Constructing models of type theory

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Homotopy Type Theory and Univalent Foundations
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Plan

1. Type theory in a category
2. Constructing new models from old
3. Product and identity types
4. Examples - gluing, realizability, polynomials...

Categorical models of type theory

A display map category is a category $\mathcal{C}$ with a class of morphisms $\mathcal{F}$

$$\begin{array}{ccc} \mathcal{F} & \hookrightarrow & \mathcal{C}^2 \\ & \searrow & \downarrow c \\ & & \mathcal{C} \end{array}$$

such that

- pullbacks of display maps exist and are in $\mathcal{F}$
- $\mathcal{F}$ contains all isomorphisms
- $\mathcal{F}$ is closed under composition
- $\mathcal{C}$ has a terminal object 1 and $\mathcal{F}$ contains all morphisms to 1.

dependent types	$\Longleftrightarrow$	display maps
$a \in A \vdash B(a)\ Type$		$A \twoheadleftarrow B$

Categorical models of type theory

The 2-category *Disp* has:

as objects display map categories,

as morphisms $(\mathcal{C}, \mathcal{F}) \to (\mathcal{D}, \mathcal{E})$ the functors $G : \mathcal{C} \to \mathcal{D}$ such that

- G preserves the terminal object
- G preserves display maps
- G preserves pullbacks of display maps,

and as 2-cells natural transformations.

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Grothendieck construction

For a pseudofunctor $P : \mathcal{C}^{op} \to Cat$, the Grothendieck construction gives a corresponding fibration

$$\begin{array}{c} \int P \\ \downarrow \psi \\ \mathcal{C}. \end{array}$$

$\int P$ has objects: pairs $(B \in \mathcal{C}, D \in P(B))$,
morphisms $(B, D) \to (B', D')$: pairs $(B \xrightarrow{g} B'$ in $\mathcal{C}, D \xrightarrow{\alpha} P(g)D'$ in $P(B))$.

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If $(\mathcal{C}, \mathcal{F})$ is a display map category and P is a pseudofunctor $\mathcal{C}^{op} \to \mathit{Disp}$, then $\int P$ has the structure of a display map category, and ψ is a morphism in Disp.

Display maps in $\int P$: morphisms (g, α) where g is a display map in $\mathcal{C}$ and α is a display map in $P(B)$.

Examples: gluing

If $(\mathcal{C}, \mathcal{F})$ is a display map category and $B \in \mathcal{C}$,
$\mathcal{F}/B$ is the full subcategory of the slice $\mathcal{C}/B$ with objects display maps.

$\mathcal{F}/B$ has a class of display maps:

$$\begin{array}{ccc} A & \xrightarrow{\in \mathcal{F}} & D \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

If $B \overset{f}{\twoheadrightarrow} C$ is a display map in $\mathcal{F}$, the pullback functor

$$f^* : \mathcal{F}/C \to \mathcal{F}/B$$

is a morphism in *Disp*.

Examples: gluing

If $(\mathcal{C}, \mathcal{F})$ and $(\mathcal{D}, \mathcal{E})$ are display map categories, and G a functor $\mathcal{C} \to \mathcal{D}$, there is a pseudofunctor $\mathcal{C}^{op} \to \mathit{Disp}$:

$$\begin{array}{ccc} B & \mapsto & \mathcal{F}/GB \\ B \xrightarrow{f} B' & \mapsto & (Gf)^* \end{array}$$

The corresponding fibration is the gluing $(\mathcal{E} \downarrow G)$ along G.
Morphisms:

$$\begin{array}{ccc} D & \twoheadrightarrow & GB \\ \downarrow & & \downarrow {\scriptstyle Gf} \\ D' & \twoheadrightarrow & GB' \end{array}$$

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Proposition (Shulman 2013)

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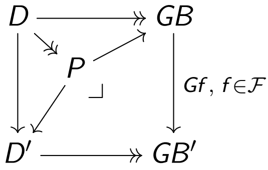
Display maps:



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Examples: product projections

Any category with finite products has a class of display maps consisting of the binary product projections:

$$A \times B \to A$$

Any finite-product preserving functor is a morphism of display map categories.

Thus any pseudofunctor $P : \mathcal{C}^{op} \to \mathit{FinProdCat}$ factors through *Disp*. This corresponds to:

If $(\mathcal{C}, \mathcal{F})$ is a display map category and $\mathcal{D} \xrightarrow{\psi} \mathcal{C}$ is a fibration such that $\mathcal{D}$ has and ψ preserves finite products, then $\mathcal{D}$ inherits the structure of a display map category.

Product types

A display map category $(\mathcal{C}, \mathcal{F})$ has product types if for any display maps

$$E \xrightarrow{g} \!\!\!\!\twoheadrightarrow A \xrightarrow{f} \!\!\!\!\twoheadrightarrow B$$

the dependent product

$$\textstyle\prod_f(g) \longrightarrow B$$

exists and is a display map.

($\iff$ for every display map f, $f^* : \mathcal{F}/B \to \mathcal{F}/A$ has a right adjoint $\prod_f$ satisfying the Beck-Chevalley condition,

$\iff$ for every display map f, f^* has a right adjoint and the inclusion $\mathcal{F}/B \hookrightarrow \mathcal{C}/B$ preserves exponentials.)

Product types

$\Pi Disp$ is the 2-category of display map categories with product types and morphisms G which preserve dependent products,

$$G(\prod_f g) \cong \prod_{Gf} Gg.$$

If $(\mathcal{C}, \mathcal{F})$ is a display map category with product types and P is a pseudofunctor $\mathcal{C}^{op} \to \Pi Disp$ such that for every $f \in \mathcal{F}$,

- $P(f)$ has a right adjoint Π_f,
- Π_f preserves display maps,
- the Beck-Chevalley condition for the Π-functors holds,

then $\int P$ has the structure of a display map category with product types, and ψ is a morphism in $\Pi Disp$.

Identity types

A morphism is anodyne if it has the left lifting property with respect to all display maps.

A display map category $(\mathcal{C}, \mathcal{F})$ has identity types if

- Every morphism in $\mathcal{C}$ factors as an anodyne map followed by a display map
- Anodyne maps are stable under pullback along display maps.

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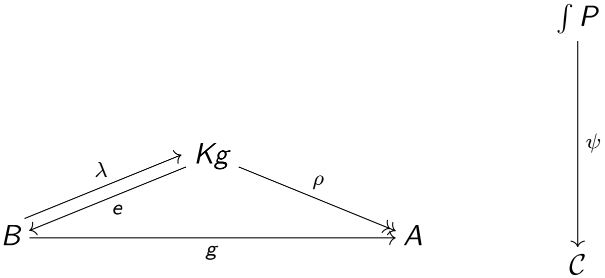
If the fibration $\psi : \int P \to \mathcal{C}$ is also an opfibration, then factorizations exist in $\int P$ (Stanculescu 2012, Harpaz & Prazma 2015).
This doesn't hold in general in the previous examples.

Identity types

To factorize $(g, \alpha) : (B, D) \to (A, C)$:



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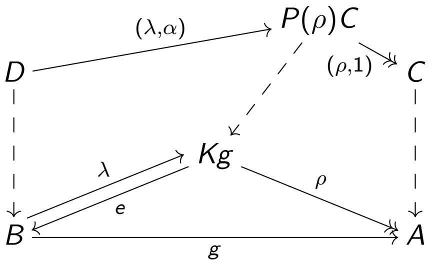

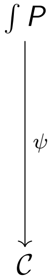

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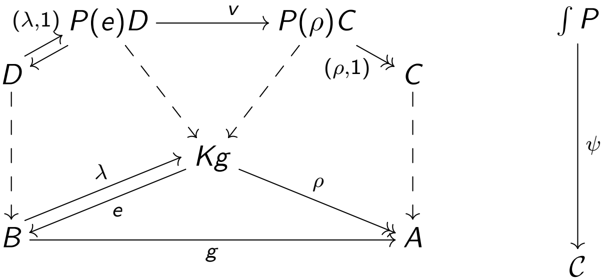

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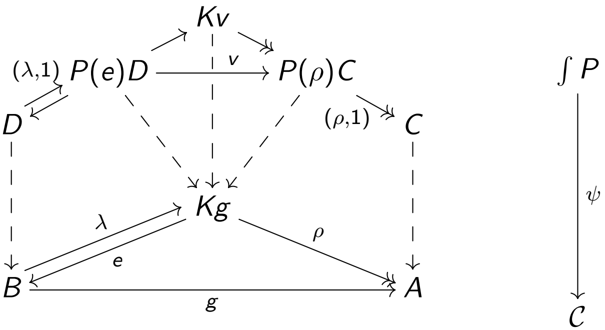

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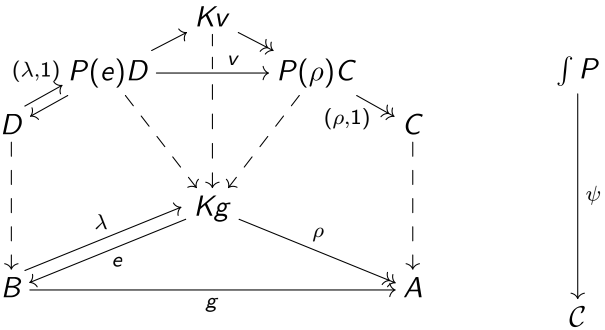

If $(\lambda, 1)$ is anodyne in $\int P$. i.e.:
Condition $(*)$: If λ is anodyne in $\mathcal{C}$, $f \in \mathcal{F}$ and $P(\lambda)f$ has a section s, then f has a section t such that $P(\lambda)t = s$.

If $(\mathcal{C}, \mathcal{F})$ is a display map category with identity types and $P : \mathcal{C}^{op} \to hFib$ satisfies $(*)$, then $\int P$ has identity types.

Function extensionality

A display map category with product and identity types satisfies function extensionality if for every display map f, the product functor $\prod_f$ preserves anodyne maps.

If function extensionality holds in $(\mathcal{C}, \mathcal{F})$ and in $P(B)$ for each $B \in C$, product and identity types are constructed as above, and the right adjoint functors Π_f preserve anodyne maps, then function extensionality holds in $\int P$.

Function extensionality

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Gluing example (Shulman 2013):
If $(\mathcal{C}, \mathcal{F})$ and $(\mathcal{D}, \mathcal{E})$ are display map categories with function extensionality, and $G : \mathcal{C} \to \mathcal{D}$ preserves display maps and anodyne maps, then $(\mathcal{E} \downarrow G)$ satisfies function extensionality.

Example: finite product projections

If the display maps in the fibres are product projections, fibrewise dependent products correspond to fibrewise exponentials.

If $(\mathcal{C}, \mathcal{F})$ is a display map category with product types and $\mathcal{D} \xrightarrow{\psi} \mathcal{C}$ is a fibration such that

- $\mathcal{D}$ has and ψ preserves finite products
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- each reindexing functor has a right adjoint satisfying BCC,

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Any morphism $f : A \to B$ has a factorization $A \xrightarrow{(1,f)} A \times B \twoheadrightarrow B$ into an anodyne followed by a display map.

If $(\mathcal{C}, \mathcal{F})$ is a display map category with identity types and $\mathcal{D} \xrightarrow{\psi} \mathcal{C}$ is a fibration satisfying condition $(*)$, then $\mathcal{D}$ has the structure of identity types.

Modified realizability sets

mod_0 is the category of non-empty modest sets
objects: $\{X = (|X| \xrightarrow{\alpha} \mathcal{P}_+\mathbb{N}),\ \alpha(x) \cap \alpha(y) = \emptyset \text{ for } x \neq y,\ |X| \neq \emptyset\}$
morphisms $X \to Y$: functions $|X| \to |Y|$ trackable by some $e \in \mathbb{N}$

mod_0 has a class of display maps with product and identity types, consisting of surjective trackable functions.

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mr_0 is the category of modified realizability sets over mod_0
objects: $\{P \subseteq |X|, X \in mod_0\}$
actual $\subseteq$ potential elements
morphisms $(P, X) \to (Q, Y)$:

$$\begin{array}{ccc} P & \rightarrowtail & |X| \\ \downarrow & & \downarrow \\ Q & \rightarrowtail & |Y| \end{array}$$

morphisms $X \to Y$ preserving actual elements

Modified realizability sets

The projection $mr_0 \to mod_0$ is a fibration which preserves finite products, exponentials and has compatible right adjoints to reindexing.
It follows that:

Proposition (Streicher 1993)

The category mr_0 has the structure of a display map category with product types.

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Proposition (Streicher 1993)

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The condition $(*)$ for identity types doesn't hold, but mr_0 can be given the structure of identity types:

$$\begin{aligned} Id(P, X) &= P \hookrightarrow (X + X \times X) \\ Id_{(P,X)}(x, y) &= 0 \hookrightarrow 1 & \text{if } x \neq y \\ &= 1 \hookrightarrow 1 + 1 & \text{if } x = y \in P \end{aligned}$$

The category mr_0 has identity types, for which function extensionality does not hold.

Polynomial models

If $(\mathcal{C}, \mathcal{F})$ is a display map category, $\mathcal{F} \to \mathcal{C}$ is a fibration. Reversing the vertical arrows gives the opposite fibration $\mathit{Poly}(\mathcal{F}) \to \mathcal{C}$.
$\mathit{Poly}(\mathcal{F})$ is the category of polynomials or containers.

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Objects:

$$B \twoheadrightarrow A \qquad\qquad \sum_{a \in A} X^{B(a)}$$

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Morphisms:

$$\begin{array}{ccccc} D_f & \xrightarrow{F} & B & \twoheadrightarrow & A \\ \downarrow & & & & \downarrow f \\ D & & \twoheadrightarrow & & C \end{array} \qquad\qquad \begin{array}{c} \sum_{a \in A} X^{B(a)} \\ \downarrow \\ \sum_{c \in C} X^{D(c)} \end{array}$$

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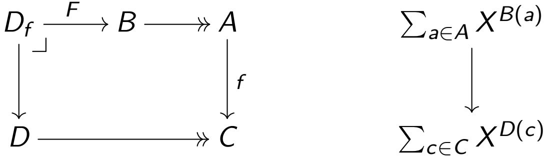

When $\mathcal{C}$ is extensive, $Poly(\mathcal{F}) \to \mathcal{C}$ has fibred finite products, and satisfies condition $(*)$. It follows that:

The category of polynomials $Poly(\mathcal{F})$ has the structure of a display map category with identity types.

Polynomial models

Display maps in $Poly(\mathcal{F})$:

$$\begin{array}{ccccc} D_f & \overset{\iota}{\hookrightarrow} & D_f + B & \twoheadrightarrow & A \\ \downarrow & & & & \downarrow\!\!\downarrow {\scriptstyle f \in \mathcal{F}} \\ D & & \twoheadrightarrow & & C \end{array}$$

Identity type $Id_{B \twoheadrightarrow A}$:

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Product types:

Proposition (Altenkirch, Levy, Staton 2010)

The category of polynomials $Poly(\mathcal{F})$ is cartesian closed, but not locally cartesian closed. $Poly(\mathcal{F}) \to \mathcal{C}$ does not preserve exponentials.

However, $Poly(\mathcal{F})$ has dependent products not preserved by the fibration.

The category of polynomials $Poly(\mathcal{F})$ is a display map category with product and identity types, for which function extensionality does not hold.

Universes

A universe in a display map category $(\mathcal{C}, \mathcal{F})$ is a display map

$$\tilde{\mathcal{U}} \xrightarrow{u} \mathcal{U}$$

such that if $\mathcal{S}$ is the class of all pullbacks of u, then

- $\mathcal{S}$ contains all isomorphisms
- $\mathcal{S}$ is closed under composition
- if $E \xrightarrow{g} A \xrightarrow{f} B$ are in $\mathcal{S}$ then so is $\prod_f(g) \twoheadrightarrow B$
- if $A \twoheadrightarrow C$ and $B \twoheadrightarrow C$ are in $\mathcal{S}$
  and f is any morphism $A \to B$ over C,
  then f factors as an anodyne map followed by a morphism in $\mathcal{S}$.

Universes

Given a universe $\tilde{\mathcal{U}} \xrightarrow{u} \mathcal{U}$ in $(\mathcal{C}, \mathcal{F})$,

$$(\tilde{\mathcal{U}}, 1) \xrightarrow{(u,1)} (\mathcal{U}, 1)$$

is a universe in $\int P$.

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Given a universe $\tilde{\mathcal{U}} \xrightarrow{u} \mathcal{U}$ in $(\mathcal{C}, \mathcal{F})$, and $\mathcal{V} \in \mathcal{C}, \tilde{\mathcal{V}} \in P(\mathcal{V})$
such that reindexings of $\tilde{\mathcal{V}}$ are closed under finite products and Π-functors,

$$(\textstyle\sum_{A:\mathcal{U}} \sum_{f:A\to\mathcal{V}} A,\ P(ev)(\tilde{\mathcal{V}})) \longrightarrow (\sum_{A:\mathcal{U}} (A \to \mathcal{V}), 1)$$

is a universe in $\int P$.
e.g. polynomials, modified realizability sets.

Universes

- More general universes?
- Univalence
- Other type constructors, e.g. W-types
- New models $\Rightarrow$
  - consistency and independence results
  - useful features of specific categories
  - theory of models...