \downarrow p \\
\star & \underset{a}{\longrightarrow} & A.
\end{array}
$$

A fibration $p : E \to A$ is a family $(E(x) : x \in A)$ of objects of $\mathcal{C}$ parametrized by a variable element $x \in A$.

A tribe is a collection of families closed under certain operations.
The local tribe $C(A)$

For an object $A$ of a tribe $C$.

The **local tribe** $C(A)$ is the full sub-category of $C/A$ whose objects $(E, p)$ are the fibrations $p : E \to A$ with codomain $A$.

A map $f : (E, p) \to (F, q)$ in $C(A)$ is a fibration if the map $f : E \to F$ is a fibration in $C$.

An object $(E, p)$ of $C(A)$ is a **dependent type** in context $x : A$.

$$x : A \vdash E(x) : Type$$

A section $t$ of $p : E \to A$ is called a **dependent term** $t(x) : E(x)$

$$x : A \vdash t(x) : E(x)$$
General contexts

Type declarations can be iterated:

\[
\begin{align*}
A &: Type \\
x &: A \vdash B(x) : Type \\
x &: A, y &: B(x) \vdash C(x, y) : Type \\
x &: A, y &: B(x), z &: C(x, y) \vdash E(x, y, z) : Type
\end{align*}
\]

\[
\begin{array}{ccc}
& E \\
\downarrow \\
A & \leftarrow & B & \leftarrow & C
\end{array}
\]

\(\Gamma = (x &: A, y &: B(x), z &: C(x, y))\) is an example of general context.
The syntactic category

An object of the syntactic category is a formal expression $[\Gamma]$ where $\Gamma$ is a (general) context.

A map $f : [x : A] \rightarrow [y : B]$ is a term

$$x : A \vdash f(x) : B$$

Two maps $f, g : [x : A] \rightarrow [y : B]$ are equal if $f(x) = g(x)$ can be proved in context $x : A$,

$$x : A \vdash f(x) = g(x) : B$$

Composition of maps is defined by substituting:

$$x : A \vdash f(x) : B, \quad y : B \vdash g(y) : C$$

$$x : A \vdash g(f(x)) : C$$
Homomorphism of tribes

A **homomorphism** of tribes is a functor $F : C \to D$ which

- takes fibrations to fibrations;
- preserves base changes of fibrations;
- preserves terminal objects.

Remark: The category of tribes is a 2-category, where a 1-cell is a homomorphism and 2-cell is a natural transformation.
Base change = change of parameters

If $f : A \to B$ is a map in a tribe $\mathcal{C}$, then the base change functor

$$f^* : \mathcal{C}(B) \to \mathcal{C}(A)$$

is a homomorphism of tribes.

In type theory, it is expressed by the following deduction rule

$$\frac{y : B \vdash E(y) : Type}{x : A \vdash E(f(x)) : Type}.$$
Restriction of context

Let $A$ be an object of a tribe $C$.

The base change functor $i_A : C \to C(A)$ along the map $A \to \star$ is a homomorphism of tribes.

By definition $i_A(E) = (E \times A, p_2)$.

The functor $i_A : C \to C(A)$ is expressed in type theory by a deduction rule called context weakening:

\[
\vdash E : Type \\
\frac{}{x : A \vdash E : Type.}
\]
Free extension

The extension $i_A : C \rightarrow C(A)$ is freely generated by a term $x_A$ of type $A$.

An analogy:

Recall that if $R$ is a commutative ring, then the polynomial extension $i : R \rightarrow R[x]$ is freely generated by the element $x$. The freeness means that for every homomorphism $f : R \rightarrow S$ and every element $s \in S$, there exists a unique homomorphism $h : R[x] \rightarrow S$ such that $hi = f$ and $h(x) = s$,

The element $x \in R[x]$ can be assigned any value. It is generic.
Generic terms

The functor $i = i_A : \mathcal{C} \to \mathcal{C}(A)$ takes the object $A$ to the object $i(A) = (A \times A, p_2)$.

The diagonal $\delta_A : A \to A \times A$ is a map $\delta_A : \star_A \to i(A)$ in $\mathcal{C}(A)$; it is thus a term $\delta_A : i(A)$.

**Theorem**

*The extension $i : \mathcal{C} \to \mathcal{C}(A)$ is freely generated by the term $\delta_A : i(A)$. Thus, $\mathcal{C}(A) = \mathcal{C}[x_A]$ with $x_A = \delta_A$.*

Hence the diagonal $\delta_A : i(A)$ is a **generic** term.
Total space and summation

The forgetful functor $\mathcal{C}(A) \to \mathcal{C}$ associates to a fibration $p : E \to A$ its *total space* $E = \sum_{x:A} E(x)$. It is thus a summation operation,

$$\Sigma_A : \mathcal{C}(A) \to \mathcal{C}.$$ 

It leads to the $\Sigma$-formation rule,

$$\begin{array}{c}
x : A \vdash E(x) : Type \\
\hline
\vdash \Sigma_{x:A} E(x) : Type
\end{array}$$

A term $t : \sum_{x:A} E(x)$ is a pair $t = (a, u)$, where $a : A$ and $u : E(a)$. 
Display maps

The projection

\[ pr_1 : \sum_{x:A} E(x) \rightarrow A \]

is called a **display map**.

The syntactic category of type theory is a tribe, where a fibration is a map isomorphic to a display map.
**Push-forward**

If $f : A \to B$ is a fibration in a tribe $C$, then the *push-forward* functor

$$f_! : C(A) \to C(B)$$

is defined by putting $f_!(E, p) = (E, fp)$.

The functor $f_!$ is left adjoint to the pullback functor $f^* : C(B) \to C(A)$.

Formally, we have

$$f_!(E)(y) = \sum_{f(x) = y} E(x).$$

for a term $y : B$. 
Function space \([A, B]\)

Our goal is now to introduce the notion of \(\pi\)-tribe. Let \(A\) be a quadrable object in a category \(C\).

Recall that the **exponential** of an object \(B \in C\) by \(A\) is an object \([A, B]\) equipped with a map \(\epsilon : [A, B] \times A \rightarrow B\) called the **evaluation** such that for every object \(C \in C\) and every map \(u : C \times A \rightarrow B\), there exists a unique map \(v : C \rightarrow [A, B]\) such that \(\epsilon(v \times A) = u\).

We write \(v = \lambda^A(u)\).
Space of sections

Let $A$ be a quadrable object in a category $C$.

The *space of sections* of an object $E = (E, p) \in C/A$ is an object $\Pi_A(E) \in C$ equipped with a map $\epsilon : \Pi_A(E) \times A \to E$ called the *evaluation* such that:

- $p\epsilon = p_2$
- for every object $C \in C$ and every map $u : C \times A \to E$ in $C/A$ there exists a unique map $v : C \to \Pi_A(E)$ such that $\epsilon(v \times A) = u$.

We write $v = \lambda^A(u)$. 
Products along a map

Let \( f : A \to B \) be a quadrable map in a category \( C \).

The **product** \( \Pi_f(E) \) of an object \( E = (E, p) \in C/A \) along a map \( f : A \to B \) is the space of sections of the map \( (E, fp) \to (A, f) \) in the category \( C/B \),

\[
\begin{array}{ccc}
E & \to & \Pi_f(E) \\
p & \downarrow & \phantom{\downarrow} \\
A & \xrightarrow{f} & B
\end{array}
\]

For every \( y : B \) we have

\[
\Pi_f(E)(y) = \prod_{f(x) = y} E(x)
\]
**π-tribes**

**Definition**
We say that a tribe $\mathcal{C}$ is $\pi$-**closed**, and that it is a $\pi$-**tribe**, if every fibration $E \to A$ has a product along any fibration $f : A \to B$ and if the structure map $\Pi_f(E) \to B$ is a fibration,

The functor $\Pi_f : \mathcal{C}(A) \to \mathcal{C}(B)$ is right adjoint to the functor $f^* : \mathcal{C}(B) \to \mathcal{C}(A)$. 

Examples of $\pi$-tribes

- A cartesian closed category, where a fibration is a projection;
- A locally cartesian category is a $\Pi$-tribe in which every map is a fibration;
- The category of small groupoids $\text{Grpd}$, where a fibration is an iso-fibration (Hofmann, Streicher);
- The category of Kan complexes $\text{Kan}$, where a fibrations is a Kan fibration (Streicher, Voevodsky);

If $C$ is a $\pi$-tribe, then so is the tribe $C(A)$ for every object $A \in C$. 
In a Π-tribe, we have the following Π-formation rule:

\[
\begin{align*}
  x : A & \vdash E(x) : Type \\
  \vdash \prod_{x : A} E(x) : Type.
\end{align*}
\]

There is also a rule for the introduction of \( \lambda \)-terms:

\[
\begin{align*}
  x : A & \vdash t(x) : E(x) \\
  \vdash (\lambda x)t(x) : \prod_{x : A} E(x)
\end{align*}
\]
Homotopical tribes

Definition
We say that a map \( u : A \to B \) in a tribe \( \mathcal{C} \) is anodyne if it has the left lifting property with respect to every fibration \( f : X \to Y \).

This means that every commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{a} & X \\
\downarrow{u} & & \downarrow{f} \\
B & \xrightarrow{b} & Y \\
\end{array}
\]

has a diagonal filler \( d : B \to X \) (\( du = a \) and \( fd = b \)).
Homotopical tribes

Definition
We say that a tribe $\mathcal{C}$ is **homotopical**, or a **h-tribe**, if the following two conditions are satisfied

- every map $f : A \to B$ admits a factorization $f = pu$ with $u$ an anodyne map and $p$ a fibration;
- the base change of an anodyne map along a fibration is anodyne.
Examples of $h$-tribes

- The category of groupoids $\text{Grpd}$, where a functor is anodyne if it is a monic equivalence (Hofmann, Streicher);
- The category of Kan complexes $\text{Kan}$, where a map is anodyne if it is a monic homotopy equivalence (Streicher, Awodey and Warren, Voevodsky);
- The syntactic category of Martin-Löf type theory, where a fibration is a map isomorphic to a display map (Gambino and Garner).

If $C$ is a $h$-tribe, then so is the tribe $C(A)$ for every object $A \in C$. 
A **path object** for an object $A \in C$ is a factorisation of the diagonal $\Delta : A \to A \times A$ as an anodyne map $r : A \to PA$ followed by a fibration $(s, t) : PA \to A \times A$,
Identity type

In Martin-Löf type theory, there is a type constructor which associates to every type $A$ a dependent type

$$x: A, y: A \vdash \text{Id}_A(x, y) : \text{Type}$$

called the **identity type** of $A$,

A term $p : \text{Id}_A(x, y)$ is regarded as a **proof** that $x = y$.

There is a term

$$x : A \vdash r(x) : \text{Id}_A(x, x)$$

called the **reflexivity term**. It is a proof that $x = x$. 
The $J$-rule

The identity type $Id_A$ is defined by putting

$$Id_A = \sum_{(x,y):A \times A} Id_A(x, y).$$

In type theory, there is an operation $J$ which takes a commutative square

$$A \xrightarrow{u} E \quad \xrightarrow{r} \quad \xrightarrow{p} \quad \xrightarrow{Id_A} \quad \xrightarrow{Id_A}$$

with $p$ a fibration, to a diagonal filler $d = J(u, p)$

$$A \xrightarrow{u} E \quad \xrightarrow{r} \quad \xrightarrow{d} \quad \xrightarrow{p} \quad \xrightarrow{Id_A} \quad \xrightarrow{Id_A}$$
Awodey and Warren: The $J$-rule shows that the reflexivity term $r : A \to Id_A$ is anodyne! Hence the identity type

$$Id_A = \sum_{(x,y) : A \times A} Id_A(x, y)$$

is a path object for $A$,
Mapping path space

The mapping path space $P(f)$ of a map $f : A \to B$ is defined by the pullback square

$$
\begin{array}{ccc}
P(f) & \xrightarrow{p_2} & PB \\
p_1 & & s \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
$$

This gives a factorization $f = pu : A \to P(f) \to B$ with $u = \langle 1_A, rf \rangle$ an anodyne map and $p = tp_2$ a fibration.

The homotopy fiber of a map $f : A \to B$ at a point $y : B$ is the fiber of the fibration $p : P(f) \to B$ at the same point,

$$
fib_f(y) = \sum_{x : A} ld_B(f(x), y).
$$
Homotopic maps

Let $C$ be a $h$-tribe.

A **homotopy** $h : f \rightsquigarrow g$ between two maps $f, g : A \to B$ in $C$ is a map $h : A \to PB$

such that $sh = f$ and $th = g$.

In type theory, $h$ is regarded as a **proof** that $f = g$,

$$x : A \vdash h(x) : Id_B(f(x), g(x)).$$
The homotopy category

Let $\mathcal{C}$ be a $h$-tribe.

**Theorem**

The homotopy relation $f \sim g$ is a congruence on the arrows of $\mathcal{C}$.

The **homotopy category** $\text{Ho}(\mathcal{C})$ is the quotient category $\mathcal{C}/\sim$.

A map $f : X \to Y$ in $\mathcal{C}$ is called a **homotopy equivalence** if it is invertible in $\text{Ho}(\mathcal{C})$.

Every anodyne map is a homotopy equivalence.

An object $X$ is **contractible** if the map $X \to \ast$ is a homotopy equivalence.
Local homotopy categories

A map $f : (E, p) \to (F, q)$ in $C/A$ is called a weak equivalence if the map $f : E \to F$ is a homotopy equivalence in $C$.

The local homotopy category $Ho(C/A)$ is defined to be the category of fraction

$$Ho(C/A) = W_A^{-1}(C/A)$$

where $W_A$ is the class of weak equivalences in $C/A$.

The inclusion $C(A) \to C/A$ induces an equivalence of categories:

$$Ho(C(A)) = Ho(C/A)$$
Recall that a square

\[
\begin{array}{c}
A \\
\downarrow \\
B
\end{array} \quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
C \quad \begin{array}{c}
D \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
B \quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
D
\end{array}
\end{array}
\end{array}
\]

is called a *homotopy pullback* if the canonical map \( A \to B \times^h_D C \) is a homotopy equivalence, where \( B \times^h_D C = (f \times g)^*(PD) \)

\[
\begin{array}{c}
B \times^h_D C \\
\downarrow \\
B \times C
\end{array} \quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
PD \quad \begin{array}{c}
f \times g \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
D \times D
\end{array}
\end{array}
\]
$h$-propositions

A map $u : A \to B$ is *homotopy monic* if the square

\[
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
\downarrow^{1_A} & & \downarrow^u \\
A & \xrightarrow{u} & B \\
\end{array}
\]

is homotopy pullback.

**Definition**

An object $A \in C$ is a *$h$-proposition* if the map $A \to \ast$ is homotopy monic.

An object $A$ is a $h$-proposition if and only if the diagonal $A \to A \times A$ is a homotopy equivalence.
The fibration $\langle s, t \rangle : PA \to A \times A$ defines an object $P(A)$ of the local tribe $\mathcal{C}(A \times A)$.

An object $A$ is
- a 0-type if $P(A)$ is a $h$-proposition in $\mathcal{C}(A \times A)$;
- a $(n + 1)$-type if $P(A)$ is a $n$-type in $\mathcal{C}(A \times A)$.

A 0-type is also called a $h$-set.

An object $A$ is a $h$-set if the diagonal $A \to A \times A$ is homotopy monic.
Homotopy initial objects

Let $\mathcal{C}$ be a $h$-tribe.

An object $\bot \in \mathcal{C}$ is **homotopy initial** if every fibration $p : E \to \bot$ has a section $\sigma : \bot \to E$,

\[
\begin{array}{c}
E \\
\downarrow p \\
\bot
\end{array}
\xleftarrow{\sigma}
\]

A homotopy initial object remains initial in the homotopy category $Ho(\mathcal{C})$. 
Homotopy coproducts

An object $A \sqcup B$ equipped with a pair of maps $i, j : A, B \to A \sqcup B$ such that for every fibration $p : E \to A \sqcup B$ and every pair of maps $f, g : A, B \to E$ such that $pf = i$ and $pg = j$,

there exists a section $\sigma : A \sqcup B \to E$ such that $\sigma i = f$ and $\sigma j = g$.

A homotopy coproduct remains a coproduct in the homotopy category $\text{Ho}(\mathcal{C})$. 

\[
\begin{array}{ccc}
A & \xrightarrow{i} & A \sqcup B & \xleftarrow{j} & B \\
\downarrow{\phantom{f}} & \phantom{p} & \downarrow{\phantom{f}} & \phantom{p} & \phantom{\downarrow{\phantom{f}}} \\
E & \xleftarrow{g} & \phantom{E} & \xrightarrow{f} & E
\end{array}
\]
Homotopy natural number object

It is a homotopy initial object \((\mathbb{N}, s, 0)\) in the category of triples \((X, f, a)\), for \(X \in C\), \(f : X \to X\) and \(a : X\).

For every fibration \(p : X \to \mathbb{N}\), such that \(pf = sp\) and \(p(a) = 0\), there exists a section \(\sigma : \mathbb{N} \to X\) such that \(\sigma s = f\sigma\) and \(\sigma(0) = a\).

A homotopy natural number object \((\mathbb{N}, s, 0)\) is not necessarily a natural number object in the homotopy category \(Ho(C)\).
Martin-Löf tribes

Definition
A tribe is a \( \pi h \)-tribe if it is both a \( \pi \)-tribe and a \( h \)-tribe.

A \( \pi h \)-tribe \( C \) satisfies the axiom of function extensionality if the product functor

\[
\Pi_f : C(A) \rightarrow C(B)
\]

preserves the homotopy relation for every fibration \( f : A \rightarrow B \).

Definition
A \textbf{ML-tribe} is a \( \pi h \)-tribe which satisfies the axiom of function extensionality.
Examples of ML-tribes

- The category of groupoids $\text{Grpd}$ (Hofmann and Streicher);
- The category of Kan complexes $\text{Kan}$ (Awodey and Warren, Voevodsky);
- The syntactic category of type theory with function extensionality (Gambino and Garner).

If $C$ is a ML-tribe, then so is the tribe $C(A)$ for every $A \in C$. 
Elementary toposes

Let $\mathcal{E}$ be a category with finite limits.

Recall that a monomorphism $t : 1 \to \Omega$ in $\mathcal{E}$ is said to be *universal* if for every monomorphism $S \to A$ there exists a unique map $f : A \to \Omega$, such that $f^{-1}(t) = S$.

\[
\begin{array}{ccc}
S & \longrightarrow & 1 \\
\downarrow & & \downarrow t \\
A & \longrightarrow & \Omega
\end{array}
\]

The pair $(\Omega, t)$ is called a *sub-object classifier*.

Lawvere and Tierney: An *elementary topos* is a locally cartesian category with a sub-object classifier $(\Omega, t)$. 
Small fibrations and universes

A class of **small fibrations** in a tribe $\mathcal{C} = (\mathcal{C}, \mathcal{F})$ is a class of maps $\mathcal{F}' \subseteq \mathcal{F}$ which contains the isomorphisms and is closed under composition and base changes.

A small fibration $q : U' \to U$ is **universal** if for every small fibration $p : E \to A$ there exists a cartesian square:

$$
\begin{array}{ccc}
E & \rightarrow & U' \\
p & \downarrow & \downarrow q \\
A & \rightarrow & U.
\end{array}
$$

A **universe** is the codomain of a universal small fibration $U' \to U$. 
Martin-Löf universes

A universe $U' \to U$ in a $\pi$-tribe $C$ is $\pi$-closed if the product of a small fibration along a small fibration is small.

A universe $U' \to U$ in a $h$-tribe $C$ is $h$-closed if the path fibration $PA \to A \times A$ can be chosen small for each object $A$.

A universe $U' \to U$ in $\pi h$-tribe $C$ is a $\pi h$-closed if it is both $\pi$-closed and $h$-closed.

We may also say that $\pi h$-closed universe is a ML-universe.
Decidability

A set \( S \) is \textit{decidable} if the relations \( x \in S \) and the equality relation \( x = y \) for \( x, y \in S \) can be decided recursively.

- The set of natural numbers \( \mathbb{N} \) is decidable;
- Not every finitely presented group is decidable (Post).

\textbf{Martin-Löf’s theorem} : The relations \( \vdash t : A \) and \( \vdash s = t : A \) are decidable in type theory without function extensionality, but with a ML-universe, with finite (homotopy) coproducts and (homotopy) natural numbers. Moreover, every globally defined term \( \vdash t : \mathbb{N} \) is definitionaly equal to a numeral \( s^n(0) : \mathbb{N} \).
Homotopical pre-sheaves

Let $\mathcal{C}$ be a ML-tribe.

**Definition**

A presheaf $F : \mathcal{C}^{\text{op}} \to \text{Set}$ **homotopical** if it respects the homotopy relation: $f \sim g \Rightarrow F(f) = F(g)$.

A homotopical presheaf is the same thing as a functor $F : \text{Ho}(\mathcal{C})^{\text{op}} \to \text{Set}$.

A homotopical presheaf $F$ is **representable** if the functor $F : \text{Ho}(\mathcal{C})^{\text{op}} \to \text{Set}$ is representable.
Let $\mathcal{C}$ be a ML-tribe.

If $E \in \mathcal{C}$, then the presheaf $F : \mathcal{C}^{\text{op}} \to \textbf{Set}$ defined by putting

$$F(A) = \begin{cases} 1, & \text{if } E_A \text{ is contractible in } \mathcal{C}(A) \\ \emptyset & \text{otherwise} \end{cases}$$

is homotopical.

It is represented by the $h$-proposition

$$\text{IsContr}(E) = \text{def } \sum_{x:E} \prod_{y:E} \text{Id}_E(x, y)$$

Compare with

$$(\exists x \in E) \ (\forall y \in E) \ x = y$$
IsEq(f)

Let $C$ be a ML-tribe.

If $f : X \to Y$ is a map in $C$, then the presheaf $F : C^{op} \to \textbf{Set}$ defined by putting

$$F(A) = \begin{cases} 1, & \text{if } f_A : X_A \to X_A \text{ is an equivalence} \\ \emptyset & \text{otherwise} \end{cases}$$

is homotopical.

It is represented by the $h$-proposition

$$\text{IsEq}(f) = \text{def } \prod_{y : Y} \text{IsCont}(\text{fib}_f(y)),$$

where $\text{fib}_f(y)$ is the homotopy fiber of $f$ at $y : Y$. 
Let $\mathcal{C}$ be a ML-tribe.

If $X, Y \in \mathcal{C}$, let us put

$$
Eq(X, Y) \overset{\text{def}}{=} \sum_{f: X \to Y} IsEq(f)
$$

For every object $A \in \mathcal{C}$, there is a bijection between the maps

$$
A \to Eq(X, Y)
$$

in $Ho(\mathcal{C})$ and the isomorphism $X_A \simeq Y_A$ in $Ho(\mathcal{C}(A))$
Let $C$ be a ML-tribe.

For every fibration $p : E \rightarrow A$ let us put

$$Eq_A(E) = \sum_{x:A} \sum_{y:A} Eq(E(x), E(y))$$

This defines a fibration $Eq_A(E) \rightarrow A \times A$.

The identity of $E(x)$ is represented by a term

$$x : A \vdash u(x) : Eq(E(x), E(x))$$

which defines the unit map $u : A \rightarrow Eq_A(E)$,
Univalent fibrations

Voevodsky:

Definition
A fibration $E \to A$ is univalent if the unit map $u : A \to Eq_A(E)$ is a homotopy equivalence.

In which case the fibration $Eq_A(E) \to A \times A$ is equivalent to the path fibration $PA \to A \times A$.

Remark: The notion of univalent fibration can be defined in any $\pi h$-tribe.
Uncompressible fibrations

A Kan fibration is univalent if and only if it is uncompressible.

To {	extit{compress}} a Kan fibration \( p : X \to A \) is to find a homotopy pullback square

\[
\begin{array}{ccc}
X & \to & Y \\
\downarrow & & \downarrow \\
A & \to & B \\
\end{array}
\]

in which \( f \) is homotopy surjective but not homotopy monic.

Every Kan fibration \( X \to A \) is the pullback of an uncompressible fibration \( X' \to A' \) along a homotopy surjection \( A \to A' \). Moreover, the fibration \( X' \to A' \) is homotopy unique.
Voevodsky tribes

Voevodsky: The tribe of Kan complexes Kan admits a univalent ML-universe $U' \rightarrow U$.

Definition
A $V$-tribe is a $\pi h$-tribe $C$ equipped with a univalent ML-universe $U' \rightarrow U$.

Voevodsky’s theorem: A $V$-tribe satisfies function extensionality, it is thus a ML-tribe.

Voevodsky’s conjecture: The relations $\vdash t : A$ and $\vdash s = t : A$ are decidable in V-type theory. Moreover, every globally defined term $\vdash t : \mathbb{N}$ is definitionaly equal to a numeral $s^n(0) : \mathbb{N}$.
What is an elementary higher topos?

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Rezk and Lurie:

**Definition**
A higher topos is a locally presentable $(\infty, 1)$-category with a classifying universe $U'_k \to U_k$ for $k$-compact morphisms for each regular cardinal $k$. 
What is an elementary higher topos?

We hope that the notion of elementary higher topos will emerge after a period of experimentations with the axioms.

In principle, the notion could be formalized with any notion of $(\infty, 1)$-category:

- a quasi-category;
- a complete Segal space;
- a Segal category;
- a simplicial category;
- a model category;
- a relative category.

A formalization may emerge from homotopy type theory.

Here we propose an axiomatization using the notion of generalized model category (to be defined next).
Generalised model categories

Let $\mathcal{E}$ be a category with terminal object $\top$ and initial object $\bot$.

Definition
A \textit{generalized model structure} on $\mathcal{E}$ is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes maps in $\mathcal{E}$ such that

- every map in $\mathcal{F}$ is quadrable and every map in $\mathcal{C}$ is co-quadrable.
- $\mathcal{W}$ satisfies 3-for-2;
- the pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorization systems;

A \textit{generalized model category} is a category $\mathcal{E}$ equipped with a generalised model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. 
A map in $\mathcal{C}$ is called a cofibration, a map in $\mathcal{W}$ is acyclic and a map in $\mathcal{F}$ a fibration. An object $A$ is cofibrant if the map $\perp \to A$ is a cofibration, an object $X$ is fibrant if the map $X \to \top$ is a fibration.

A generalized model structure is right proper (resp. left proper) if the base (resp. cobase) change of a weak equivalence along a fibration (resp. a cofibration) is a weak equivalence. A generalized model structure is proper if it is both left and right proper.

A generalized model structure is smooth if every object is cofibrant, if it is right proper, if the base change of a cofibration along a fibration is a cofibration and if the product of a fibration along a fibration exists.

Remark: A smooth generalized model structure is proper and the product of a fibration along a fibration is a fibration.
EH-topos?

Definition ($\beta$-version): An **EH-topos** is a smooth generalised model category $\mathcal{E}$ equipped a univalent ML-universe $U' \to U$.

Examples:

- The category of simplicial sets $\textbf{sSet}$ (Voevodsky);
- The category of simplicial presheaves over any elegant Reedy category (Shulman).
- The category of symmetric cubical sets (Coquand).
- The category of presheaves over any elegant (local) test category (Cisinski).
Critics

Critic 1: We may want a hierarchy of universes $U_0 : U_1 : U_2 : \cdots$.

Critic 2: We may want a fibrant-cofibrant natural number object $\mathbb{N}$.

Critic 3: Every fibration should factor as a homotopy surjection followed by a monic fibration.

Critic 4: The initial object should be strict.

Critic 5: The inclusions $i_1 : X \to X \sqcup Y$ and $i_2 : Y \to X \sqcup Y$ should be fibration for every pair of objects $(X, Y)$.

Critic 5’: The functor $(i_1^*, i_2^*) : \mathcal{E}/(X \sqcup Y) \to \mathcal{E}/X \times \mathcal{E}/Y$ should be an equivalence of generalized model categories.
More critics

Critic 6: If \( u : A \to B \) is a cofibration between fibrant objects and \( p : E \to B \) is a fibration, then the map

\[
u^* : \prod_B(E) \to \prod_A(u^*(E))\]

induced by \( u \) should be a fibration. Moreover, \( u^* \) should be acyclic when \( u \) is acyclic.

Critic 6': Condition 6 should be true in every slice category \( \mathcal{E}/C \).
Epilogue

What is mathematics?

Georg Cantor: "The essence of mathematics lies in its freedom"

Bertrand Russell: "Mathematics is the subject in which we never know what we are talking about, nor whether what we are saying is true"

Godfrey H. Hardy: "Beauty is the first test; there is no permanent place in the world for ugly mathematics"

John von Neumann: "In mathematics you don't understand things. You just get used to them"
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THANK YOU FOR YOUR ATTENTION!