LECTURES ON ORBIFOLDS AND GROUP COHOMOLOGY

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Abstract. The topics discussed in these notes include basic properties and definitions of orbifolds, and aspects of their cohomology and K–theory. Connections to group cohomology and equivariant algebraic topology appear in the context of orbifolds and their associated invariants. These notes are based on lectures given by the first author at the summer school on *Orbifolds and Transformation Groups*, held at Hangzhou China in June/July 2008.

Contents

1. Introduction

Orbifolds and their invariants play an important role in mathematics. The study of basic examples of quotients by Lie groups acting with finite isotropy on smooth compact manifolds leads to applications of ideas and techniques ranging from differential geometry and topology to algebraic geometry, group cohomology, homotopy theory and mathematical physics.

In these lecture notes we present some basic definitions and properties of orbifolds emphasizing their connections to algebraic topology and group cohomology. The language of

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groupoids provides a convenient mechanism for connecting these apparently distinct topics, and the global perspective this provides yields useful insight. In particular techniques from classical transformation groups can be used to construct interesting examples and formulate calculations in terms of better understood invariants from algebraic topology, such as cohomology and K–theory. Plenty of examples are provided both as a source of motivation and as a way to facilitate the understanding of the theory. We also discuss a *stringy* product in orbifold K–theory that was recently introduced in [5], which is motivated by the Chen–Ruan product in orbifold cohomology.

These notes are intended for graduate students interested in the general topic of orbifolds and their invariants. They reproduce the lectures given at the *2008 Hangzhou Summer School on Orbifolds and Transformation Groups* by the first author. Thus they are not meant to be complete or fully rigorous; rather their goal is to motivate casual readers to learn more about the subjects discussed here by consulting the literature; we offer the book [2] and the references therein as a good place to start. Moreover for these notes this book will be the standard reference, and we will omit referring to it to avoid repetition.

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2. Classical orbifolds

In this section we give a definition of an orbifold from a geometric point of view which is close to the original one (see [23] for Satake's definition of V-manifold).

Definition 2.1. Let X be a topological space and fix $n \geqslant 0$.

(1) An n-dimensional *orbifold chart* on X is given by a connected open subset $\widetilde{U} \subseteq \mathbb{R}^n$, a finite group G of effective smooth automorphisms of $\widetilde{U}$, and a map $\varphi : \widetilde{U} \to X$ such that φ is G-invariant and induces an homeomorphism of $\widetilde{U}/G$ onto an open subset $U \subseteq X$.
(2) An *embedding* $\lambda : (\widetilde{U}, G, \varphi) \to (\widetilde{V}, H, \psi)$ between two charts is a smooth embedding $\lambda : \widetilde{U} \to \widetilde{V}$ such that $\psi \circ \lambda = \varphi$.
(3) An *orbifold atlas* on X is a family $\mathfrak{U} = \left\{ (\widetilde{U}, G, \varphi) \right\}$ of charts which cover X and are locally compatible: given two charts $(\widetilde{U}, G, \varphi)$ with $U = \varphi(\widetilde{U})$ and $(\widetilde{V}, H, \psi)$ with $V = \psi(\widetilde{V})$, and a point $x \in U \cap V$, there exists an open neighborhood $W \subseteq U \cap V$ of x and a chart $(\widetilde{W}, K, \phi)$ with $\phi(\widetilde{W}) = W$ and such that there are two embeddings $\lambda : (\widetilde{W}, K, \phi) \to (\widetilde{V}, H, \psi)$ and $\mu : (\widetilde{W}, K, \phi) \to (\widetilde{U}, G, \varphi)$.

(4) An atlas $\mathfrak{U}$ *refines* another atlas $\mathfrak{W}$ if for every chart in $\mathfrak{U}$ there exists an embedding into some chart of $\mathfrak{W}$. Two orbifold atlases are *equivalent* if they have a common refinement.

Definition 2.2. A (*classical*) *orbifold* $\mathfrak{X}$ of dimension n is a paracompact Hausdorff space X equipped with an equivalence class $[\mathfrak{U}]$ of n-dimensional orbifold atlases.

Remark. We collect here some technical facts about orbifolds that are supposed to give a better understanding of the above definition:

(1) For every chart $(\widetilde{U}, G, \varphi)$ of an orbifold $\mathfrak{X}$, the group G acts freely on a dense open subset of $\widetilde{U}$.
(2) By local smoothness, every orbifold has an atlas consisting of linear charts $(\mathbb{R}^n, G, \varphi)$ where $G \subset O(n)$ (see [9]).
(3) An embedding $\lambda : (\widetilde{U}, G, \varphi) \to (\widetilde{V}, H, \psi)$ between two charts induces an injection $\lambda : G \to H$.
(4) Every atlas is contained in a unique maximal one and two atlases are equivalent if and only if they are contained in the same maximal one.
(5) If all the G-actions of an atlas are free, then $\mathfrak{X}$ is a honest manifold.

Given the remarks above, we can think of an orbifold as a "space with isolated singularities"; a notion that we make more precise with the next two definitions:

Definition 2.3. Let $x \in X$ with $\mathfrak{X} = (X, \mathfrak{U})$ an orbifold. The *local group* at x is the group $G_x = \{g \in G | gu = u\}$ where $(\widetilde{U}, G, \varphi)$ is any local chart with $\varphi(u) = x$. The group G_x is well defined up to conjugation. For an orbifold $\mathfrak{X} = (X, \mathfrak{U})$ its *singular set* is the subspace $\Sigma(\mathfrak{X}) = \{x \in X | G_x \neq 0\}$. A point in $\Sigma(\mathfrak{X})$ is a *singular point* of the orbifold $\mathfrak{X}$.

Let us now turn our attention to the notion of a map between two orbifolds (which turns out to be a more subtle concept that one might expect, as we will see later). We give a first definition in the current geometric setting:

Definition 2.4. Let $\mathfrak{X} = (X, \mathfrak{U})$ and $\mathfrak{Y} = (Y, \mathfrak{V})$ be two orbifolds. A map $f : X \to Y$ is a *smooth map between orbifolds* if for any point $x \in X$ there are charts $(\widetilde{U}, G, \varphi)$ around x and $(\widetilde{V}, H, \psi)$ around $f(x)$, with the property that f maps $\varphi(\widetilde{U})$ into $\psi(\widetilde{V})$ and can be lifted to a smooth map $\widetilde{f} : \widetilde{U} \to \widetilde{V}$ with $\psi\widetilde{f} = f\varphi$. Two orbifolds are *diffeomorphic* if there are smooth maps $f : X \to Y$ and $g : Y \to X$ with $fg = Id_Y$ and $gf = Id_X$.

A way to construct orbifolds is to take the quotient of a manifold by some nice group action. Let M be a smooth manifold and G a compact Lie group acting smoothly, effectively and almost freely on M (i.e. with finite isotropy). For each element $x \in M$ there is a chart $U \cong \mathbb{R}^n$ of M around x which is G_x invariant. The triples $(U, G_x, \pi : U \to U/G_x)$ are the orbifold charts.

Definition 2.5. A *quotient orbifold* is an orbifold given as the quotient of a smooth, effective, almost free action of a compact Lie group G on a smooth manifold M. If the group G is finite, the associated orbifold is called a *global quotient*.

Remark. If a compact Lie group G acts smoothly and almost freely on a manifold M, then we have a group extension:

$$1 \to G_0 \to G \to G_{eff} \to 1$$

where G_0 is finite and G_{eff} acts effectively. Even though $M/G = M/G_{eff}$, the original G-action does not give a classical orbifold. This will be one of the motivations for a more general definition of an orbifold, not involving the effective condition (see definition 5.15).

3. Examples of Orbifolds

(a) Consider a finite subgroup $G \subset GL_n(\mathbb{Z})$; it acts smoothly on the torus $X = \mathbb{R}^n/\mathbb{Z}^n$ giving rise to a so called *toroidal orbifold* $X \to X/G$ (see [3] for a discussion of their properties). Many important examples are of this form.

• The matrix $-I \in GL_4(\mathbb{Z})$ defines a $\mathbb{Z}/2$-action given by $\tau(z_1, z_2, z_3, z_4) = (z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1})$. The quotient $\mathbb{T}^4/G$ is the Kummer surface and it has sixteen isolated singularities.

• The group $\mathbb{Z}/4$ acts on $\mathbb{C}^3$ via $\tau(z_1, z_2, z_3) = (-z_1, iz_2, iz_3)$. There is a lattice $M \subset \mathbb{C}^3$ on which the action has the form:

$$\tau(a_1, a_2, a_3, a_4, a_5, a_6) = (a_1^{-1}, a_2^{-1}, a_4, a_3^{-1}, a_6, a_5^{-1}).$$

This gives rise to a $\mathbb{Z}/4$-action on $\mathbb{T}^6$ which has 16 isolated fixed points and $[\mathbb{T}^6]^{\mathbb{Z}/2}$ consists of 16 copies of $\mathbb{T}^2$. This example arises in the work of Vafa and Witten and has also been studied by Joyce who has shown that it has 5 different desingularizations (see [16]).

• The action of $(\mathbb{Z}/2)^2$ on $\mathbb{T}^6$ defined on generators by:

$$\sigma_1(z_1, z_2, z_3, z_4, z_5, z_6) = (z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1}, z_5, z_6)$$

$$\sigma_2(z_1, z_2, z_3, z_4, z_5, z_6) = (z_1^{-1}, z_2^{-1}, z_3, z_4, z_5^{-1}, z_6^{-1})$$

defines a toroidal orbifold $\mathbb{T}^6/(\mathbb{Z}/2)^2$ with $\{\pm 1\}^6$ as the set of fixed points and $(\mathbb{T}^6)^{<\sigma_1>} \cong (\mathbb{T}^6)^{<\sigma_2>} \cong \mathbb{T}^2 \times \{\pm 1\}^4$. Joyce showed that in contrast to the previous example, this orbifold has many desingularizations (see [16]).

(b) There are also beautiful examples defined using algebraic equations. Let Y be the degree 5 hypersurface of $\mathbb{C}P^4$ defined by the equation:

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \varphi z_0 z_1 z_2 z_3 z_4 = 0$$

where φ is a generic constant. The group $G = (\mathbb{Z}/5)^3$ acts on Y via:

$$e_1(z_0, z_1, z_2, z_3, z_4) = (\lambda z_0, z_1, z_2, z_3, \lambda^{-1} z_4)$$

$$e_2(z_0, z_1, z_2, z_3, z_4) = (z_0, \lambda z_1, z_2, z_3, \lambda^{-1} z_4)$$

$$e_3(z_0, z_1, z_2, z_3, z_4) = (z_0, z_1, \lambda z_2, z_3, \lambda^{-1} z_4)$$

where λ is a fifth root of the unity and the e_i's are the obvious generators of G. The quotient Y/G is the very popular *mirror quintic*.

(c) Another family of examples arises from the natural action of the permutation group S_n on the product $M^n = M \times \cdots \times M$ of n copies of a smooth manifold M. The quotient space $SP^n(M) = M^n/S_n$ is called the symmetric product and is of great interest in algebraic geometry and topology.

(d) Yet another family of important examples arises from quotient singularities of the form $\mathbb{C}^n/G$ for a subgroup $G \subset GL_n(\mathbb{C})$. They have the stucture of an algebraic variety arising from the algebra of G-invariant polynomials in $\mathbb{C}^n$. They appear in the context of the McKay correspondence (see [22]).

(e) For a choice of $n{+}1$ coprime integers $a_0, ..., a_n$; the circle group S^1 acts on $S^{2n+1} \subset \mathbb{C}^{n+1}$ as follows:

$$\lambda(z_0, ..., z_n) = (\lambda^{a_0} z_0, ..., \lambda^{a_n} z_n)$$

for every $\lambda \in S^1$. Since the integers are coprime, the action is effective and the quotient orbifold $W\mathbb{P}(a_0, ..., a_n) = S^{2n+1}/S^1$ is called the *weighted projective space*. The case $W\mathbb{P}(1, 2)$ has the shape of a teardrop. The $W\mathbb{P}$'s are examples of orbifolds which are NOT global quotients.

4. Orbifolds and manifolds

Similarly to the case of manifolds, we can construct a tangent bundle over an orbifold. The tangent bundle of an orbifold carries the following properties reminiscent of the manifold structure:

Proposition 4.1. *The tangent bundle $T\mathfrak{X} = (TX, T\mathfrak{U})$ of an n-dimensional orbifold has the structure of a $2n$-dimensional orbifold and the projection $p : T\mathfrak{X} \to \mathfrak{X}$ defines a smooth map of orbifolds with fibers $p^{-1}(x) = T_{\tilde{x}}\widetilde{U}/G_{\tilde{x}}$.*

Remark. The tangent bundle is an important object because it allows us to define some of the manifold structures over orbifolds. We can construct for example the dual bundle $T^*\mathfrak{X}$ of $T\mathfrak{X}$, the frame bundle $Fr(\mathfrak{X})$ and the exterior power $\bigwedge T^*\mathfrak{X}$. In this way we can also define Riemannian metrics, almost complex structures, orientability, differential forms and De Rham cohomology.

The objects above satisfy, among others, the following properties:

Proposition 4.2. *The orbifold De Rham cohomology with real coefficients depends only on the underlying space, i.e $H^*_{DR}(\mathfrak{X}, \mathbb{R}) \cong H^*(X, \mathbb{R})$. If $\mathfrak{X}$ is an orientable orbifold, then $H^*_{DR}(\mathfrak{X})$ is a Poincare duality algebra, in particular for a proper, almost free action of a compact Lie group G on a smooth manifold M, the De Rham cohomology of the quotient orbifold satisfies Poincare duality.*

This in particular says that De Rham cohomology is not the most appropriate for orbifolds since, in the case of a group action on a manifold for example, it only carries information about the quotient, forgetting the group action giving rise to it. Another interesting result is:

Theorem 4.3. *For a given orbifold $\mathfrak{X}$, its frame bundle $Fr(\mathfrak{X})$ is a smooth manifold with a smooth, effective and almost free $O(n)$-action. In this way $\mathfrak{X}$ is naturally isomorphic to the resulting quotient orbifold $Fr(\mathfrak{X})/O(n)$.*

Remark. The theorem above says in particular that every classical orbifold is a quotient orbifold. The manifold and the group action from which we can obtain a given orbifold are not unique.

5. Orbifolds and groupoids

We now set some categorical notions which will be used to re-define and generalize the concept of an orbifold. Our work will be justified by theorem 5.14 (see [20]).

Definition 5.1. A *groupoid* is a small category in which every morphism is an isomorphism.

Definition 5.2. A *topological groupoid* $\mathfrak{G}$ is a groupoid whose sets of objects G_0 and arrows G_1 are endowed with a topology in such a way that the five following maps are continuous:

(1) $s : G_1 \to G_0$, where $s(g)$ is the source of g,
(2) $t : G_1 \to G_0$, where $t(g)$ is the target of g,
(3) $m : G_{1s}\times_t G_1 = \{(h,g) \in G_1 \times G_1 | s(h) = t(g)\} \to G_1$, where $m(h,g) = h \circ g$ is the composition,
(4) $u : G_0 \to G_1$, where $u(x)$ is the identity of x,
(5) $i : G_1 \to G_1$, where $i(g)$ is the inverse of g.

Definition 5.3. A *Lie groupoid* $\mathfrak{G}$ is a topological groupoid where G_0 and G_1 are smooth manifolds with s, t smooth submersions and m, u and i smooth maps.

Example 5.4. Let G be Lie group acting smoothly from the left on a smooth manifold M. One defines a Lie groupoid $G \ltimes M$ by setting $(G \ltimes M)_0 = M$ and $(G \ltimes M)_1 = G \times M$. The source map $s : G \times M \to M$ is the projection, the target map $t : G \times M \to M$ is the action and the composition is defined by the product in the group G: if $(g, x) \in G \times M$ and $(g', x') \in G \times M$ are such that $s(g, x) = x = g'x' = t(g', x')$ then $(g, x) \circ (g', x') = (gg', x')$.

Definition 5.5. Let $\mathfrak{G}$ be a Lie groupoid. For a point $x \in G_0$ the set of all arrows from x to x form group denoted by G_x and called the *isotropy group* at x. The set $ts^{-1}(x)$ of targets or arrows out of x is called the *orbit* of x. The *orbit space* $|\mathfrak{G}|$ of $\mathfrak{G}$ is the quotient space of G_0 under the relation: $x \sim y$ iff x and y are in the same orbit, i.e. iff there is an arrow going from x to y.

Remark. Since $G_x = s^{-1}(x) \cap t^{-1}(x)$ and s and t are submersions, we have that G_x is a Lie group.

Before establishing the connection between orbifolds and groupoids, we need more definitions:

Definition 5.6. Let $\mathfrak{G}$ be a Lie groupoid.

(1) $\mathfrak{G}$ is *proper* if $(s, t) : G_1 \to G_0 \times G_0$ is proper (recall that a map is proper if the pre-image of every compact is compact),
(2) $\mathfrak{G}$ is a *foliation groupoid* if each isotropy group is discrete,
(3) $\mathfrak{G}$ is *étale* if s and t are local diffeomorphisms. In this case we define the *dimension* of $\mathfrak{G}$ as follow: $dim(\mathfrak{G}) = dim(G_0) = dim(G_1)$.

We remark immediately that every étale groupoid is a foliation groupoid. Furthermore if $\mathfrak{G}$ is a proper foliation groupoid, then all the isotropy groups are finite. It follows that given a proper, étale groupoid $\mathfrak{G}$, for any point $x \in G_0$ there exists a neighborhood U_x of x with the following property: for any $g \in G_x$ let $\sigma : U_x \to W_g$ be a local inverse to s with

$t : W_g \to U_x$ a diffeomorphism. Then $t \circ \sigma : U_x \to U_x$ is a well defined diffeomorphism of U_x. Thus we have a map $G_x \to Diff(U_x)$.

Definition 5.7. A proper, étale groupoid $\mathfrak{G}$ is *effective* if the above group homomorphism $G_x \to Diff(U_x)$ is injective.

We begin now the discussion concerning morphisms.

Definition 5.8. A *homomorphism* between two Lie groupoids $\mathfrak{G}$ and $\mathfrak{H}$ is a functor $\phi : \mathfrak{G} \to \mathfrak{H}$ such that the two maps $\phi_0 : G_0 \to H_0$ and $\phi_1 : G_1 \to H_1$ are smooth.

A *natural transformation* between two homomorphisms $\phi, \psi : \mathfrak{G} \to \mathfrak{H}$ is a categorical natural transformation $\alpha : \psi \to \phi$ between the two functors such that $\alpha : H_0 \to G_1$ is a smooth map.

Definition 5.9. A homomorphism $\phi : \mathfrak{G} \to \mathfrak{H}$ between Lie groupoids is called an *equivalence* if:

(1) The map $t \circ \pi_1 : G_1{}_s\times_\phi H_0 \to G_0$ defined on the fibered product of manifolds $\{(g, y) \in G_1 \times H_0 | s(g) = \phi(y)\}$ is a surjective submersion.
(2) The square:

$$\begin{array}{ccc} H_1 & \xrightarrow{\ \phi\ } & G_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ H_0 \times H_0 & \xrightarrow[\phi\times\phi]{} & G_0 \times G_0 \end{array}$$

is a fibered product of manifolds.

This crucial but rather mysterious definition hides the fact that an equivalence is in particular an equivalence of categories. This will be important in the sequel.

Definition 5.10. Two Lie groupoids $\mathfrak{G}$ and $\mathfrak{G}'$ are *Morita equivalent* if there exists a third groupoid $\mathfrak{H}$ and two equivalences:

$$\mathfrak{G} \leftarrow \mathfrak{H} \to \mathfrak{G}'$$

Remark. If two Lie groupoids $\mathfrak{G}$ and $\mathfrak{G}'$ are Morita equivalent, then $\mathfrak{G}$ is proper (resp. foliation) iff $\mathfrak{G}'$ is proper (resp. foliation). However, being étale is NOT invariant under Morita equivalence.

The above definitions are related by the following theorem:

Theorem 5.11. *A Lie groupoid is a foliation groupoid iff it is Morita equivalent to an étale groupoid and a Lie groupoid is a proper, foliation groupoid iff it is Morita equivalent to a proper, étale groupoid (see* [21]*).*

Example 5.12. Consider as usual a compact Lie group G acting on a smooth manifold M with finite stabilizers G_x. Then $G \ltimes M$ is a Lie groupoid. By the slice theorem for smooth actions, for all $x \in M$ we have a slice $V_x \subset M$ for which the action defines a diffeomorphism $K \times_{G_x} V_x \to M$ onto an open neighborhood U_x of x. Then $G_x \ltimes V_x$ is an étale groupoid Morita equivalent to $G \ltimes U_x$. Patching these étale groupoids together, yields an étale groupoid Morita equivalent to $G \ltimes M$.

We now turn to the relation between groupoids and classical orbifolds.

Proposition 5.13. *Let $\mathfrak{G}$ be a proper, effective, étale groupoid. Then its orbit space $X = |\mathfrak{G}|$ carries an orbifold structure constructed from the groupoid $\mathfrak{G}$.*

We are now in the following situation: to every classical orbifold $\mathfrak{X}$ we can associate a proper, effective, étale groupoid $\mathfrak{G}_{\mathfrak{X}} = O(n) \ltimes Fr(\mathfrak{X})$ and to every proper, effective, étale groupoid $\mathfrak{G}$ we can associate an orbifold $|\mathfrak{G}|$. Moreover, by construction, we have that $|\mathfrak{G}_{\mathfrak{X}}| \cong \mathfrak{X}$. The key theorem relating orbifolds to groupoids is the following.

Theorem 5.14. *Two proper, effective, étale orbifolds $\mathfrak{G}$ and $\mathfrak{G}'$ give rise to the same classical orbifold up to isomorphism if and only if they are Morita equivalent (see* [20]*).*

We can now generalize the notion of an orbifold, leaving out the effective condition (see end of section 2).

Definition 5.15. An *orbifold groupoid* is a proper, étale, Lie groupoid. An *orbifold structure* on a paracompact Hausdorff space X consists of an orbifold groupoid $\mathfrak{G}$ and a homeomorphism $f : |\mathfrak{G}| \to X$. If $\phi : \mathfrak{H} \to \mathfrak{G}$ is an equivalence, then $|\phi| : |\mathfrak{H}| \to |\mathfrak{G}|$ is a homeomorphism and we say that the composition $f \circ |\phi| : |\mathfrak{H}| \to X$ defines an *equivalent* orbifold structure on X. An *orbifold* $\mathfrak{X}$ is a paracompact, Hausdorff space X with an equivalence class of orbifold structures. A specific structure $f : |\mathfrak{G}| \to X$ is called a *presentation* of the orbifold $\mathfrak{X}$.

Example 5.16. Under this new definition the weighted projective space $W\mathbb{P}(a_0, ..., a_n)$ is an orbifold for any choice of the integers $a_0, ..., a_n$.

To define maps between orbifolds we have to keep in mind the fact that an orbifold may have different presentations and we want to be allowed to take refinements before defining our map. From the groupoid point of view this means considering maps that factor through Morita equivalence.

Definition 5.17. A *generalized map* between two Lie groupoids $\mathfrak{G}$ and $\mathfrak{H}$ is a pair of maps:

$$\mathfrak{H} \xleftarrow{\varepsilon} \mathfrak{H}' \xrightarrow{\phi} \mathfrak{G}$$

such that ϕ is a homomorphism of groupoids and ε is an equivalence of groupoids.

A *map* between two orbifolds $\mathfrak{Y} \to \mathfrak{X}$ presented by $\mathfrak{G}_\mathfrak{Y}$ and $\mathfrak{G}_\mathfrak{X}$ is a continuous map of underlying spaces $Y \to X$ together with an orbifold morphism $\mathfrak{G}_\mathfrak{Y} \to \mathfrak{G}_\mathfrak{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{G}_\mathfrak{Y} & \longrightarrow & \mathfrak{G}_\mathfrak{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

Now, for a given orbifold $\mathfrak{X}$ we choose a presentation $f : \mathfrak{G}_\mathfrak{X} \to X$. We can then consider the simplicial object given by the nerve $N_\bullet \mathfrak{G}_\mathfrak{X}$ of the underlying category. Taking the geometric realization $B\mathfrak{G}_\mathfrak{X} = |N_\bullet \mathfrak{G}_\mathfrak{X}|$ we get a *classifying space* arising from the orbifold structure. Since Morita equivalent Lie groupoids are in particular equivalent categories, we obtain:

Proposition 5.18. *If $\mathfrak{G}$ and $\mathfrak{G}'$ are Morita equivalent Lie groupoids, then $B\mathfrak{G}$ and $B\mathfrak{G}'$ are homotopy equivalent. For the action groupoid $G \ltimes M$, its classifying space $B(G \ltimes M)$ is homotopy equivalent to the Borel construction $EG \times_G M$.*

Based on this we can define:

Definition 5.19. Let $\mathfrak{X}$ be an orbifold and $\mathfrak{G}$ any groupoid representing $\mathfrak{X}$ via a given homeomorphism $f : |\mathfrak{G}| \to X$.

(1) The n^{th} *orbifold homotopy group* of $\mathfrak{X}$ based at $x \in X$ is the group:

$$\pi_n(\mathfrak{X}, x) = \pi_n(B\mathfrak{G}, \tilde{x})$$

where $\tilde{x} \in G_0$ maps to x under the composition $G_0 \to |\mathfrak{G}| \to X$.

(2) Let R be a commutative ring with unit. The *singular cohomology* of $\mathfrak{X}$ with coefficients in R is $H^*(\mathfrak{X}, R) = H^*(B\mathfrak{G}, R)$.

Remark. The map $G_0 \to |\mathfrak{G}|$ gives rise to a map $\pi : B\mathfrak{G} \to |\mathfrak{G}|$. For the action groupoid $B(G \ltimes M) \simeq EG \times_G M$ and the map above is the projection $EG \times_G M \to M/G$ which induces an isomorphism $\pi^* : H^*(|\mathfrak{G}|, \mathbb{Q}) \to H^*(B\mathfrak{G}, \mathbb{Q})$. The above definition of fundamental group recovers Thurston's orbifold fundamental group, which plays a role for orbifold covers analogous to that of the usual fundamental group (see [24]). The notion of orbifold homotopy groups has been generalized by Leida in [18], where he introduces *extended orbifold homotopy groups* as a more complete invariant, which encodes notions from equivariant homotopy theory.

We end the section with some calculation involving two examples already given (see section 2).

Example 5.20. For a toroidal orbifold, $B\mathfrak{X} \simeq EG \times_G \mathbb{T}^n$ which is a $K(\Gamma, 1)$ with $\Gamma = \mathbb{Z}^n \ltimes G$. Therefore $\pi_1(\mathfrak{X}) = \Gamma$, $\pi_k(\mathfrak{X}) = 0$ for $k > 1$ and $H^*(\mathfrak{X}) = H^*(\Gamma)$. In particular for the Kummer surface one can calculate the cohomology of the corresponding group and obtain:

$$H^i(B\mathfrak{X}, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ 0, & \text{if i is odd;} \\ \mathbb{Z}^6 \oplus (\mathbb{Z}/2)^5, & \text{if } i = 2; \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^{15}, & \text{if } i = 4; \\ \mathbb{Z}^{16}, & \text{if } i > 4 \text{ even.} \end{cases}$$

Example 5.21. For a weighted projective space given by the action of S^1 on S^3, we have as usual $B\mathfrak{X} \simeq ES^1 \times_{S^1} S^3$ which gives a fibration $S^3 \to B\mathfrak{X} \to \mathbb{C}P^\infty$. We deduce then that $\pi_1(\mathfrak{X}) = 0$, $\pi_2(\mathfrak{X}) = \mathbb{Z}$ and $\pi_k(\mathfrak{X}) = \pi_k(S^3)$ for $k > 2$.

Using the spectral sequence of the above fibration, we can compute the cohomology groups. Observe first that:

$$E_2^{**} = H^*(\mathbb{C}P^\infty) \otimes H^*(S^3) = \mathbb{Z}\,[z] \otimes E(e_3)$$

Assume now that our weighted projective space is of the form $W\mathbb{P}(p, q)$ with p, q distinct primes. Then the image of e_3 under the differential of the second page of the spectral sequence is: $d(e_3) = pqz^2$. (This follows from the fact that only 1, $\mathbb{Z}/p$, $\mathbb{Z}/q \subset S^1$ have fixed points). Therefore:

$$H^i(B\mathfrak{X}, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2; \\ \mathbb{Z}/pq, & \text{if } i > 2 \text{ is even;} \\ 0, & \text{if } i > 0 \text{ is odd.} \end{cases}$$

Note that $|\mathfrak{G}| = S^2$ and that $B\mathfrak{X} \simeq_{\mathbb{Q}} |\mathfrak{G}|$ as expected.

6. The Orbifold Euler Characteristic and K–theory

We now introduce the notion of an *orbifold Euler characteristic*. This number was first introduced by physicists in the late 1980's (see [12] and [13]).

Definition 6.1. Let M be a smooth manifold with a smooth, effective G-action, where G is a finite group. The *orbifold Euler characteristic* for $M \to M/G$ is defined as:

$$\chi_{orb}(M/G) = (1/|G|) \sum_{gh=hg} \chi(M^{<g,h>})$$

After some manipulations, this number becomes:

$$\chi_{orb}(M/G) = \sum_{(g)} \chi(M^{<g>}/Z_G(g))$$

where $Z_G(g)$ is the centralizer of g in G and the sum runs over the congugacy classes of elements g in G.

Given M a manifold with a G-action, consider

$$Y = \{(x, g) \in M \times G | gx = x\}.$$

If $h \in G$, then we define $h(x, g) = (hx, hgh^{-1})$. The new quotient $Y \to Y/G$ is called the *inertia orbifold* and it is usually denoted by $\Lambda\mathfrak{X}$, where $\mathfrak{X}$ stands for the original orbifold $(M/G, \mathfrak{U})$.

It turns out that

$$|\Lambda\mathfrak{X}| \cong \coprod_{(g)} M^{<g>}/Z_G(g).$$

The pieces corresponding to $g \neq 1$ are called the *twisted sectors*. Note that for $g = 1$ we get M/G. In this context we are interested in the inertia orbifold because it yields the following property:

$$\chi_{orb}(\mathfrak{X}) = \chi(|\Lambda\mathfrak{X}|).$$

Example 6.2. For the Kummer surface $\mathbb{T}^4 \to \mathbb{T}^4/(\mathbb{Z}/2)$ we have:

$$\chi_{orb}(\mathfrak{X}) = \chi(\mathbb{T}^4/(\mathbb{Z}/2)) + 16 = 8 + 16 = 24.$$

We now give a K-theoretic interpretation of the orbifold Euler characteristic defined above. For a finite group G, the standard G-equivariant K-theory admits the following decomposition (see [7])

$$K^*_G(M) \otimes \mathbb{C} \cong \oplus_{(g)} K^*(M^{<g>}/Z_G(g)) \otimes \mathbb{C}$$

so that:

$$\chi_{orb}(\mathfrak{X}) = dim(K_G^0(M) \otimes \mathbb{C}) - dim(K_G^1(M) \otimes \mathbb{C}).$$

In order to give a more general interpretation using orbifold K-theory, we need the notion of vector bundles for orbifolds.

Definition 6.3. Let $\mathfrak{G}$ be an orbifold groupoid. A *right* $\mathfrak{G}$*-space* is a manifold E with a $\mathfrak{G}$-action; which is given by two maps: $\pi : E \to G_0$ and $\mu : E \times_{G_0} G_1 \to E$, where $E \times_{G_0} G_1 = \{(e, g) \in E \times G | \pi(e) = t(g)\}$. We write $\mu(e, g) = eg$ and the maps satisfy the usual properties:

(1) $\pi(eg) = (s(g))$,
(2) $e1_x = e$,
(3) $(eg)h = e(gh)$ whenever defined.

Definition 6.4. A *vector bundle over a groupoid* $\mathfrak{G}$ is a $\mathfrak{G}$-space E for which $\pi : E \to G_0$ is a vector bundle and the action of $\mathfrak{G}$ on E is fibrewise linear, i.e. any arrow $g : x \to y$ induces a linear isomorphism $g^{-1} : E_y \to E_x$.

We write $Vect(\mathfrak{G})$ for the category of vector bundles on $\mathfrak{G}$ and we observe that if $\mathfrak{G}$ is Morita equivalent to $\mathfrak{H}$, then $Vect(\mathfrak{G})$ is equivalent to $Vect(\mathfrak{H})$

Proposition 6.5. *If* $\mathfrak{X}$ *is a quotient orbifold* $M \to M/G$*, then* $Vect(\mathfrak{X})$ *is equivalent to* $Vect_G(M)$*, the category of* G*-equivariant vector bundles on* M*. In particular:*

$$K_{orb}(\mathfrak{X}) \cong K_G(M)$$

Note that if a quotient is represented in two ways, $M \to M/G$ and $N \to N/H$ then $K_H(N) \cong K_{orb}(\mathfrak{X}) \cong K_G(M)$.

For a classical orbifold we have $K_{orb}(\mathfrak{X}) \cong K_{O(n)}(Fr(\mathfrak{X}))$ so that the next goal is to compute $K_{orb}(\mathfrak{X})$ for a quotient orbifold. Assume that G is a compact Lie group acting smoothly on M, therefore admitting an equivariant cellular decomposition. We have the following spectral sequence converging to $K_G^*(M)$:

$$E_1^{pq} = \begin{cases} \oplus_{[\Delta_p]} R(G_{\Delta_p}), & \text{if } q \text{ is even;} \\ 0, & \text{if } q \text{ is odd.} \end{cases}$$

Thus we have: $\chi_{orb}(M) = \sum_{[\Delta_p]} (-1)^p rk(R(G_{\Delta_p}))$, where $[\Delta_p]$ is the equivalence class of the p-cell Δ_p and $R(G_{\Delta_p})$ is the representation ring of the isotropy subgroup G_{Δ_p}. Using an equivariant Chern character, the following decomposition of orbifold K–theory can be obtained for quotient orbifolds $M \to M/G$:

Theorem 6.6. *If $\mathfrak{X}$ is a quotient orbifold $M \to M/G$, where G is a compact Lie group acting smoothly and almost freely on M, then there is a multiplicative isomorphism:*

$$K_{orb}(\mathfrak{X}) \cong \prod_{(C)} (K(M^C/Z_G(C)) \otimes \mathbb{Q}(\zeta_{|C|}))^{W_G(C)}$$

where the product is taken over conjugacy classes of finite cyclic subgroups $C \subset G$, $\zeta_{|C|}$ is a primitive $|C|$–th root of unity and $W_G(C) = N_G(C)/Z_G(C)$ (see [4]*).*

Corollary 6.7. *It follows that: $K_{orb}(\mathfrak{X}) \otimes \mathbb{Q} \cong K(|\Lambda\mathfrak{X}|) \otimes \mathbb{Q}$.*

Example 6.8. For the Kummer surface $\mathbb{T}^4 \to \mathbb{T}^4/(\mathbb{Z}/2)$ we have that $(\mathbb{T}^4)^{\mathbb{Z}/2} = \{\pm 1\}^4$ and therefore

$$K_{orb}(\mathfrak{X}) \cong_{\mathbb{Q}} K(|\mathfrak{X}|) \times K(\{\pm 1\}^4) \cong K(\mathbb{T}^4/(\mathbb{Z}/2)) \times \mathbb{Q}^{16}$$

Example 6.9. Consider $\mathfrak{X} = W\mathbb{P}(p,q)$ with p, q two distinct primes. Then $K_{orb}(\mathfrak{X}) = K_{S^1}(S^3)$ where the action of S^1 on S^3 is given by $\lambda(v,w) = (\lambda^p v, \lambda^q w)$. The quotient space has two singular points $[(1,0)]$ and $[(0,1)]$ with isotropy $\mathbb{Z}/p$ and $\mathbb{Z}/q$ respectively. We have $(S^3)^{\mathbb{Z}/p} = \{(v,0) | v \in S^1\}$ and $(S^3)^{\mathbb{Z}/q} = \{(0,w) | w \in S^1\}$. Moreover $(S^3)^{\mathbb{Z}/p}/S^1 = [(1,0)]$ and $(S^3)^{\mathbb{Z}/q}/S^1 = [(0,1)]$, so that:

$$K_{orb}(\mathfrak{X}) \cong_{\mathbb{Q}} \mathbb{Q}(\zeta_p) \times \mathbb{Q}(\zeta_q) \times K(S^1)$$

and $\chi_{orb}(\mathfrak{X}) = p - 1 + q - 1 + 2 = p + q$.

Example 6.10. There are some important examples known as *arithmetic orbifolds* that can be analyzed in this context. Let $G(\mathbb{R})$ be a semisimple arithmetic group, $K \subset G(\mathbb{R})$ a maximal compact subgroup and $\Gamma \subset G(\mathbb{Q})$ an arithmetic subgroup. Then Γ acts properly on $X = G(\mathbb{R})/K$. Moreover X^H is contractible if H is finite and $X^H = \emptyset$ otherwise. Replacing X with the Borel–Serre compactification, we can assume that these spaces are of finite type, and that we can apply the methods outlined here. Then we have

$$\chi_{orb}(X/\Gamma) = \sum_{(\gamma)} \chi(BZ_\Gamma(\gamma))$$

where the sum runs over the conjugacy classes of finite order elements of Γ; in fact we have the following computation which appears in [1] (see also [19]):

$$K^0_\Gamma(X) \otimes \mathbb{Q} = \oplus_{(\gamma)} H^{ev}(Z_\Gamma(\gamma), \mathbb{Q})$$

$$K^0_\Gamma(X) \otimes \mathbb{Q} = \oplus_{(\gamma)} H^{odd}(Z_\Gamma(\gamma), \mathbb{Q})$$

where $Z_\Gamma(\gamma)$ denotes the centralizer of γ in Γ.

Example 6.11. Set $\Gamma = G_1 *_H G_2$ for some subgroup $H \subset G_1$, $H \subset G_2$. Here we assume that G_1 and G_2 are finite. The spectral sequence here is:

$$K^0_\Gamma(X) \to R(G_1) \oplus R(G_2) \to R(H) \to K^1_\Gamma(X)$$

and

$$\chi_{orb}(X/\Gamma) = rk(R(G_1)) + rk(R(G_2)) - rk(R(H)),$$

where for a finite group Q, $R(Q)$ denotes its ring of complex characters. We can identify $K^0_\Gamma(X) = \lim\limits_{H \subset \Gamma} R(H)$ where the limit is taken over finite subgroups $H \subset \Gamma$ subject to inclusions and conjugation (so-called stable elements). It has rank equal to $n(\Gamma)$, the number of distinct conjugacy classes of elements of finite order in Γ. Now, from our sequence above we have that:

$$\chi_{orb}(X/\Gamma) = n(G_1) + n(G_2) - n(H) = n(\Gamma) - dim K^1_\Gamma(X) \otimes \mathbb{Q}$$

and this yields the following

Theorem 6.12. *Let $\Gamma = G_1 *_H G_2$ denote an amalgamated product of finite groups. Then the number of distinct conjugacy classes of elements of finite order in Γ is given by*

$$n(\Gamma) = n(G_1) + n(G_2) - n(H) + \sum_{(\gamma)} dim_{\mathbb{Q}} H^1(Z_\Gamma(\gamma), \mathbb{Q})$$

where the sum runs over the conjugacy classes of elements of finite order in Γ.

In particular for $\Gamma = SL_2(\mathbb{Z}) = (\mathbb{Z}/6) *_{(\mathbb{Z}/2)} (\mathbb{Z}/4)$ we have $n(\Gamma) = 6 + 4 - 2 = 8$, since all the centralizers $Z_\Gamma(\gamma)$ are finite. For $\Gamma = S_3 *_{(\mathbb{Z}/3)} S_3$ we have $n(\Gamma) = 4$, since there exists an element $\gamma \in \Gamma$ of finite order with $H^1(Z_\Gamma(\gamma)) \cong \mathbb{Q}$.

7. Stringy Products in K–theory

We start by considering a basic example: let G be a finite group and $K_G(G)$ its G-equivariant complex K-theory with respect to the conjugation action. From the decomposition into orbits of the form $G/Z_G(g)$, we see that $K_G(G) \cong \sum_{(g)} R(Z_G(g))$ where the sum runs over the conjugacy classes of elements g in G. We are interested in the following special product on $K_G(G)$: a bundle over G can be thought of as a complex vector space with a G-action, $V = \sum_{g\in G} V_g$, such that $hV_g = V_{hgh^{-1}}$. Given two bundles V and W, we define their product as:

$$(V * W)_g = \sum_{st=g} V_s \otimes W_t$$

This product was first introduced by Lusztig and also appears in the work of physicists (see [10]).

Example 7.1. Consider $G = \mathbb{Z}/2$. In this case $K_G(G)$ has a $\mathbb{Z}$-basis s_0, s_1, t_0, t_1 with products given by:

	s_0	s_1	t_0	t_1
s_0	s_0	s_1	t_0	t_1
s_1	s_1	s_0	t_1	t_0
t_0	t_0	t_1	s_0	s_1
t_1	t_1	t_0	s_1	s_0

Another way of understanding this product is by using the multiplication map. We have three G-equivariant maps $e_1, e_2, e_{12} : G \times G \to G$, given by $e_1(g, h) = g$, $e_2(g, h) = h$ and $e_{12}(g, h) = gh$. Then we have:

$$\alpha * \beta = (e_{12})_*(e_1^*(\alpha) \cdot e_2^*(\beta))$$

This is also know as the Pontryagin product, better known for $H_*(G)$, G a topological group. Our next goal is to extend this to an orbifold setting and to develop a twisted version of it; we will follow the exposition in [5].

In order to generalize the previous concepts to orbifolds, we need to introduce some definitions.

Definition 7.2. Given a groupoid $\mathfrak{G}$, its *inertia groupoid* $\Lambda\mathfrak{G}$ is given by

$$\Lambda\mathfrak{G}_0 = \{g \in G_1 | s(g) = t(g)\}$$

and

$$\Lambda\mathfrak{G}_1 = \{(a, v) \in G_1 \times G_1 | s(a) = t(a) = s(v)\}$$

with $s(a, v) = a$ and $t(a, v) = v^{-1}av$.

More generally we define the *groupoid of k-sectors* $\mathfrak{G}^k$ as:

$$\mathfrak{G}^k_0 = \left\{(a_1, ..., a_k) \in G_1^k | s(a_i) = t(a_i) \ \forall \ i = 1, ..., k\right\}$$

$$\mathfrak{G}^k_1 = \left\{(a_1, ..., a_k, u) \in G_1^{k+1} | s(a_i) = t(a_i) \ \forall \ i = 1, ..., k \ \textit{and} \ t(a_k) = s(u)\right\}$$

and with $s(a_1, ..., a_k, u) = (a_1, ..., a_k)$ and $t(a_1, ..., a_k, u) = (u^{-1}a_1u, ..., u^{-1}a_ku)$.

As in the group case explained above, we have three maps $e_1, e_2, e_{12} : \mathfrak{G}^2 \to \Lambda\mathfrak{G}$ defined by $e_1(a_1, a_2) = a_1$ and $e_1(a_1, a_2, u) = (a_1, u)$; $e_2(a_1, a_2) = a_2$ and $e_2(a_1, a_2, u) = (a_2, u)$; $e_{12}(a_1, a_2) = a_1a_2$ and $e_{12}(a_1, a_2, u) = (a_1a_2, u)$.

Remark. If U/G is a local chart for $\mathfrak{G}$, then $(\coprod_{g\in G} U^g)/G$ is a local chart for $\Lambda\mathfrak{G}$ and $(\coprod_{g_1,g_2\in G} U^{g_1} \cap U^{g_2})/G$ is a local chart for the 2-sectors. The three embeddings above correspond to the three inclusions $(U^{g_1} \cap U^{g_2}) \subset U^{g_1}$, $(U^{g_1} \cap U^{g_2}) \subset U^{g_2}$ and $(U^{g_1} \cap U^{g_2}) \subset U^{g_1g_2}$.

Consider the pull back:

$$\begin{array}{ccc} \mathfrak{H} & \longrightarrow & \mathfrak{G}^2 \\ \downarrow & & \downarrow{\scriptstyle e_1} \\ \mathfrak{G}^2 & \xrightarrow[e_{12}]{} & \Lambda\mathfrak{G} \end{array}$$

In fact we can identify $\mathfrak{H}$ with $\mathfrak{G}^3$ i.e. we can think of the two copies of $\mathfrak{G}^2$ as suborbifolds of $\Lambda\mathfrak{G}$ under e_1, e_{12} and with intersection $\mathfrak{G}^3$:

$$\begin{array}{ccc} \mathfrak{G}^3 & \xrightarrow{\pi_2} & \mathfrak{G}^2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle e_1} \\ \mathfrak{G}^2 & \xrightarrow[e_{12}]{} & \Lambda\mathfrak{G} \end{array}$$

The problem is that the intersection is not transverse in general. Let

$$\nu = ((e_{12}\pi_1)_* T(\Lambda\mathfrak{G}))/(\pi_1^* T\mathfrak{G}^2 \oplus \pi_2^* T\mathfrak{G}^2)$$

This is called the *excess bundle* of the intersection. We have an additional ingredient: there exists a bundle E defined on $\mathfrak{G}^k$ such that $E_{\mathfrak{G}^3} = \pi_1^* E_{\mathfrak{G}^2} \oplus \pi_2^* E_{\mathfrak{G}^2} \oplus \nu$, which is called the *obstruction bundle*. We are now ready to define our product in this more general setting.

Definition 7.3. For $\mathfrak{G}$ an almost complex orbifold and for $\alpha, \beta \in K(\Lambda\mathfrak{G})$, we define $\alpha * \beta = (e_{12})_*(e_1^*(\alpha) \cdot e_2^*(\beta) \cdot e(E_{\mathfrak{G}^2}))$ where $e(E_{\mathfrak{G}^2})$ is the Euler class.

Theorem 7.4. *This defines an associative product on* $K(\Lambda\mathfrak{G})$.

The basic ingredients to prove the theorem are (see [5]) the obstruction bundle and Quillen's clean intersection formula: if $i_1 : \mathfrak{H}_1 \to \mathfrak{G}$ and $i_2 : \mathfrak{H}_2 \to \mathfrak{G}$ are suborbifolds forming a clean intersection, then for $u \in K(\mathfrak{H}_1)$ we have $i_2^* i_{1*} u = \pi_{2*}(\pi_1^* u \cdot e(\nu))$. This yields the K–theoretic version of the Chen–Ruan product.

8. Twisted version

Let's go back to the basic example above and consider a finite group G. In this case sometimes the twisting can be understood very explicitly. Let $\alpha \in Z^2(G, U(1))$ denote a 2-cocycle, which gives rise to a central extension:

$$1 \longrightarrow S^1 \longrightarrow \widetilde{G}_\alpha \longrightarrow G \longrightarrow 1$$

If M is a closed manifold with a G-action, we can consider all $\widetilde{G}_\alpha$-bundles over M such that the action restricts on S^1 to scalar multiplication on the fibers. This gives rise to ${}^\alpha K_G(M)$ in the usual way. For this construction, the usual pairing gives:

$${}^\alpha K_G(M) \otimes {}^\beta K_G(M) \to {}^{\alpha+\beta} K_G(M)$$

i.e. the level jumps. Our goal will be to define a product on ${}^\alpha K(\Lambda\mathfrak{G})$ (for suitable twisting) which is based on the product defined previously in the untwisted case.

What we have to do now, is to define a twisting where the levels match up to give the wanted product. Dijkgraaf and Witten (see [11]) describe the inverse transgression, which for G a compact Lie group inverts the usual map $H^3(G) \to H^4(BG)$. We look first at the case when G is finite. In this context their construction is of the form

$$Z^3(G, U(1)) \to Z^2(Z_G(h), U(1))$$

$\alpha \mapsto \theta(\alpha)_h$ where:

$$\theta(\phi)_h(g_1, g_2) = \phi(g_1, g_2, h) - \phi(g_1, h, g_2) + \phi(h, g_1, g_2)$$

for $g_1, g_2 \in Z_G(h)$. We extend this to orbifolds by defining a cochain map on continous $U(1)$–valued cochains:

$$\theta : C^{k+1}(\mathfrak{G}, U(1)) \to C^k(\wedge\mathfrak{G}, U(1))$$

as follows:

$$\theta(\phi)(a, u_1, \cdots, u_k) = (-1)^k \phi(a, u_1, \cdots, u_k) + \sum_{i=1}^{k} (-1)^{i+k} \phi(u_1, \cdots, u_i, a_i, u_{i+1}, \cdots, u_k)$$

where $a_i = (u_1 \cdots u_i)^{-1} a u_1 \cdots u_i$. We have remarked that a twisting cocycle for K-theory should be in $Z^2(-, U(1))$. Here $\theta : Z^3(\mathfrak{G}, U(1)) \to Z^2(\Lambda\mathfrak{G}, U(1))$ gives rise to a twisting cocycle for $K(\Lambda\mathfrak{G})$. In order to solve our twisting problem, we have to analyze the compositions of $\theta : C^k(\mathfrak{G}) \to C^{k-1}(\Lambda\mathfrak{G})$ with the three maps $e_1^*, e_2^*, e_{12}^* : C^{k-1}(\Lambda\mathfrak{G}) \to C^{k-1}(\mathfrak{G}^2)$.

Definition 8.1. Define $\mu : C^{k+2}(\mathfrak{G}, U(1)) \to C^k(\mathfrak{G}^2, U(1))$ by:

$$\mu(\phi)(a, b, u_1, \cdots, u_k) =$$

$$\phi(a, b, u_1, \cdots, u_k) + \sum_{\{(i,j)\ |\ 0 \le i \le j \le k\ \ (i,j) \ne (0,0)\}} (-1)^{i+j} \phi(u_1, \cdots, u_i, a_i, u_{i+1}, \cdots, u_j, b_j, u_{j+1}, \cdots, u_k).$$

A key multiplicative formula is given by the equation:

$$\mu\delta + \delta\mu \ = \ e_1^*\theta + e_2^*\theta - e_{12}^*\theta.$$

Theorem 8.2. *Let ϕ be a 2-gerbe on an orbifold groupoid $\mathfrak{G}$ and let $\alpha, \beta \in {}^{\theta(\phi)}K(\Lambda\mathfrak{G})$. Then, if we define:*

$$\alpha * \beta = e_{12*}(e_1^*(\alpha) \cdot e_2^*(\beta) \cdot e(E_{\mathfrak{G}^2}))$$

this element lies in ${}^{\theta(\phi)}K(\Lambda\mathfrak{G})$ and makes it into an associative algebra.

We make some clarifying remarks:

(1) Here we use the definition of twisted K-theory developed in [17] (see also [6]) using Fredholm operators.
(2) The Euler class is still untwisted.
(3) Recall that we have the three maps:

$$e_1^* : \ {}^{\theta(\phi)}K(\Lambda\mathfrak{G}) \to \ {}^{e_1^*\theta(\phi)}K(\mathfrak{G}^2)$$
$$e_2^* : \ {}^{\theta(\phi)}K(\Lambda\mathfrak{G}) \to \ {}^{e_2^*\theta(\phi)}K(\mathfrak{G}^2)$$
$$e_{12}^* : \ {}^{\theta(\phi)}K(\Lambda\mathfrak{G}) \to \ {}^{e_{12}^*\theta(\phi)}K(\mathfrak{G}^2)$$

So that $e_1^*(\alpha)e_2^*(\beta) \in {}^{e_1^*\theta(\phi)+e_2^*\theta(\phi)}K(\mathfrak{G}^2)$. Since ϕ is a cocycle, the key formula above becomes

$$\delta\mu(\phi) + e_{12}^*\theta(\phi) \ = \ e_1^*\theta(\phi) + e_2^*\theta(\phi)$$

where $\delta\mu(\phi)$ is a coboundary. Therefore we have a canonical isomorphism

$${}^{\delta\mu(\phi)+e_{12}^*\theta(\phi)}K(\mathfrak{G}^2) \cong \ {}^{e_{12}^*\theta(\phi)}K(\mathfrak{G}^2).$$

Now applying the pushforward yields:

$${}^{e_{12}^*\theta(\phi)}K(\mathfrak{G}^2) \xrightarrow[e_{12*}]{} {}^{\theta(\phi)}K(\Lambda\mathfrak{G})$$

to obtain an element in ${}^{\theta(\phi)}K(\Lambda\mathfrak{G})$.
(4) The cocycle $\phi \in Z^3(\mathfrak{G}, U(1))$ defines a cohomology class in $H^3(B\mathfrak{G}, U(1)) \cong H^4(B\mathfrak{G}, \mathbb{Z})$ and the inverse trangsression lies in $H^2(B\Lambda\mathfrak{G}, U(1)) \cong H^3(B\Lambda\mathfrak{G}, \mathbb{Z})$.

We now go back to the finite group case to construct some examples. For a finite group G and an element $g \in G$, we have a group homomorphism

$$\rho_g : \mathbb{Z} \times Z_G(g) \to G$$

given by $(T^i, x) \mapsto g^i x$ where T generates $\mathbb{Z}$.

Proposition 8.3. *If $[\phi] \in H^4(G, \mathbb{Z})$ and $e \in H^1(\mathbb{Z}, \mathbb{Z})$ is the canonical generator, then*

$$\rho_g^*([\phi]) = [\theta(\phi)_g] \otimes e + [res(\phi)] \otimes 1$$

where $\theta(\phi)_g$ is the inverse transgression.

Remark. The formula given in [11] is in fact a particular case of a "shuffle product" type construction, which gives a (co)chain level construction for the multiplication map $\rho_g : C_*(\mathbb{Z} \times Z_G(g)) \to C_*(G)$.

We restrict now our attention to G an abelian group. In this case $Z_G(g) = G$ and our compatibility condition is simply $\theta(\phi)_{gh} = \theta(\phi)_g + \theta(\phi)_h$, i.e. a homomorphism $G \to H^3(G, \mathbb{Z})$. The easiest non-trivial examples are elementary abelian 2-groups $G_n = (\mathbb{Z}/2)^n$. Indeed the group $\mathbb{Z}/2 \times \mathbb{Z}/2$ is the smallest group such that its fourth and third cohomology groups are non-zero. This property holds for all the larger groups of this type. Noting that $2 \cdot \overline{H}^*(G_n, \mathbb{Z}) = 0$ we obtain an inclusion $H^*(G_n, \mathbb{Z}) \to H^*(G_n, \mathbb{F}_2)$ given by the modulo 2 reduction map. The image is precisely the kernel of the connecting homomorphism $\delta : H^*(G_n, \mathbb{F}_2) \to H^{*+1}(G_n, \mathbb{Z})$. However, as the modulo 2 reduction map is injective, this is the same as the kernel of $Sq^1 : H^*(G_n, \mathbb{F}_2) \to H^{*+1}(G_n, \mathbb{F}_2)$.

We have that $H^*(G_n, \mathbb{F}_2) \cong \mathbb{F}_2 [x_1, ..., x_n]$ with all the generators of degree 1 and with $Sq^1 x_i = x_i^2$. We can now compute explicitly for G_2 and G_3. We use the sets of letters x, y and x, y, z for the variables. Consider first the G_2 case and the map $\theta^* : H^4(G_2, \mathbb{Z}) \to H^3(G_2, \mathbb{Z})$. All the classes in $H^4(G_2, \mathbb{Z})$ are squares and θ^* is determined on $H^1(G_2, \mathbb{Z})$, hence the coefficient for the unit e must be zero and so $\theta^* = 0$. This means that ${}^{\theta(\phi)}K_{G_2}(G_2) \cong K_{G_2}(G_2)$ for all $\phi \in Z^3(G_2, U(1))$.

We now consider the case G_3 with $H^*(G_3, \mathbb{F}_2) = \mathbb{F}_2 [x, y, z]$ and the cohomology class

$$Sq^1(xyz) = x^2yz + xy^2z + xyz^2$$

which by construction defines an integral class.

Proposition 8.4. *Let $g = x^a y^b z^c$, then*

$$\theta^*(x^2yz + xy^2z + xyz^2)_g = a(y^2z + z^2y) + b(x^2z + xz^2) + c(x^2y + xy^2)$$

and this defines an isomorphism $G_3 \cong H^3(G_3, \mathbb{Z})$.

Hence we see that on this class, the inverse transgression gives rise to non-trivial classes on all components except the untwisted sectors, as expected and desired. For $\phi = x^2yz + xy^2z + xyz^2$ we see that ${}^{\theta(\phi)}K_{G_3}(G_3)$ has an interesting stringy product, which is very different from the untwisted one and the rank of $K_{G_3}(G_3)$ is 64 while the rank of ${}^{\theta(\phi)}K_{G_3}(G_3)$ is 22. Calculations for quotients by abelian groups can be found in [8]. An interesting related product can be found in [15]. For much more on such products in the context of compact Lie groups, we refer the reader to [14].

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