Categorical Homotopy Type Theory

André Joyal

UQÀM

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Warning

The present slides include corrections and modifications that were made during the week following my talk. Thanks to Steve Awodey, David Spivak, Thierry Coquand, Nicola Gambino, and Michael Shulman.

The emergence of Homotopy Type Theory

Gestation:

- **Russell**: *Mathematical logic based on the theory of types* (1908)
- **Church**: *A formulation of the simple theory of types* (1940)
- **Lawvere**: *Equality in hyperdoctrines and comprehension schema as an adjoint functor* (1968)
- **Martin-Löf**: *Intuitionistic theory of types* (1971, 1975, 1984)
- **Hofmann, Streicher**: *The groupoid interpretation of type theory* (1995)

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- **Hofmann, Streicher**: *The groupoid interpretation of type theory* (1995)

Birth:

- **Awodey, Warren**: *Homotopy theoretic models of identity types* (2006∼2007)
- **Voevodsky**: *Notes on type systems* (2006∼2009)

Suggested readings

Recent work in homotopy type theory
Slides of a talk by Steve Awodey at the AMS meeting January 2014

Notes on homotopy λ-calculus
Vladimir Voevodsky

Homotopy Type Theory
A book by the participants to the Univalent Foundation Program held at the IAS in 2012-13

Axiomatic Homotopy Theory

Henry Whitehead (1950):

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Examples of axiomatic systems

- Triangulated categories (Verdier 1963);
- Homotopical algebra (Quillen 1967);
- Homotopy theories (Heller 1988)
- Theory of derivators (Grothendieck 198?)
- Homotopy type theory
- Elementary higher topos?

Some features of Hott

Hott replaces

- sets by spaces,
- isomorphisms by equivalences,
- proofs of equality $x = y$ by paths $x \rightsquigarrow y$,
- the relation $x = y$ by the homotopy relation $x \sim y$,
- equivalences $X \simeq Y$ by paths $X \rightsquigarrow Y$.

The formal system of Hott is decidable in a precise way.

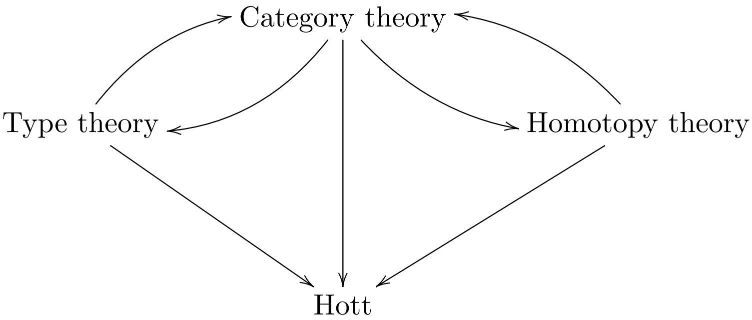
Potential applications

- to constructive mathematics,
- to proof verification and proof assistant,
- to homotopy theory.

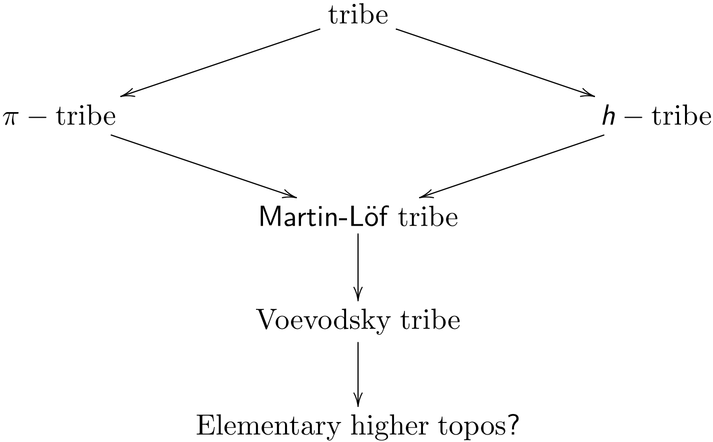
A wish list:

- to higher topos theory,
- higher category theory,
- derived algebraic geometry.

Category theory as a bridge



Overview of the talk



Quadrable objects and maps

An object X of a category $\mathcal{C}$ is **quadrable** if the cartesian product $A \times X$ exists for every object $A \in \mathcal{C}$.

A map $p : X \to B$ is **quadrable** if the object (X, p) of the category $\mathcal{C}/B$ is quadrable. This means that the pullback square

$$\begin{array}{ccc} A \times_B X & \xrightarrow{\;p_2\;} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{\;f\;} & B \end{array}$$

exists for every map $f : A \to B$.

The projection p_1 is called the **base change** of $p : X \to B$ *along* $f : A \to B$.

Tribes

Let $\mathcal{C}$ be a category with terminal object $\star$.

Definition

A **tribe structure** on $\mathcal{C}$ is a class of maps $\mathcal{F} \subseteq \mathcal{C}$ satisfying the following conditions:

- $\mathcal{F}$ contains the isomorphisms and is closed under composition;
- every map in $\mathcal{F}$ is quadrable and $\mathcal{F}$ is closed under base changes;
- the map $X \to \star$ belongs to $\mathcal{F}$ for every object $X \in \mathcal{C}$.

A **tribe** is a category $\mathcal{C}$ with terminal object equipped with a tribe structure $\mathcal{F}$. A map in $\mathcal{F}$ is called a **fibration**.

Examples of tribes

- A category with finite products, if the fibrations are the projections;
- The category of small groupoids **Grpd** if the fibrations are the iso-fibrations;
- The category of Kan complexes **Kan** if the fibrations are the Kan fibrations;
- The category of fibrant objects of a Quillen model category.

Types and terms

An object E of a tribe $\mathcal{C}$ is called a **type**. Notation:

$$\vdash E : Type$$

A map $t : \star \to E$ in $\mathcal{C}$ is called a **term** of type E. Notation:

$$\vdash t : E$$

Fibrations and families

The **fiber** $E(a)$ of a fibration $p : E \to A$ at a point $a : A$ is defined by the pullback square

$$\begin{array}{ccc} E(a) & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ \star & \xrightarrow{\;a\;} & A. \end{array}$$

A fibration $p : E \to A$ is a **family** $(E(x) : x \in A)$ of objects of $\mathcal{C}$ parametrized by a variable element $x \in A$.

A tribe is a collection of families closed under certain operations.

The local tribe $\mathcal{C}(A)$

For an object A of a tribe $\mathcal{C}$.

The **local tribe** $\mathcal{C}(A)$ is the full sub-category of $\mathcal{C}/A$ whose objects (E, p) are the fibrations $p : E \to A$ with codomain A.

A map $f : (E, p) \to (F, q)$ in $\mathcal{C}(A)$ is a fibration if the map $f : E \to F$ is a fibration in $\mathcal{C}$.

An object (E, p) of $\mathcal{C}(A)$ is a **dependent type** in **context** $x : A$.

$$x : A \vdash E(x) : Type$$

A section t of $p : E \to A$ is called a **dependent term** $t(x) : E(x)$

$$x : A \vdash t(x) : E(x)$$

General contexts

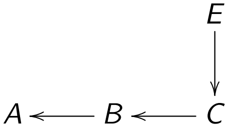
Type declarations can be iterated:

$$A : Type$$

$$x : A \vdash B(x) : Type$$

$$x : A, y : B(x) \vdash C(x, y) : Type$$

$$x : A, y : B(x), z : C(x, y) \vdash E(x, y, z) : Type$$



$\Gamma = (x : A, y : B(x), z : C(x, y))$ is an example of *general context*.

The syntactic category

An object of the syntactic category is a formal expression $[\Gamma]$ where Γ is a (general) context.

A map $f : [x : A] \to [y : B]$ is a term

$$x : A \vdash f(x) : B$$

Two maps $f, g : [x : A] \to [y : B]$ are equal if $f(x) = g(x)$ can be proved in context $x : A$,

$$x : A \vdash f(x) = g(x) : B$$

Composition of maps is defined by substituting:

$$\frac{x : A \vdash f(x) : B, \qquad y : B \vdash g(y) : C}{x : A \vdash g(f(x)) : C}$$

Homomorphism of tribes

A **homomorphism** of tribes is a functor $F : \mathcal{C} \to \mathcal{D}$ which

- takes fibrations to fibrations;
- preserves base changes of fibrations;
- preserves terminal objects.

Remark: The category of tribes is a 2-category, where a 1-cell is a homomorphism and 2-cell is a natural transformation.

Base change=change of parameters

If $f : A \to B$ is a map in a tribe $\mathcal{C}$, then the base change functor

$$f^\star : \mathcal{C}(B) \to \mathcal{C}(A)$$

is a homomorphism of tribes.

In type theory, it is expressed by the following *deduction rule*

$$\frac{y : B \vdash E(y) : Type}{x : A \vdash E(f(x)) : Type.}$$

Restriction of context

Let A be an object of a tribe $\mathcal{C}$.

The base change functor $i_A : \mathcal{C} \to \mathcal{C}(A)$ along the map $A \to \star$ is a homomorphism of tribes.

By definition $i_A(E) = (E \times A, p_2)$.

The functor $i_A : \mathcal{C} \to \mathcal{C}(A)$ is expressed in type theory by a deduction rule called *context weakening*:

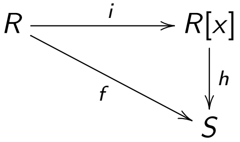
$$\frac{\vdash E : Type}{x : A \vdash E : Type.}$$

Free extension

The extension $i_A : \mathcal{C} \to \mathcal{C}(A)$ is freely generated by a term x_A of type A.

An analogy:

Recall that if R is a commutative ring, then the polynomial extension $i : R \to R[x]$ is freely generated by the element x. The freeness means that for every homomorphism $f : R \to S$ and every element $s \in S$, there exists a unique homomorphism $h : R[x] \to S$ such that $hi = f$ and $h(x) = s$,



The element $x \in R[x]$ can be assigned any value. It is **generic**.

Generic terms

The functor $i = i_A : \mathcal{C} \to \mathcal{C}(A)$ takes the object A to the object $i(A) = (A \times A, p_2)$.

The diagonal $\delta_A : A \to A \times A$ is a map $\delta_A : \star_A \to i(A)$ in $\mathcal{C}(A)$; it is thus a term $\delta_A : i(A)$.

Theorem

The extension $i : \mathcal{C} \to \mathcal{C}(A)$ is freely generated by the term $\delta_A : i(A)$. Thus, $\mathcal{C}(A) = \mathcal{C}[x_A]$ with $x_A = \delta_A$.

Hence the diagonal $\delta_A : i(A)$ is a **generic** term.

Total space and summation

The forgetful functor $\mathcal{C}(A) \to \mathcal{C}$ associates to a fibration $p : E \to A$ its *total space* $E = \sum_{x:A} E(x)$. It is thus a summation operation,

$$\Sigma_A : \mathcal{C}(A) \to \mathcal{C}.$$

It leads to the Σ-*formation* rule,

$$\frac{x : A \vdash E(x) : \mathit{Type}}{\vdash \sum_{x:A} E(x) : \mathit{Type}}$$

A term $t : \sum_{x:A} E(x)$ is a pair $t = (a, u)$, where $a : A$ and $u : E(a)$.

Display maps

The projection

$$pr_1 : \sum_{x:A} E(x) \to A$$

is called a **display map**.

The syntactic category of type theory is a tribe, where a fibration is a map isomorphic to a display map

Push-forward

If $f : A \to B$ is a fibration in a tribe $\mathcal{C}$, then the *push-forward* functor

$$f_! : \mathcal{C}(A) \to \mathcal{C}(B)$$

is defined by putting $f_!(E, p) = (E, fp)$.

The functor $f_!$ is left adjoint to the pullback functor $f^\star : \mathcal{C}(B) \to \mathcal{C}(A)$.

Formally, we have

$$f_!(E)(y) = \sum_{f(x)=y} E(x).$$

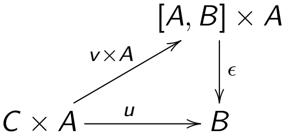
for a term $y : B$.

Function space $[A, B]$

Our goal is now to introduce the notion of π-tribe.

Let A be a quadrable object in a category $\mathcal{C}$.

Recall that the **exponential** of an object $B \in \mathcal{C}$ by A is an object $[A, B]$ equipped with a map $\epsilon : [A, B] \times A \to B$ called the *evaluation* such that for every object $C \in \mathcal{C}$ and every map $u : C \times A \to B$, there exists a unique map $v : C \to [A, B]$ such that $\epsilon(v \times A) = u$.



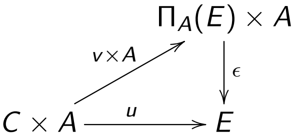
We write $v = \lambda^A(u)$.

Space of sections

Let A be a quadrable object in a category $\mathcal{C}$.

The *space of sections* of an object $E = (E, p) \in \mathcal{C}/A$ is an object $\Pi_A(E) \in \mathcal{C}$ equipped with a map $\epsilon : \Pi_A(E) \times A \to E$ called the *evaluation* such that:

- $p\epsilon = p_2$
- for every object $C \in \mathcal{C}$ and every map $u : C \times A \to E$ in $\mathcal{C}/A$ there exists a unique map $v : C \to \Pi_A(E)$ such that $\epsilon(v \times A) = u$.

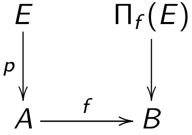


We write $v = \lambda^A(u)$.

Products along a map

Let $f : A \to B$ be a quadrable map in a category $\mathcal{C}$.

The **product** $\Pi_f(E)$ of an object $E = (E, p) \in \mathcal{C}/A$ along a map $f : A \to B$ is the space of sections of the map $(E, fp) \to (A, f)$ in the category $\mathcal{C}/B$,



For every $y : B$ we have

$$\Pi_f(E)(y) = \prod_{f(x)=y} E(x)$$

π-tribes

Definition

We say that a tribe $\mathcal{C}$ is π-**closed**, and that it is a π-**tribe**, if every fibration $E \to A$ has a product along any fibration $f : A \to B$ and if the structure map $\Pi_f(E) \to B$ is a fibration,

The functor $\Pi_f : \mathcal{C}(A) \to \mathcal{C}(B)$ is right adjoint to the functor $f^\star : \mathcal{C}(B) \to \mathcal{C}(A)$.

Examples of π-tribes

- A cartesian closed category, where a fibration is a projection;
- A locally cartesian category is a Π-tribe in which every map is a fibration;
- The category of small groupoids **Grpd**, where a fibration is an iso-fibration (Hofmann, Streicher);
- The category of Kan complexes **Kan**, where a fibrations is a Kan fibration (Streicher, Voevodsky);

If $\mathcal{C}$ is a π-tribe, then so is the tribe $\mathcal{C}(A)$ for every object $A \in \mathcal{C}$.

Π-formation rule

In a Π-tribe, we have the following Π-*formation* rule:

$$\frac{x : A \vdash E(x) : Type}{\vdash \prod_{x:A} E(x) : Type.}$$

There is also a rule for the introduction of λ-terms:

$$\frac{x : A \vdash t(x) : E(x)}{\vdash (\lambda x)t(x) : \prod_{x:A} E(x)}$$

Homotopical tribes

Definition

We say that a map $u : A \to B$ in a tribe $\mathcal{C}$ is **anodyne** if it has the left lifting property with respect to every fibration $f : X \to Y$.

This means that every commutative square

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{b} & Y \end{array}$$

has a diagonal filler $d : B \to X$ ($du = a$ and $fd = b$).

Homotopical tribes

Definition

We say that a tribe $\mathcal{C}$ is **homotopical**, or a **h-tribe**, if the following two conditions are satisfied

- every map $f : A \to B$ admits a factorization $f = pu$ with u an anodyne map and p a fibration;
- the base change of an anodyne map along a fibration is anodyne.

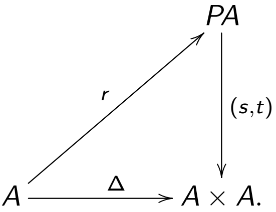
Examples of h-tribes

- The category of groupoids **Grpd**, where a functor is anodyne if it is a monic equivalence (Hofmann, Streicher);
- The category of Kan complexes **Kan**, where a map is anodyne if it is a monic homotopy equivalence (Streicher, Awodey and Warren, Voevodsky);
- The syntactic category of Martin-Löf type theory, where a fibration is a map isomorphic to a display map (Gambino and Garner).

If $\mathcal{C}$ is a h-tribe, then so is the tribe $\mathcal{C}(A)$ for every object $A \in \mathcal{C}$.

Path object

A **path object** for an object $A \in \mathcal{C}$ is a factorisation of the diagonal $\Delta : A \to A \times A$ as an anodyne map $r : A \to PA$ followed by a fibration $(s, t) : PA \to A \times A$,



Identity type

In Martin-Löf type theory, there is a type constructor which associates to every type A a dependent type

$$x : A, y : A \ \vdash Id_A(x, y) : Type$$

called the **identity type** of A,

A term $p : Id_A(x, y)$ is regarded as a **proof** that $x = y$.

There is a term

$$x : A \ \vdash r(x) : Id_A(x, x)$$

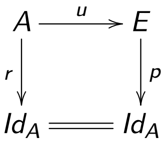
called the **reflexivity term**. It is a proof that $x = x$.

The J-rule

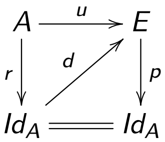
The *identity type* Id_A is defined by putting

$$Id_A = \sum_{(x,y):A\times A} Id_A(x,y).$$

In type theory, there is an operation J which takes a commutative square



with p a fibration, to a diagonal filler $d = J(u,p)$

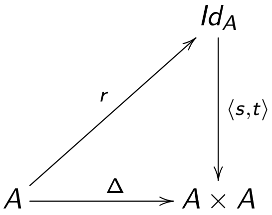


Identity type as a path object

Awodey and Warren: The J-rule shows that the reflexivity term $r : A \to Id_A$ is anodyne! Hence the identity type

$$Id_A = \sum_{(x,y):A\times A} Id_A(x,y)$$

is a path object for A,



Mapping path space

The **mapping path space** $P(f)$ of a map $f : A \to B$ is defined by the pullback square

$$\begin{array}{ccc} P(f) & \xrightarrow{p_2} & PB \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle s} \\ A & \xrightarrow{f} & B. \end{array}$$

This gives a factorization $f = pu : A \to P(f) \to B$ with $u = \langle 1_A, rf \rangle$ an anodyne map and $p = tp_2$ a fibration.

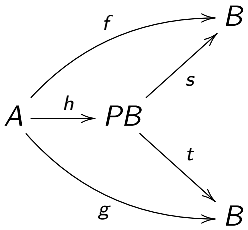
The **homotopy fiber** of a map $f : A \to B$ at a point $y : B$ is the fiber of the fibration $p : P(f) \to B$ at the same point,

$$\mathrm{fib}_f(y) = \sum_{x:A} Id_B(f(x), y).$$

Homotopic maps

Let $\mathcal{C}$ be a h-tribe.

A **homotopy** $h : f \rightsquigarrow g$ between two maps $f, g : A \to B$ in $\mathcal{C}$ is a map $h : A \to PB$



such that $sh = f$ and $th = g$.

In type theory, h is regarded as a **proof** that $f = g$,

$$x : A \vdash h(x) : Id_B(f(x), g(x)).$$

The homotopy category

Let $\mathcal{C}$ be a h-tribe.

Theorem

The homotopy relation $f \sim g$ is a congruence on the arrows of $\mathcal{C}$.

The **homotopy category** $Ho(\mathcal{C})$ is the quotient category $\mathcal{C}/\sim$.

A map $f : X \to Y$ in $\mathcal{C}$ is called a **homotopy equivalence** if it is invertible in $Ho(\mathcal{C})$.

Every anodyne map is a homotopy equivalence.

An object X is **contractible** if the map $X \to \star$ is a homotopy equivalence.

Local homotopy categories

A map $f : (E, p) \to (F, q)$ in $\mathcal{C}/A$ is called a *weak equivalence* if the map $f : E \to F$ is a homotopy equivalence in $\mathcal{C}$.

The **local homotopy category** $Ho(\mathcal{C}/A)$ is defined to be the category of fraction

$$Ho(\mathcal{C}/A) = W_A^{-1}(\mathcal{C}/A)$$

where W_A is the class of weak equivalences in $\mathcal{C}/A$.

The inclusion $\mathcal{C}(A) \to \mathcal{C}/A$ induces an equivalence of categories:

$$Ho(\mathcal{C}(A)) = Ho(\mathcal{C}/A)$$

Homotopy pullback

Recall that a square

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

is called a *homotopy pullback* if the canonical map $A \to B \times_D^h C$ is a homotopy equivalence, where $B \times_D^h C = (f \times g)^\star(PD)$

$$\begin{array}{ccc} B \times_D^h C & \longrightarrow & PD \\ \downarrow & & \downarrow \\ B \times C & \xrightarrow{f \times g} & D \times D \end{array}$$

h-propositions

A map $u : A \to B$ is *homotopy monic* if the square

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle u} \\ A & \xrightarrow{u} & B \end{array}$$

is homotopy pullback.

Definition

An object $A \in \mathcal{C}$ is a *h-proposition* if the map $A \to \star$ is homotopy monic.

An object A is a h-proposition if and only if the diagonal $A \to A \times A$ is a homotopy equivalence.

n-types

The fibration $\langle s, t \rangle : PA \to A \times A$ defines an object $P(A)$ of the local tribe $\mathcal{C}(A \times A)$.

An object A is

- a 0-**type** if $P(A)$ is a h-proposition in $\mathcal{C}(A \times A)$;
- a $(n+1)$-**type** if $P(A)$ is a n-type in $\mathcal{C}(A \times A)$.

A 0-type is also called a *h-set*.

An object A is a *h-set* if the diagonal $A \to A \times A$ is homotopy monic.

Homotopy initial objects

Let $\mathcal{C}$ be a h-tribe.

An object $\bot \in \mathcal{C}$ is **homotopy initial** if every fibration $p : E \to \bot$ has a section $\sigma : \bot \to E$,

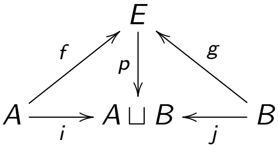
$$\begin{array}{c} E \\ {\scriptstyle p}\big\downarrow \big\uparrow{\scriptstyle \sigma} \\ \bot . \end{array}$$

A homotopy initial object remains initial in the homotopy category $Ho(\mathcal{C})$.

Homotopy coproducts

An object $A \sqcup B$ equipped with a pair of maps $i, j : A, B \to A \sqcup B$

such that for every fibration $p : E \to A \sqcup B$ and every pair of maps $f, g : A, B \to E$ such that $pf = i$ and $pg = j$,



there exists a section $\sigma : A \sqcup B \to E$ such that $\sigma i = f$ and $\sigma j = g$.

A homotopy coproduct remains a coproduct in the homotopy category $Ho(\mathcal{C})$.

Homotopy natural number object

It is a homotopy initial object $(\mathbb{N}, s, 0)$ in the category of triples (X, f, a), for $X \in \mathcal{C}$, $f : X \to X$ and $a : X$.

For every fibration $p : X \to \mathbb{N}$, such that $pf = sp$ and $p(a) = 0$

$$\begin{array}{ccccc} \star & \xrightarrow{a} & X & \xrightarrow{f} & X \\ \| & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \star & \xrightarrow{0} & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \end{array}$$

there exists a section $\sigma : \mathbb{N} \to X$ such that $\sigma s = f\sigma$ and $\sigma(0) = a$.

A homotopy natural number object $(\mathbb{N}, s, 0)$ is not necessarily a natural number object in the homotopy category $Ho(\mathcal{C})$.

Martin-Löf tribes

Definition

A tribe is a πh-**tribe** if it is both a π-tribe and a h-tribe.

A πh-tribe $\mathcal{C}$ satisfies the axiom of *function extensionality* if the product functor

$$\Pi_f : \mathcal{C}(A) \to \mathcal{C}(B)$$

preserves the homotopy relation for every fibration $f : A \to B$.

Definition

A **ML-tribe** is a πh-tribe which satisfies the axiom of function extensionality.

Examples of ML-tribes

- The category of groupoids **Grpd** (Hofmann and Streicher);
- The category of Kan complexes **Kan** (Awodey and Warren, Voevodsky);
- The syntactic category of type theory with function extensionality (Gambino and Garner).

If $\mathcal{C}$ is a ML-tribe, then so is the tribe $\mathcal{C}(A)$ for every $A \in \mathcal{C}$.

Elementary toposes

Let $\mathcal{E}$ be a category with finite limits

Recall that a monomorphism $t : 1 \to \Omega$ in $\mathcal{E}$ is said to be *universal* if for every monomorphism $S \to A$ there exists a unique map $f : A \to \Omega$, such that $f^{-1}(t) = S$,

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \downarrow & & \downarrow t \\ A & \xrightarrow{f} & \Omega \end{array}$$

The pair (Ω, t) is called a *sub-object classifier*.

Lawvere and Tierney: An *elementary topos* is a locally cartesian category with a sub-object classifier (Ω, t).

Small fibrations and universes

A class of **small fibrations** in a tribe $\mathcal{C} = (\mathcal{C}, \mathcal{F})$ is a class of maps $\mathcal{F}' \subseteq \mathcal{F}$ which contains the isomorphisms and is closed under composition and base changes.

A small fibration $q : U' \to U$ is **universal** if for every small fibration $p : E \to A$ there exists a cartesian square:

$$\begin{array}{ccc} E & \longrightarrow & U' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \longrightarrow & U. \end{array}$$

A **universe** is the codomain of a universal small fibration $U' \to U$.

Martin-Löf universes

A universe $U' \to U$ in a π-tribe $\mathcal{C}$ is π-**closed** if the product of a small fibration along a small fibration is small.

A universe $U' \to U$ in a h-tribe $\mathcal{C}$ is h-**closed** if the path fibration $PA \to A \times A$ can be chosen small for each object A.

A universe $U' \to U$ in πh-tribe $\mathcal{C}$ is a πh-**closed** if it is both π-closed and h-closed.

We may also say that πh-**closed** universe is a **ML-universe**.

Decidability

A set S is *decidable* if the relations $x \in S$ and the equality relation $x = y$ for $x, y \in S$ can be decided recursively.

- The set of natural numbers $\mathbb{N}$ is decidable;
- Not every finitely presented group is decidable (Post).

Martin-Löf's theorem :The relations $\vdash t : A$ and $\vdash s = t : A$ are decidable in type theory without function extensionality, but with a ML-universe, with finite (homotopy) coproducts and (homotopy) natural numbers. Moreover, every globally defined term $\vdash t : \mathbb{N}$ is definitionaly equal to a numeral $s^n(0) : \mathbb{N}$.

Homotopical pre-sheaves

Let $\mathcal{C}$ be a ML-tribe.

Definition

A presheaf $F : \mathcal{C}^{op} \to Set$ **homotopical** if it respects the homotopy relation: $f \sim g \Rightarrow F(f) = F(g)$.

A homotopical presheaf is the same thing as a functor $F : Ho(\mathcal{C})^{op} \to Set$.

A homotopical presheaf F is **representable** if the functor $F : Ho(\mathcal{C})^{op} \to \mathbf{Set}$ is representable.

$IsContr(X)$

Let $\mathcal{C}$ be a ML-tribe.

If $E \in \mathcal{C}$, then the presheaf $F : \mathcal{C}^{op} \to \mathbf{Set}$ defined by putting

$$F(A) = \begin{cases} 1, & \text{if } E_A \text{ is contractible in } \mathcal{C}(A) \\ \emptyset & \text{otherwise} \end{cases}$$

is homotopical.

It is represented by the *h*-proposition

$$IsContr(E) =_{\mathrm{def}} \sum_{x:E} \prod_{y:E} Id_E(x, y)$$

Compare with

$$(\exists x \in E)\ (\forall y \in E)\ x = y$$

IsEq(f)

Let $\mathcal{C}$ be a ML-tribe.

If $f : X \to Y$ is a map in $\mathcal{C}$, then the presheaf $F : \mathcal{C}^{op} \to \mathbf{Set}$ defined by putting

$$F(A) = \begin{cases} 1, & \text{if } f_A : X_A \to X_A \text{ is an equivalence} \\ \emptyset & \text{otherwise} \end{cases}$$

is homotopical.

It is represented by the *h*-proposition

$$IsEq(f) =_{\mathrm{def}} \prod_{y:Y} IsCont(\mathrm{fib}_f(y)),$$

where $\mathrm{fib}_f(y)$ is the homotopy fiber of f at $y : Y$.

$Eq(X, Y)$

Let $\mathcal{C}$ be a ML-tribe.

If $X, Y \in \mathcal{C}$, let us put

$$Eq(X, Y) =_{\text{def}} \sum_{f:X \to Y} IsEq(f)$$

For every object $A \in \mathcal{C}$, there is a bijection between the maps

$$A \to Eq(X, Y)$$

in $Ho(\mathcal{C})$ and the isomorphism $X_A \simeq Y_A$ in $Ho(\mathcal{C}(A))$

$Eq_A(E)$

Let $\mathcal{C}$ be a ML-tribe.

For every fibration $p : E \to A$ let us put

$$Eq_A(E) = \sum_{x:A} \sum_{y:A} Eq(E(x), E(y))$$

This defines a fibration $Eq_A(E) \to A \times A$.

The identity of $E(x)$ is represented by a term

$$x : A \vdash u(x) : Eq(E(x), E(x))$$

which defines the *unit map* $u : A \to Eq_A(E)$,

Univalent fibrations

Voevodsky:

Definition

A fibration $E \to A$ is **univalent** if the unit map $u : A \to Eq_A(E)$ is a homotopy equivalence.

In which case the fibration $Eq_A(E) \to A \times A$ is equivalent to the path fibration $PA \to A \times A$.

$$\begin{array}{ccc} PA & \xrightarrow{\simeq} & Eq_A(E) \\ & {\scriptstyle \langle s,t \rangle} \searrow \quad \swarrow {\scriptstyle (s,t)} & \\ & A \times A & \end{array}$$

Remark: The notion of univalent fibration can be defined in any πh-tribe.

Uncompressible fibrations

A Kan fibration is univalent if and only if it is uncompressible.

To *compress* a Kan fibration $p : X \to A$ is to find a homotopy pullback square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

in which f is homotopy surjective but not homotopy monic.

Every Kan fibration $X \to A$ is the pullback of an uncompressible fibration $X' \to A'$ along a homotopy surjection $A \to A'$. Moreover, the fibration $X' \to A'$ is homotopy unique.

Voevodsky tribes

Voevodsky: The tribe of Kan complexes **Kan** admits a univalent ML-universe $U' \to U$.

Definition

A **V-tribe** is a πh-tribe $\mathcal{C}$ equipped with a univalent *ML*-universe $U' \to U$.

Voevodsky's theorem: A V-tribe satisfies function extensionality, it is thus a ML-tribe.

Voevodsky's conjecture : The relations $\vdash t : A$ and $\vdash s = t : A$ are decidable in V-type theory. Moreover, every globally defined term $\vdash t : \mathbb{N}$ is definitionaly equal to a numeral $s^n(0) : \mathbb{N}$.

What is an elementary higher topos?

Grothendieck topos	Elementary topos
Higher topos	EH-topos?

Rezk and Lurie:

Definition

A higher topos is a locally presentable $(\infty, 1)$-category with a classifying universe $U'_k \to U_k$ for k-compact morphisms for each regular cardinal k.

What is an elementary higher topos?

We hope that the notion of elementary higher topos will emerge after a period of experimentations with the axioms.

In principle, the notion could be formalized with any notion of $(\infty, 1)$-category:

- a quasi-category;
- a complete Segal space;
- a Segal category;
- a simplicial category;
- a model category;
- a relative category.

A formalization may emerge from homotopy type theory.

Here we propose an axiomatization using the notion of generalized model category (to be defined next).

Generalised model categories

Let $\mathcal{E}$ be a category with terminal object $\top$ and initial object $\bot$.

Definition

A *generalized model structure* on $\mathcal{E}$ is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes maps in $\mathcal{E}$ such that

- every map in $\mathcal{F}$ is quadrable and every map in $\mathcal{C}$ is co-quadrable.
- $\mathcal{W}$ satisfies 3-for-2;
- the pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorization systems;

A *generalized model category* is a category $\mathcal{E}$ equipped with a generalised model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

A map in $\mathcal{C}$ is called a *cofibration*, a map in $\mathcal{W}$ is *acyclic* and a map in $\mathcal{F}$ a *fibration*. An object A is *cofibrant* if the map $\perp \to A$ is a cofibration, an object X is *fibrant* if the map $X \to \top$ is a fibration.

A generalized model structure is *right proper* (resp. *left proper*) if the base (resp. cobase) change of a weak equivalence along a fibration (resp. a cofibation) is a weak equivalence. A generalized model structure is *proper* if it is both left and right proper.

A generalized model structure is **smooth** if every object is cofibrant, if it is right proper, if the base change of a cofibration along a fibration is a cofibration and if the product of a fibration along a fibration exists.

Remark: A smooth generalized model structure is proper and the product of a fibration along a fibration is a fibration.

EH-topos?

Definition (β-version):
An **EH-topos** is a smooth generalised model category $\mathcal{E}$ equipped a univalent ML-universe $U' \to U$.

Examples:

- The category of simplicial sets **sSet** (Voevodsky);
- The category of simplicial presheaves over any elegant Reedy category (Shulman).
- The category of symmetric cubical sets (Coquand).
- The category of presheaves over any elegant (local) test category (Cisinski).

Critics

Critic 1: We may want a hierarchy of universes $U_0 : U_1 : U_2 : \cdots$.

Critic 2: We may want a fibrant-cofibrant natural number object $\mathbb{N}$.

Critic 3: Every fibration should factor as a homotopy surjection followed by a monic fibration.

Critic 4: The initial object should be strict.

Critic 5: The inclusions $i_1 : X \to X \sqcup Y$ and $i_2 : Y \to X \sqcup Y$ should be fibration for every pair of objects (X, Y).

Critic 5': The functor $(i_1^\star, i_2^\star) : \mathcal{E}/(X \sqcup Y) \to \mathcal{E}/X \times \mathcal{E}/Y$ should be an equivalence of generalized model categories.

More critics

Critic 6: If $u : A \to B$ is a cofibration between fibrant objects and $p : E \to B$ is a fibration, then the map

$$u^\star : \Pi_B(E) \to \Pi_A(u^\star(E))$$

induced by u should be a fibration. Moreover, $u^\star$ should be acyclic when u is acyclic.

Critic 6': Condition 6 should be true in every slice category $\mathcal{E}/C$.

Epilogue

What is mathematics?

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"In mathematics you don't understand things. You just get used to them"

THANK YOU FOR YOUR ATTENTION!