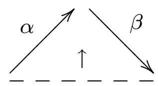
16/6/1983

Dear Professor Grothendieck,

Professor Brown has asked me to reply independently to your letter, etc. as he feels my comments are of a different nature to his. I feel that to understand my reactions ot your notes, it will help if I explain some of my own work in this area and hence I am sending you some of my papers on subjects which may be of relevance to your general ideas. I am also sending you a copy of a note by Thomason discussing the model category structure of Cat.

As I understand it, your chief motivation for studying modelisers is to understand the structure of some as yet hypothetical non-strict ∞-groupoids and thus to define the corresponding "stacks", all this with the eventual aim of developing a homotopy theory of toposes. I believe that, in fact, the ultimate in non-strict or lax ∞-groupoid structures is already essentially well known (even well loved) although not by that name. The objects to which I am referring are Kan complexes (in either simplicial or cubical languages). Here composition is not even strictly defined - given $\alpha, \beta \in X_1$, X a Kan complex, one forms the composite by filling the horn



in any way whatsoever. Two such fillers are homotopic, associativity is only defined up to homotopy, and so on,, In higher dimensions it is possible to define composites of n-simplices if one considers an n-simplex σ to have boundary mad up of two hemispheres $\partial_{even}\sigma$ and $\partial_{odd}\sigma$ with σ giving an n-cell linking them. Here one has to show that the even ordered cells do compose, likewise the odd-ordered ones, and whilst I have no proof of this, I am convinced it is so. Of course in claiming that lax ∞- groupoids are exactly equivalent to kan complexes I am only appealing to intuition, nothing more, but I feel that the idea *is* worth pursuing.

Seen in this light results on n-Picard categories, and n-truncated chain complexes are "revealed" as being due to the equivalence of both structure with (Kan) simplicial abelian groups truncated at level n.

When it comes to stacks, it is worth remembering that so far the homotopy theory of toposes à là Artin-Mazur depends on the hypercovering construction and that hypercoverings are exctly Kan simplicial sheaves in the topos, which are augmented towards the final object, e, and which are aspherical as augmented simplicial sheaves. Thus it is just possible that one already has stacks in their "ultimate lax" form and that they are just hypercoverings. I will return to this later in discussing the homotopy theory of toposes, but will finish here by noting that perhaps simplicial sheaves have to be replaced by "simplicial coherent sheaves" with "coherent" here meaning "glueing defined up to specified homotopies etc." (and not anything of an algebraic nature).

If my idea is correct and Kan complexes are lax ∞-groupoids. your notes take on a slight;y different meaning. If one wants algebraic models for homotopy types, Kan is already there, but now we want to know *all* other algebraic models so as to choose the invariants of homotopy type which best suit any particular geometric problem. As an example of a possible rigid version of Kan complexes, I will describe for you an amusing construction giving a modified singular complex for a space, X.

The singular complex, $Sing(X)$, is incredibly big as you remark in one of your letters. Each time one has a singular simplex

$$\sigma : \Delta^n \to X$$

, one has as well all the reparametrisations of σ. My idea is to identify all these together as follows:

Let $Mon(\mathbf{\Delta})$ be the simplicial monoid of automorphisms of the models of simplices - explicitly

$$Mon(\mathbf{\Delta})_0 = \{id\Delta^0 \to \Delta^0\}$$

and

$$Mon(\mathbf{\Delta}) = \{f : \Delta^n \to \Delta \mid fd_i = d_i\tilde{f}_i, \text{ for } 0 \leq i \leq n, \text{ where } \tilde{f}_i \in Mon(\mathbf{\Delta})_{n-1}\}$$

Thus $f \in Mon(\mathbf{\Delta})$ if it reparametrises Δ^n in such a way that vertices remain fixed and each face isn itself reparametrised by the restriction of f. Now $Mon(\mathbf{\Delta})$ operates by composition on $Sing(X)$ (and also on the simplicial hom-set $\underline{Hom}(X,Y)$, where $\underline{Hom}(X,Y)_n = Top(X \times \Delta^n, Y)$) and we can define a new "singular complex" by

$$TSing(X) = Sing(X)/\text{Action of } Mon(\mathbf{\Delta})$$

(Likewise one gets

$$T\underline{Hom}(X,Y) = \underline{Hom}(X,Y)/\text{Action of } Mon(\mathbf{\Delta}).)$$

$Sing(X)$ is a Kan complex, because of the existence of deformation retractions of Δ^n onto each of its horns. One can easily check that two such retractions differ by an element of $Mon(\mathbf{\Delta})$, hence the resulting fillers of $Sing(X)$ determine the same element of $TSing(X)$. We call these fillers *thin* (as in the Brown-Higgins T-complex theory.) Thus in $TSing(X)$ and $T\underline{Hom}(X,Y)$ every horn has a unique thin filler, however one does not have a T-complex structure as one can easily check. This "thin" structure makes $TSing(X)$ into an ∞-category with lax involution, lax inverses, but strict associativity in the composition. I say 'makes' but I should add that this is easy to check in low dimensions, but the machinery to check it in higher dimensions is lacking - so it should really be "seems to make". If it works one has another nice model for an ∞-groupoid structure, more rigid than Kan complexes, but still arising naturally from the topological context.

Cordially yours

Tim Porter