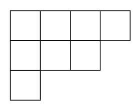
A PROOF OF THE HCF

1. Introduction

For our purposes, a partition, written, $\lambda = (n_0, \ldots, n_d)$ is simply a finite list of weakly decreasing integers. (Indexing the n_i starting on 0 will make a number of later computations neater.) A Young diagram of shape λ is an array of boxes with n_i boxes in the i^{th} row (again indexing beginning with 0). A Young diagram of shape λ' or λ "conjugate" is an array of boxes with n_i boxes in the i^{th} column. For instance, a Young diagram of shape $(4, 3, 1)$ (or of shape $(3, 2, 2, 1)'$) is drawn as follows:



A semi-standard Young tableau or SSYT of parameters (N, λ) (where N is a nonnegative integer) is a way of filling the boxes of a Young diagram of shape λ with positive integers no greater than N so that the entries are weakly increasing across rows, and strictly increasing down columns. For λ as above, the following is an SSYT of parameters $(3,\lambda)$:

1	1	2	3
2	2	3	
3			

The hook h_{ij} of box x_{ij} is the rectilinear path from x_{ij} up to the top row of λ and then across to the rightmost box in the top row. By the hook length of x_{ij}, expressed, $|h_{ij}|$, we refer to the number of boxes in h_{ij}. The content of box x_{ij}, denoted by $|x_{ij}|$, is the number, $N + i - j$. For instance, for λ as above, $|h_{01}| = 5$ and, if $N = 3$, then $|x_{01}| = 3 + 0 - 1 = 2$.

In the following, we prove the Hook Content Formula (HCF), namely that: The number of SSYT of parameters (N, λ) is given by:

$$SSYT(N, \lambda) = \prod_{x_{ij} \in \lambda} \frac{|x_{ij}|}{|h_{ij}|}$$

[2]

2. The Set-Up

Suppose $d \geq 0$, and we are given $N \geq 0$, and a list of $(d+1)$ nonnegative integers, $n_0, \ldots, n_d$. Let $v_1, \ldots, v_N$ be a list of N, $(d+1)$–dimensional vectors, each of which is composed of 1's and 0's and such that:

$$\sum_{k=1}^{N} v_k = \begin{bmatrix} n_0 \\ \vdots \\ n_d \end{bmatrix}.$$

Define $C(N, n_0, \ldots, n_d)$ to be the number of choices of such a list such that for each integer $L \in [1, N]$, the vector $\sum_{k=1}^{L} v_k$ is weakly decreasing from top to bottom.

If there exists $i \in \{1, \ldots, d\}$ such that $n_{i-1} < n_i$, then $C(N, n_0, \ldots, n_d) = 0$. If not, it follows that $n_0 \geq \cdots \geq n_d$. If, in addition, $n_d \neq 0$, then one can sheck that:

$$C(N, n_0, \ldots, n_d) = SSYT(N, (n_0, \ldots, n_d)').$$

In this case (i.e., $n_0 \geq \cdots \geq n_d$ and $n_d \neq 0$) each $n_i \geq 1$, so, if $N \geq 1$ as well, we may write:

$$C(N, n_0, \ldots, n_d) = \sum_{(j_0, \ldots, j_d): j_i \in \{0,1\}} C(N-1, n_0 - j_0, \ldots, n_d - j_d),$$

where one should think of each possible vector of the form $\begin{bmatrix} j_0 \\ \vdots \\ j_d \end{bmatrix}$ as a possible value for the final vector v_N, in the list $v_1, \ldots, v_N$.

3. Counting $C(N, n_0, \ldots, n_d)$

Let $d \geq 0$ and let $\vec{N} = (N, n_0, \ldots, n_d)$ be a $(d+2)$–tuple of nonnegative integers. If, for all $i \in \{1, \ldots, d\}$, we have $n_{i-1} + 1 \geq n_i$, define

$$\dagger(\vec{N}) = 1.$$

Otherwise, define

$$\dagger(\vec{N}) = 0.$$

Moreover, let:

$$F(N, n_0, \ldots, n_d) = \prod_{i=0}^{d} \frac{(N+i)!}{(N+i-n_i)!} \times \frac{V(m_0, \ldots, m_d)}{m_0! \cdots m_d!},$$

where $m_i = n_i + d - i$ for each i. We use the conventions that $0! = 1$, and for $n > 0$,

$$\frac{1}{(-n)!} = \lim_{k \to \infty} \prod_{m=n}^{k} \left(\frac{1}{-m} \right) = 0.$$

Theorem 3.1. *Let $d \geq 0$ and let $\vec{N} = (N, n_0, \ldots, n_d)$ be a $(d+2)$–tuple of non-negative integers. We have:*

$$\dagger(\vec{N}) = 0 \implies C(\vec{N}) = 0$$
$$\dagger(\vec{N}) = 1 \implies C(\vec{N}) = F(\vec{N}).$$

Proof. We prove the theorem by induction on N. Let $N = 0$. First, suppose that $\dagger(\vec{N}) = 0$. It follows that some $n_i \neq 0$, whence, $C(\vec{N}) = 0$. On the other hand, suppose $\dagger(\vec{N}) = 1$. If each $n_i = 0$, then $C(\vec{N}) = 1 = F(\vec{N})$. Otherwise some $n_i > 0$, and $C(\vec{N}) = 0$. Moreover, in this case, either $n_0 > 0$ so that the left hand factor of $F(\vec{N})$ vanishes, or, for some $i \in \{1, \ldots, d\}$, we have $n_{i-1} < n_i$ (i.e., $n_{i-1} + 1 = n_i$), so that the right hand factor of $F(\vec{N})$ vanishes. Regardless, $F(\vec{N}) = 0$. Thus, Theorem 3.1 holds for $N = 0$.

Now suppose $N \geq 1$ and that 3.1 holds for $N - 1$. If $\dagger(\vec{N}) = 0$ it is clear that $C(\vec{N}) = 0$ as claimed, as $\begin{bmatrix} n_0 \\ \vdots \\ n_d \end{bmatrix}$ itself is not weakly decreasing from top to bottom.

Therefore, assume $\dagger(\vec{N}) = 1$. First, suppose that for some $i \in \{1, \ldots, d\}$, we have $n_{i-1} < n_i$ (i.e., $n_{i-1} + 1 = n_i$). Clearly $C(\vec{N}) = 0$, and $F(\vec{N}) = 0$ as well, because the right hand factor of $F(\vec{N})$ vanishes. Hence, we may suppose that for each $i \in \{1, \ldots, d\}$, we have $n_{i-1} \geq n_i$. Since both $C(\vec{N})$ and $F(\vec{N})$ are invariant under the addition or removal of terminal 0's from $(n_0, \ldots, n_d)$, we may assume WLOG that $n_d \neq 0$. This implies that each $n_i \geq 1$, which, in conjunction with the fact that $N \geq 1$, allows us to write:

$$\begin{aligned} C(N, n_0, \ldots, n_d) &= \sum_{(j_0, \ldots, j_d): j_i \in \{0,1\}} C(N - 1, n_0 - j_0, \ldots, n_d - j_d) \\ &= \sum_{(j_0, \ldots, j_d): j_i \in \{0,1\}} F(N - 1, n_0 - j_0, \ldots, n_d - j_d), \end{aligned}$$

where the last equality follows by the inductive hypothesis and the fact that, for each $(j_0, \ldots, j_d)$, we have $\dagger(N-1, n_0-j_0, \ldots, n_d-j_d) = 1$. Hence if we can establish the identity:

$$F(N, n_0, \ldots, n_d) = \sum_{(j_0, \ldots, j_d): j_i \in \{0,1\}} F(N - 1, n_0 - j_0, \ldots, n_d - j_d). \tag{3.1}$$

we are done. □

4. A Useful Algebraic Result

In the following lemma, we make use of the Vandermonde polynomial, V, which is defined as:

$$V(x_0, \ldots, x_d) = \prod_{0 \leq i < j \leq d} (x_i - x_j).$$

In order to establish (3.1) we will need the following lemma:

Lemma 4.1. *Write $\vec{X} = (X, x_0, \ldots, x_n)$ and define:*

$$G(\vec{X}, t) = \sum_{(j_0, \ldots, j_n): j_i \in \{0,1\}} \left(\left[\prod_{i=0}^{n} (X - x_i)^{1-j_i} x_i^{j_i} \right] V(x_0 - j_0 t, \ldots, x_n - j_n t) \right).$$

We have:

$$G(\vec{X}, t) = \left[\prod_{r=0}^{n} (X - rt) \right] V(x_0, \ldots, x_n), \tag{4.1}$$

Proof. Before we start, we let $\vec{J} = (j_0, \ldots, j_n)$ and write:

$$\psi(\vec{X}, t, \vec{J}) = \left[\prod_{i=0}^{n} (X - x_i)^{1-j_i} x_i^{j_i}\right] V(x_0 - j_0 t, \ldots, x_n - j_n t),$$

so that we have:

$$G(\vec{X}, t) = \sum_{\vec{J} \in \{0,1\}^{n+1}} \psi(\vec{X}, t, \vec{J}).$$

4.1. First, we show that $G(\vec{X}, t)$ is antisymmetric with respect to transposition of any two of the variables $(x_0, \ldots, x_n)$. Indeed, fix k and l such that $0 \leq k < l \leq n$ and let $\vec{J_{kl}} = (j_0, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{l-1}, j_{l+1}, \ldots, j_n) \in \{0,1\}^{n-1}$ denote the values of j_i in $\vec{J}$ for $i \neq k, l$. Then,

$$G(\vec{X}, t) = \sum_{\vec{J_{kl}} \in \{0,1\}^{n-1}} \left(\sum_{(j_k, j_l) \in \{0,1\} \times \{0,1\}} \psi(\vec{X}, t, \vec{J_{kl}}, j_k, j_l) \right).$$

$G(\vec{X}, t)$ is antisymmetric because the expression inside the large parenthesis above is always antisymmetric. To see the latter, let $\vec{X}'$ be the vector obtained from $\vec{X}$ by switching x_k and x_l. Then, for any fixed value of $\vec{J_{kl}} \in \{0,1\}^{n-1}$,

$$\begin{aligned}
\psi(\vec{X}, t, \vec{J_{kl}}, 0, 0) &= -\psi(\vec{X}', t, \vec{J_{kl}}, 0, 0) \\
\psi(\vec{X}, t, \vec{J_{kl}}, 0, 1) &= -\psi(\vec{X}', t, \vec{J_{kl}}, 1, 0) \\
\psi(\vec{X}, t, \vec{J_{kl}}, 1, 0) &= -\psi(\vec{X}', t, \vec{J_{kl}}, 0, 1) \\
\psi(\vec{X}, t, \vec{J_{kl}}, 1, 1) &= -\psi(\vec{X}', t, \vec{J_{kl}}, 1, 1).
\end{aligned}$$

4.2. Now we show that $G(\vec{X}, t)$ is homogenous of degree $\frac{n(n+1)}{2}$ with respect to the variables $(x_0, \ldots, x_n)$. Clearly any monomial in its monomial expansion must have degree at least $\frac{n(n+1)}{2}$ by antisymmetry. Suppose, therefore, that some monomial, m, in this expansion has degree larger than $\frac{n(n+1)}{2}$ in $(x_0, \ldots, x_n)$. It follows that m has degree greater than n (but no greater than $n+1$) in some variable x_i. We may assume, WLOG, this variable is x_0, that is, that m has degree $n+1$ in x_0.

Let $\vec{J_0} = (j_1, \ldots, j_n) \in \{0,1\}^n$ denote the values of j_i in $\vec{J}$ for $i \neq 0$, so that:

$$G(\vec{X}, t) = \sum_{\vec{J_0} \in \{0,1\}^n} \left(\sum_{j_0 \in \{0,1\}} \psi(\vec{X}, t, \vec{J_0}, j_0) \right). \tag{4.2}$$

$G(\vec{X}, t)$ has no monomials of degree $n+1$ in x_0 because the expression inside the parenthesis above never has any. Indeed, fix $\vec{J_0} = (j_1, \ldots, j_n)$. Then the entire degree $n+1$ (with respect to x_0) part of $\psi(\vec{X}, t, \vec{J_0}, 0)$ is given by:

$$-x_0 \left[\prod_{i=1}^{n} (X - x_i)^{1-j_i} x_i^{j_i}\right] x_0^n \left[\sum_{\sigma \in S_n} (-1)^{sgn(\sigma)} \left(\prod_{i=1}^{n} (x_i - j_i t)^{n-\sigma(i)}\right)\right],$$

whereas the entire degree $n+1$ (with respect to x_0) part of $\psi(\vec{X}, t, \vec{J_0}, 1)$ is given by:

$$x_0 \left[\prod_{i=1}^{n} (X - x_i)^{1-j_i} x_i^{j_i} \right] x_0^n \left[\sum_{\sigma \in S_n} (-1)^{sgn(\sigma)} \left(\prod_{i=1}^{n} (x_i - j_i t)^{n-\sigma(i)} \right) \right].$$

Hence the degree $n+1$ (with respect to x_0) of the expression inside the parenthesis in (4.2) is 0, so it has no monomials of degree $n+1$ in x_0. We have established that each monomial in the monomial expansion of $G(\vec{X}, t)$ has total degree $\frac{n(n+1)}{2}$ in the variables $(x_0, \ldots, x_n)$.

4.3. Since $G(\vec{X}, t)$ is antisymmetric, homogenous of degree $\frac{n(n+1)}{2}$, with respect to $(x_0, \ldots, x_n)$, it is divisible by $V(x_0, \ldots, x_n)$, and the quotient has degree 0 in $(x_0, \ldots, x_n)$. That is, we may write:

$$G(\vec{X}, t) = H(X, t) \times V(x_0, \ldots, x_n),$$

for a function H, that only depends on the two variables, X and t. In this section, we compute $H(X, t)$.

For $\vec{J} \in \{0,1\}^{n+1}$, $\sigma \in S_{n+1}$, write:

$$\begin{aligned} \Pi(\vec{J}) &= \prod_{i=0}^{n} (X - x_i)^{1-j_i} x_i^{j_i} \\ V_\sigma(\vec{J}) &= (-1)^{sgn(\sigma)} (x_0 - j_0 t_0)^{n-\sigma(0)} \cdots (x_n - j_n t_n)^{n-\sigma(n)}, \end{aligned}$$

so that we have:

$$G(\vec{X}, t) = \sum_{\vec{J} \in \{0,1\}^{n+1}} \left[\sum_{\sigma \in S_{n+1}} \Pi(\vec{J}) V_\sigma(\vec{J}) \right].$$

We make the following definitions:

(1) Let $p \in \mathbb{Z}[X, t, x_0, \ldots, x_n]$. Consider p as a polynomial in $x_0, \ldots, x_n$ with coefficients in $\mathbb{Z}[X, t]$. Define $\delta(p) \in \mathbb{Z}[X, t]$ to be the coefficient of $x_0^n \cdots x_n^0$ in p.
(2) Let $p_i \in \mathbb{Z}[X, t, x_i]$. Consider p_i as a polynomial in x_i with coefficients in $\mathbb{Z}[X, t]$. Define $\delta_i(p_i) \in \mathbb{Z}[X, t]$ to be the coefficient of x_i^{n-i} in p_i.
(3) Define $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. (Kronecker delta function.)

Using these definitions, we may write:

$$H(X, t) = \delta(G(\vec{X}, t)) = \sum_{\vec{J} \in \{0,1\}^{n+1}} \left[\sum_{\sigma \in S_{n+1}} \delta\Big(\Pi(\vec{J}) V_\sigma(\vec{J}) \Big) \right].$$

Since $\Pi(\vec{J})$ is linear in each x_i, it follows that $\delta\big(\Pi(\vec{J})V_\sigma(\vec{J})\big) = 0$ unless $\sigma(i) \in \{i, i+1\}$ for each i. The only such permutation is the identity, so,

$$\begin{aligned} H(X,t) &= \sum_{\vec{J}\in\{0,1\}^{n+1}} \delta\Bigg(\Pi(\vec{J})(x_0 - j_0 t_0)^n \cdots (x_n - j_n t_n)^0\Bigg) \\ &= \sum_{\vec{J}\in\{0,1\}^{n+1}} \delta\Bigg(\prod_{i=0}^{n}\Big[(X - x_i)^{1-j_i} x_i^{j_i}(x_i - j_i t)^{n-i}\Big]\Bigg) \\ &= \sum_{\vec{J}\in\{0,1\}^{n+1}} \Bigg[\prod_{i=0}^{n} \delta_i\Big((X - x_i)^{1-j_i} x_i^{j_i}(x_i - j_i t)^{n-i}\Big)\Bigg] \\ &= \sum_{\vec{J}\in\{0,1\}^{n+1}} \Bigg[\prod_{i=0}^{n} X^{(\delta_{0j_i})}\big((i-n)t\big)^{(\delta_{1j_i})}\Bigg]. \end{aligned}$$

From this expression, we see that each monomial in the monomial expansion of $H(X,t)$ has total degree $n+1$ in the variables X, t. Thus we may write:

$$H(X,t) = \sum_{m=0}^{n+1} a_m X^{(n+1-m)} t^m, \quad \text{where,} \quad a_m = \sum_{\vec{J}:\sum(j_i)=m} \Bigg[\prod_{j_i=1}(i-n)\Bigg] = \sum_{I\in\binom{\{0,\ldots,n\}}{m}} \Bigg[\prod_{i\in I}(-i)\Bigg],$$

and where $\binom{\{0,\ldots,n\}}{m}$ is the set of all m element subsets of $\{0,\ldots,n\}$. Moreover, if we write out the following expansion:

$$\prod_{r=0}^{n}(X - rt) = \sum_{m=0}^{n+1} b_m X^{(n+1-m)} t^m, \quad \text{we find:} \quad b_m = \sum_{I\in\binom{\{0,\ldots,n\}}{m}} \Bigg[\prod_{i\in I}(-i)\Bigg],$$

$$\Longrightarrow \quad H(X,t) = \prod_{r=0}^{n}(X - rt),$$

and the lemma is complete. □

5. Conclusion of Theorem 3.1

We apply (4.1) with $n = d$, $X = N + d$, $x_i = m_i$, and $t = 1$, noting that $([N+d] - m_i)$ may be replaced by $(N + i - n_i)$ by the definition of m_i. This gives:

$$\begin{aligned} &\frac{(N+d)!}{(N-1)!} V(m_0, \ldots, m_d) \\ = &\sum_{(j_0,\ldots,j_d)\in\{0,1\}^{d+1}} \Bigg(\Bigg[\prod_{i=0}^{d}(N + i - n_i)^{1-j_i} m_i^{j_i}\Bigg] V(m_0 - j_0, \ldots, m_d - j_d)\Bigg). \end{aligned}$$

Multiplying both sides by:

$$\Bigg[\prod_{i=0}^{d} \frac{(N+i-1)!}{(N+i-n_i)!}\Bigg] \frac{1}{m_0! \cdots m_d!},$$

we have:

$$\begin{aligned}
&\left[\prod_{i=0}^{d}\frac{(N+i)!}{(N+i-n_i)!}\right]\frac{V(m_0,\ldots,m_d)}{m_0!\cdots m_d!}\\
=&\sum_{(j_0,\ldots,j_d)\in\{0,1\}^{d+1}}\left(\left[\prod_{i=0}^{d}\frac{(N+i-1)!}{(N+i-n_i)!}(N+i-n_i)^{1-j_i}m_i^{j_i}\right]\frac{V(m_0-j_0,\ldots,m_d-j_d)}{m_0!\cdots m_d!}\right)\\
=&\sum_{(j_0,\ldots,j_d)\in\{0,1\}^{d+1}}\left(\left[\prod_{i=0}^{d}\frac{(N+i-1)!}{(N+i-n_i-1+j_i)!}\right]\frac{V(m_0-j_0,\ldots,m_d-j_d)}{(m_0-j_0)!\cdots(m_d-j_d)!}\right).
\end{aligned}$$

Equating the first and third lines above gives:

$$F(N,n_0,\ldots,n_d)=\sum_{(j_0,\ldots,j_d)\in\{0,1\}^{d+1}}F(N-1,n_0-j_1,\ldots,n_d-j_d).$$

This establishes Theorem 3.1.

6. The Hook Content Formula

Let $\mu=(n_0,\ldots,n_d)'$ be a Young diagram. Then, by definition, we must have $n_0\geq\cdots\geq n_d\geq 1$. As noted earlier, this implies that if N is any nonnegative integer and we let $\vec{N}=(N,n_0,\ldots,n_d)$, then $SSYT(N,\mu)=C(\vec{N})$. Moreover, since $\dagger(\vec{N})=1$, it follows by Theorem 3.1 that $C(\vec{N})=F(\vec{N})$, whence:

$$SSYT(N,\mu)=\prod_{i=0}^{d}\frac{(N+i)!}{(N+i-n_i)!}\times\frac{V(m_0,\ldots,m_d)}{m_0!\cdots m_d!}. \tag{6.1}$$

(Again, we use the notation $m_i=n_i+d-i$).

On the other hand, the Hook Content Formula states that:

$$SSYT(N,\mu)=\prod_{x_{ij}\in\mu}\frac{|x_{ij}|}{|h_{ij}|}=\left[\prod_{x_{ij}\in\mu}(N+i-j)\right]\left[\prod_{x_{ij}\in\mu}\frac{1}{|h_{ij}|}\right],$$

[2]. It follows from the fact that $\mu=(n_0,\ldots,n_d)'$ that:

$$\prod_{x_{ij}\in\mu}(N+i-j)=\prod_{i=0}^{d}\frac{(N+i)!}{(N+i-n_i)!},$$

so, the hook content formula will follow from (6.1) if we show that:

$$(*)\ \prod_{x_{ij}\in\mu}|h_{ij}|=\frac{m_0!\cdots m_d!}{V(m_0,\ldots,m_d)},\ \text{or equivalently,}\ \prod_{x_{ij}\in\lambda}|h_{ij}|=\frac{m_0!\cdots m_d!}{V(m_0,\ldots,m_d)},$$

[1] for $\lambda=(n_0,\ldots,n_d)=\mu'$, since the product of hook lengths is invariant under conjugation.

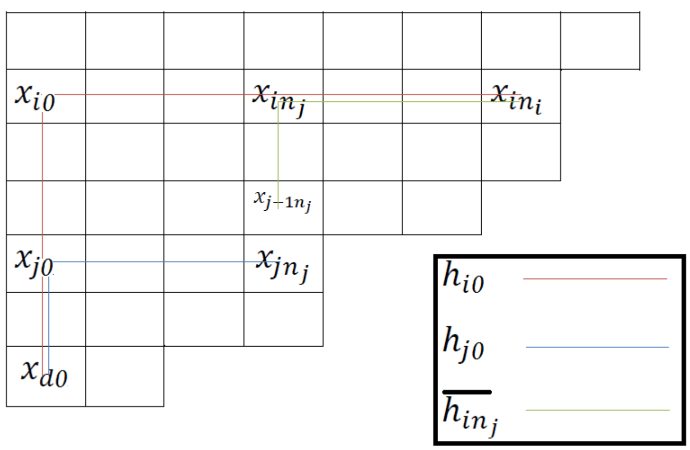
To demonstrate the latter equality, first note that it may be rewritten as:

$$\prod_{x_{ij}\in\lambda}|h_{ij}|=\prod_{i=0}^{d}\left[\frac{m_i!}{\prod_{j=i+1}^{d}(m_i-m_j)}\right].$$

To establish the equation above, thereby proving $(*)$, we show that, for each i, the product of the hook lengths of the squares in row i of λ (denote λ_i) is given by:

$$\prod_{x_{ij}\in\lambda_i} |h_{ij}| = \frac{m_i!}{\prod\limits_{j=i+1}^{d}(m_i - m_j)} \tag{6.2}$$

[2, Ch. 7, p. 374]. First, note that, for any j such that $i < j \leq d$, the value of $m_i - m_j$ does not coincide with the hook length of any of the squares in λ_i. To see this, let $\overline{h_{in_j}}$ be the hook obtained by removing from h_{in_j} all the squares below x_{j-1n_j}. The path from x_{d0} to x_{in_i} along h_{i0} includes m_i squares. It follows that the path from x_{d0} to x_{in_i} that begins along h_{j0} and concludes along $\overline{h_{in_j}}$ must also include m_i squares. Moreover, the path from x_{d0} to x_{jn_j} along h_{j0} includes m_j squares. From this it follows that the length of $\overline{h_{in_j}}$ is given by $m_i - m_j$. One easily observes that for $k \leq n_j$, $|h_{ik}| > |\overline{h_{in_j}}|$, and for $k > n_j$, $|h_{ik}| < |\overline{h_{in_j}}|$. Hence, no square in row i of λ has hook length equal to $|\overline{h_{in_j}}| = m_i - m_j$.



Let $H_i = \{|h_{ij}| : x_{ij} \in \lambda_i\}$, be the set of hook lengths in row i of λ (each hook in a row has a distinct length), let $K_i = \{(m_i - m_j) : i < j \leq d\}$, and let $M_i = \{1, \ldots, m_i\}$. Now $H_i \subseteq M_i$, $K_i \subseteq M_i$, and:

$$\#(H_i) + \#(K_i) = (n_i) + (d - i) = m_i = \#(M_i).$$

Further, by the argument above H_i and K_i are disjoint, so $H_i \dot{\cup} K_i = M_i$, whence:

$$\prod_{x_{ij}\in\lambda_i} |h_{ij}| = \prod_{h\in H_i}(h) = \frac{\prod_{m\in M_i}(m)}{\prod_{k\in K_i}(k)} = \frac{m_i!}{\prod\limits_{j=i+1}^{d}(m_i - m_j)},$$

and (6.2) has been proven. This establishes $(*)$, and The Hook Content Formula now follows.

References

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[2] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.