POWER OPERATIONS AND ABSOLUTE GEOMETRY

JACK MORAVA AND REKHA SANTHANAM

Abstract. This is a sketch and summary of **work in progress** on the relation of power operations in homotopy theory (ie over Spec S^0) to those in absolute geometry (ie over Spec $\mathbb{F}_1$). In its current form it is nothing but a **tissue of conjecture**! Comments and suggestions are welcome.

1. Some comonads

1.0 Group completion defines a map

$$\coprod_{n \geq 0} B\Sigma_n \to \mathbb{Z} \times QS \ ,$$

where $QS := B\Sigma_\infty^+$ is an E_∞ ring space. We'll write $S^0 \to QS_+$ for its basepoint, and

$$+, \times : QS_+ \wedge QS_+ \to QS_+$$

for its two compositions.

Some conventions: X_+ results from an **un**pointed space X by adding a disjoint basepoint, and

$$S^0[X_+] := S[X]$$

denotes its suspension spectrum (eg $S[\mathrm{pt}] = S^0$). Thus $S[QS]$ is an E_∞ ring-spectrum, augmented by maps

$$S^0 \to S^0[QS_+] := S[QS] \to S^0 \ .$$

We'll also write $DX := [S[X], S^0]$ for the Spanier-Whitehead functional dual spectrum.

1.1 Claim $D(QS)$ is a coring object, in the sense of Hess, in the category of commutative S^0-algebras. In particular, there are maps

$$D(QS) \to D(QS_+ \wedge QS_+) \to D(QS) \wedge D(QS)$$

defined by the product of the maps dual to the left and right units

$$QS_+ = S^0 \wedge QS_+, \ QS_+ \wedge S^0 \to QS_+ \wedge QS_+ \to QS_+ \wedge QS_+ \ .$$

Date: 23 March 2012.
This work was supported by the NSF.

1.2 Claim

$$A \mapsto \mathrm{Hom}_{S^0-\mathrm{alg}}(D(QS), A) := \mathsf{Q}(A)$$

defines a functor from the category of commutative S^0-algebras to itself.

Remark The analog of this assertion, after passing to homotopy groups, is **false**, because of the lack of a Künneth theorem for stable homotopy groups. When we have such a theorem, we can often work instead with Hopf rings [10, 11].

1.3 Claim If X is a space, then the unstable Hurewicz homomorphism (of ring spectra)

$$DX = [S[X], S^0] \to \mathrm{Hom}_{S^0-\mathrm{alg}}(D(QS), DX) = \mathsf{Q}(DX)$$

makes DX into a Q-coalgebra.

This provides DX with some kind of (co)descent data [in the sense of [5]], as a Q-comonad coalgebra ...

2. Consequences of the Segal conjecture

2.1 If G is a finite group, let $\langle G \rangle$ denote the set of its conjugacy classes of subgroups. If $H \subset G$ is one such, then

$$W_G(H) := N_G(H)/H = (\text{normalizer of } H \text{ in } G)/H$$

is its 'Weyl group'. [This might be abbreviated in context, eg to $W(H)$.] Recall that if G acts on X, then the subspace X^H (of points with isotropy group H) inherits an action of $W_G(H)$.

2.1 The **Segal conjecture** says that

$$\bigvee_{H \in \langle G \rangle} S[BW_G(H)] \to D(BG)$$

becomes a homotopy equivalence after a suitable completion [2 §5.1, 7].

Definition

$$\mathsf{B}_* = \bigvee_{H \in \langle \Sigma_* \rangle} S[BW_{\Sigma_*}(H)]$$

is a graded-commutative ringspectrum, with multiplication defined by the group homomorphisms

$$W_{\Sigma_m}(H) \times W_{\Sigma_n}(H') \to W_{\Sigma_{n+m}}(H \times H') \; .$$

2.2 Claim B_* is a coRIG - object, in the category of commutative ringspectra; with structure maps

$$\sum \mathrm{trans}^H_{H' \times H''} \, \delta_H^{H' \times H''} : S[BW_{m+n}(H)] \to \bigvee S[BW_m(H')] \wedge S[BW_n(H'')]$$

defined by stable transfers [6 Ch I §7]; and similarly for $\times$.

2.3 Claim

$$A \mapsto \mathsf{B}_*(A) := \mathrm{Hom}_{S^0-\mathrm{alg}}(\mathsf{B}_*, A)$$

is a graded-commutative RIG-object in the category of commutative S^0-algebras.

2.4 Claim If X is a space, there is a total power operation [4 §3.2, 8, 9, 12 §1.7]

$$\natural : DX \to \prod \mathsf{B}_*(DX)$$

sending $\alpha : X_+ \to S^0$ to the sequence

$$H \mapsto \natural^H(\alpha) : S[BW_{\Sigma_n}(H)] \to DX$$

of compositions

$$BW_{\Sigma_n}(H) \wedge X_+ \xrightarrow{\mathrm{tr}} BN_{\Sigma_n}(H) \wedge X_+ \longrightarrow (X_+^{\wedge n})^H//N_{\Sigma_n}(H) \longrightarrow X_+^{\wedge n}//\Sigma_n \xrightarrow{\alpha^n} S^0$$

(defined by geometrically realizing the maps

$$[X/N_{\Sigma_n}(H)] \to [(X^n)^H/N_{\Sigma_n}(H)] \to [X^n/\Sigma_n]$$

of transformation groupoids, and composing with the stable transfer associated to the quotient $N \to N/H = W$).

3. A group completion conjecture

3.0 If we regard B_* as the sections of a bundle $S[BW_*] \to \langle \Sigma_* \rangle$, then there is a natural identification

$$\mathsf{B}_* \otimes \mathbb{Q} = \mathrm{Fns}(\langle \Sigma_* \rangle, \mathbb{Q})$$

as coRIG algebras. Burnside's matrix of marks is upper-triangular (in the inclusion order on subgroups), with nonzero diagonal entries, so the ring homomorphism

$$A(G) \to \mathrm{Fns}(\langle G \rangle, \mathbb{Z}) : G/K \mapsto [H \mapsto \#[G/K]^H]$$

[3 Ch I §1.2] becomes an isomorphism after rationalizing.

3.1 $H \subset \Sigma_n \subset \Sigma_{n+1}$ defines a homomorphism

$$u : W_{\Sigma_n}(H) \to W_{\Sigma_{n+1}}(H)$$

which makes B_* into a (graded) $\mathbb{Z}[u]$-algebra.

Claim The localization $\mathsf{B} = \mathsf{B}_*[u^{-1}]$ is a co**ring** object in S^0-algebras.

3.2 Conjecture There is a lift

$$\begin{array}{ccc} & & \mathsf{B}(DX) \\ & \overset{\natural}{\nearrow} & \vdots \downarrow \\ DX & \longrightarrow & \mathsf{Q}(DX) \end{array} ,$$

and (in fact ?) spaces are just B-comonad coalgebras in the category of commutative S^0-algebras [analogous to Borger's interpretation [1] of $\mathbb{F}_1$-objects as algebras with a coaction of the Λ-ring comonad].

3.3 Is $\mathrm{Fns}(\langle \Sigma_* \rangle, \mathbb{Q}[u, u^{-1}])$ the free β-ring on one generator, tensored with $\mathbb{Q}$?

References

1. J Borger, Lambda-rings and the field with one element, `arXiv:0906.3146`
2. G Carlsson, C Douglas, B Dundas, Higher topological cyclic homology and the Segal conjecture for tori, `arXiv:0803.2745`
3. T tom Dieck, **Transformation groups**, de Gruyter Studies in Math 8 (1987)
4. P Guillot, Adams operations in cohomotopy, `arXiv:math/0612327`
5. K Hess, A general framework for homotopic descent and codescent, `arXiv:1001.1556`
6. I Mac Donald **Symmetric functions and Hall polynomials**, 2nd edition, OUP (1995)
7. H Miller, appendix (441 - 445) to DC Ravenel, The Segal conjecture for cyclic groups and its consequences, AJM 106 (1984)
8. C Rezk, The congruence criterion for power operations in Morava E-theory, Homology, Homotopy and Applications, Vol. 11 (2009) 327 - 379; available at `arXiv:0902.2499`
9. G Segal, Operations in stable homotopy theory, in **New developments in topology**, pp. 105 - 110, LMS Lecture Notes 11, CUP (1974)
10. NP Strickland, Morava E-theory of symmetric groups, `arXiv:math/9801125`
11. A Stacey, S Whitehouse, The hunting of the Hopf ring, Homology, Homotopy Appl. 11 (2009) 75 - 132
12. E Vallejo, The free β-ring on one generator, JPAA 86 (1993) 95 - 108

The Johns Hopkins University, Baltimore,Maryland
E-mail address: `jack@math.jhu.edu`

Indian Institute of Technology, Kanpur, Bharat
E-mail address: `reksan@iitk.ac.in`