

# 3

## Orbifold K-theory

### 3.1 Introduction

Orbifold K-theory is the K-theory associated to orbifold vector bundles. This can be developed in the full generality of groupoids, but as we have seen in Chapter 1, any effective orbifold can be expressed as the quotient of a smooth manifold by an almost free action of a compact Lie group. Therefore, we can use methods from equivariant topology to study the K-theory of effective orbifolds. In particular, using an appropriate equivariant Chern character, we obtain a decomposition theorem for *orbifold K-theory* as a ring. A byproduct of our orbifold K-theory is a natural notion of orbifold Euler number for a general effective orbifold. What we lose in generality is gained in simplicity and clarity of exposition. Given that all known interesting examples of orbifolds do indeed arise as quotients, we feel that our presentation is fairly broad and will allow the reader to connect orbifold invariants with classical tools from algebraic topology. In order to compute orbifold K-theory, we make use of equivariant Bredon cohomology with coefficients in the representation ring functor. This equivariant theory is the natural target for equivariant Chern characters, and seems to be an important technical device for the study of orbifolds.

A key physical concept in orbifold string theory is twisting by discrete torsion. An important goal of this chapter is to introduce twisting for orbifold K-theory. We introduce *twisted orbifold K-theory* using an explicit geometric model. In the case when the orbifold is a global quotient  $\mathcal{X} = Y/G$ , where  $G$  is a finite group, our construction can be understood as a twisted version of equivariant K-theory, where the twisting is done using a fixed element  $\alpha \in H^2(G; \mathbb{S}^1)$ . The basic idea is to use the associated central extension, and to consider equivariant bundles with respect to this larger group which cover the  $G$ -action on  $Y$ . A computation of the associated twisted theory can be

explicitly obtained (over the complex numbers) using ingredients from the classical theory of projective representations.

More generally we can define a twisted orbifold K-theory associated to the universal orbifold cover; in this generality it can be computed in terms of twisted Bredon cohomology. This can be understood as the  $E_2$ -term of the twisted version of a spectral sequence converging to twisted orbifold K-theory, where in all known instances the higher differentials are trivial in characteristic zero (this is a standard observation in the case of the Atiyah–Hirzebruch spectral sequence). Finally, we should also mention that orbifold K-theory seems like the ideal setting for comparing invariants of an orbifold to that of its resolutions. A basic conjecture in this direction is the following.

**Conjecture 3.1** (K-Orbifold String Theory Conjecture) *If  $\mathcal{X}$  is a complex orbifold and  $Y \rightarrow \mathcal{X}$  is a crepant resolution, then there is a natural additive isomorphism*

$$K(Y) \otimes \mathbb{C} \cong K_{\text{orb}}(\mathcal{X}) \otimes \mathbb{C}$$

*between the orbifold K-theory of  $\mathcal{X}$  and the ordinary K-theory of its crepant resolution  $Y$ .*

Note, for example that if  $\mathcal{X}$  is a complex 3-orbifold with isotropy groups in  $SL_3(\mathbb{C})$ , then it admits a crepant resolution – this condition is automatically satisfied by Calabi–Yau orbifolds.

### 3.2 Orbifolds, group actions, and Bredon cohomology

Our basic idea in studying orbifold K-theory is to apply methods from equivariant topology. In this section, we recall some basic properties of orbifolds and describe how they relate to group actions.

We have seen that if a compact Lie group  $G$  acts smoothly, effectively, and almost freely on a manifold  $M$ , then the quotient  $M/G$  is an effective orbifold. More generally,  $\mathcal{X} = M/G$  is an orbifold for any smooth Lie group action if the following conditions are satisfied:

- For any  $x \in M$ , the isotropy subgroup  $G_x$  is finite.
- For any  $x \in M$  there is a *smooth slice*  $S_x$  at  $x$ .
- For any two points  $x, y \in M$  such that  $y \notin Gx$ , there are slices  $S_x$  and  $S_y$  such that  $GS_x \cap GS_y = \emptyset$ .

If  $G$  is compact, an almost free  $G$ -action automatically satisfies the second and third conditions. Examples arising from proper actions of discrete groups will also appear in our work.

In Chapter 1, we used frame bundles to show that every effective orbifold  $\mathcal{X}$  has an action groupoid presentation  $G \ltimes M$ , where in fact we may take  $G = O(n)$  to be an orthogonal group. Furthermore, we conjectured (Conjecture 1.55) that in fact every orbifold has such a presentation. Therefore, it is no great loss of generality if we restrict our attention to quotient orbifolds of the form  $G \ltimes M$  for (possibly non-effective) almost free actions of a Lie group  $G$  on a smooth manifold  $M$ .

We will assume for simplicity that our orbifolds are compact. In the case of quotient orbifolds  $M/G$  with  $G$  a compact Lie group, this is equivalent to the compactness of  $M$  itself (see [31, p. 38]); a fact the we will use. In order to apply methods from algebraic topology in the study of orbifolds, we recall a well-known result about manifolds with smooth actions of compact Lie groups (see [71]):

**Theorem 3.2** *If a compact Lie group  $G$  acts on a smooth, compact manifold  $M$ , then the manifold is triangulable as a finite  $G$ -CW complex.*

Hence any such manifold will have a cellular  $G$ -action such that the orbit space  $M/G$  has only finitely many cells.

For the rest of this chapter, we will focus on quotient orbifolds  $M/G$ , which as we have seen are quite general. We will consider actions of both compact and discrete groups, using  $G$  to denote a compact Lie group and  $\Gamma$  to denote a discrete group.

In Section 2.3, we defined singular cohomology and characteristic classes for orbifolds. In the case of a quotient  $G \ltimes M$ , the orbifold cohomology coincided with the usual equivariant cohomology  $H^*(EG \times_G M; R)$ . This became the natural home for characteristic classes associated to the orbifold  $M/G$ . However, if  $R$  is a ring such that the order  $|G_x|$  of each isotropy group is invertible in  $R$ , then there is an algebra isomorphism  $H_{orb}^*(G \ltimes M; R) \cong H^*(M/G; R)$ , obtained from a Leray spectral sequence. An appropriate ring  $R$  can be constructed from the integers by inverting the least common multiple of the orders of all the local transformation groups; the rational numbers  $\mathbb{Q}$  are of course also a suitable choice. Thus if  $G \ltimes M$  has all isotropy groups of odd order, we may think of its Stiefel–Whitney classes  $w_i(G \ltimes M)$  as classes in  $H^*(M/G; \mathbb{F}_2)$ . Similarly, if  $G \ltimes M$  is complex, we have Chern classes  $c_i(G \ltimes M) \in H^*(M/G; R)$  for an appropriate ring  $R$ .

More generally, what we see is that with integral coefficients, the equivariant cohomology of  $M$  will have interesting *torsion* classes. Unfortunately, integral computations are notoriously difficult, especially when finite group cohomology is involved. The mod  $p$  equivariant cohomology of  $M$  will contain

interesting information about the action; in particular, its Krull dimension will be equal to the maximal  $p$ -rank of the isotropy subgroups (see [128]). However, for our geometric applications it is convenient to use an equivariant cohomology theory which has substantial *torsion-free* information. That is where K-theory<sup>1</sup> naturally comes in, as instead of cohomology, the basic object is a representation ring.

Less well known than ordinary equivariant cohomology is the *Bredon cohomology* associated to a group action. It is in fact the most adequate equivariant cohomology theory available. We briefly sketch its definition for the case of compact Lie groups, and refer the reader to [30], [101], [63], and [73, appendix].

Let  $\text{Or}(G)$  denote the homotopy category whose objects are the orbit spaces  $G/H$  for subgroups  $H \subseteq G$ , and whose morphisms  $\text{Hom}_{\text{Or}(G)}(G/H, G/K)$  are  $G$ -homotopy classes of  $G$ -maps between these orbits. A *coefficient system* for Bredon cohomology is a functor  $F : \text{Or}(G)^{\text{op}} \rightarrow \text{Ab}$ . For any  $G$ -CW complex  $M$ , define

$$C_*^G(M) : \text{Or}(G) \rightarrow \text{Ab}_*$$

by setting

$$C_*^G(M)(G/H) = C_*(M^H / \mathbf{W}H_0). \quad (3.1)$$

Here  $C_*(-)$  denotes the cellular chain complex functor, and  $\mathbf{W}H_0$  is the identity component of  $\mathbf{N}H/H$ . We now define

$$C_G^*(M; F) = \text{Hom}_{\text{Or}(G)}(C_*^G(M), F) \quad (3.2)$$

and  $H_G^*(M; F) = H(C_G^*(M; F))$ . One can see that for each  $n \geq 0$ , the group  $C_G^n(M; F)$  is the direct product, over all orbits  $G/H \times D^n$  of  $n$ -cells in  $M$ , of the groups  $F(G/H)$ . Moreover,  $C_G^*(M; F)$  is determined on  $\text{Or}(G, M)$ , the full subcategory consisting of the orbit types appearing in  $M$ . From the definitions, there will be a spectral sequence (see [63])

$$E_2 = \text{Ext}_{\text{Or}(G)}^*(\underline{H}_*(M), F) \Rightarrow H_G^*(M; F), \quad (3.3)$$

where  $\underline{H}_*(M)(G/H) = H_*(M^H / \mathbf{W}H_0; \mathbb{Z})$ .

In our applications, the isotropy groups will always be finite. Our basic example will involve the complex representation ring functor  $R(-)$  on  $\text{Or}(G, M)$ ; i.e.,  $G/H \mapsto R(H)$ . In this case, the fact that  $R(H)$  is a ring for each  $H$  implies that Bredon cohomology will have a natural ring structure (constructed using the diagonal).

<sup>1</sup> For background on equivariant K-theory, the reader may consult [142], [101].

We will also use the rationalized functor  $R_{\mathbb{Q}} = R(-) \otimes \mathbb{Q}$ . For  $G$  finite, it is shown in [144] that  $R_{\mathbb{Q}}$  is an injective functor; similarly, when  $\Gamma$  is a discrete group it is shown in [101] that  $R_{\mathbb{Q}}$  is injective for proper actions with finite isotropy. This result will also hold for  $G$ -CW complexes with finite isotropy, where  $G$  is a compact Lie group. This follows by adapting the methods in [144] and is described in [63]. The key technical ingredient is the surjectivity of the homomorphism  $R_{\mathbb{Q}}(H) \rightarrow \lim_{K \in F_p(H)} R_{\mathbb{Q}}(K)$ , where  $H$  is any finite subgroup of  $G$  and  $F_p(H)$  is the family of all proper subgroups in  $H$ . Thus, we have the following basic isomorphism:  $H_G^*(M; R_{\mathbb{Q}}) \cong \text{Hom}_{\text{Or}(G)}(\underline{H}_*(M); R_{\mathbb{Q}})$ .

Suppose that  $\mathcal{X} = M/G$  is a quotient orbifold. Using equivariant K-theory, we will show that the Bredon cohomology  $H_G^*(M; R_{\mathbb{Q}})$  is independent of the presentation  $M/G$ , and canonically associated with the orbifold  $\mathcal{X}$  itself. A direct proof with more general coefficients would be of some interest. In the case of an effective orbifold, we can canonically associate to it the Bredon cohomology of its frame bundle; motivated by this, we introduce the following definition.

**Definition 3.3** Let  $\mathcal{X}$  be a effective orbifold. The *orbifold Bredon cohomology of  $\mathcal{X}$  with  $R_{\mathbb{Q}}$ -coefficients* is  $H_{\text{orb}}^*(\mathcal{X}; R_{\mathbb{Q}}) = H_{O(n)}^*(\text{Fr}(\mathcal{X}); R_{\mathbb{Q}})$ .

### 3.3 Orbifold bundles and equivariant K-theory

In Chapter 2, we introduced the notion of orbifold vector bundles using the language of groupoids. That is, we saw that orbibundles on an orbifold  $\mathcal{X}$  could be described as  $\mathcal{G}$ -vector bundles, where  $\mathcal{G}$  is an orbifold groupoid presentation of  $\mathcal{X}$ . It is apparent that they behave naturally under vector space constructions such as sums, tensor products, exterior products, and so forth.

**Definition 3.4** Given a compact orbifold groupoid  $\mathcal{G}$ , let  $K_{\text{orb}}(\mathcal{G})$  to be the Grothendieck ring of isomorphism classes of  $\mathcal{G}$ -vector bundles on  $\mathcal{G}$ . When  $\mathcal{X}$  is an orbifold, we define  $K_{\text{orb}}(\mathcal{X})$  to be  $K_{\text{orb}}(\mathcal{G})$ , where  $\mathcal{G}$  is any groupoid presentation of  $\mathcal{X}$ .

Recall that under an orbifold morphism  $F : \mathcal{H} \rightarrow \mathcal{G}$ , one can verify that orbifold bundles over  $\mathcal{G}$  pull back to orbifold bundles over  $\mathcal{H}$ . We have the following proposition.

**Proposition 3.5** *Each orbifold morphism  $F : \mathcal{H} \rightarrow \mathcal{G}$  induces a ring homomorphism  $F^* : K_{\text{orb}}(\mathcal{G}) \rightarrow K_{\text{orb}}(\mathcal{H})$ .*

In particular, for Morita equivalent groupoids  $\mathcal{G}$  and  $\mathcal{H}$  we see that  $K_{\text{orb}}(\mathcal{G}) \cong K_{\text{orb}}(\mathcal{H})$ . Thus,  $K_{\text{orb}}(\mathcal{X})$  is well defined.

Of course, an important example of an orbifold morphism is the projection map  $p : M \rightarrow M/G$ , where  $G$  is a compact Lie group acting almost freely on the manifold  $M$ . In this case, if  $E$  is an orbifold vector bundle over  $M/G$ , then  $p^*E$  is a smooth vector bundle over  $M$ . It is obvious that  $p^*E$  is  $G$ -equivariant. Conversely, if  $F$  is a  $G$ -equivariant bundle over  $M$ ,  $F/G \rightarrow M/G$  is an orbifold vector bundle over  $\mathcal{X} = M/G$ . Therefore, we have a canonical identification between  $K_{\text{orb}}(\mathcal{X})$  and  $K_G(M) = K_{\text{orb}}(G \ltimes M)$ .

**Proposition 3.6** *Let  $\mathcal{X} = M/G$  be a quotient orbifold. Then the projection map  $p : M \rightarrow M/G$  induces an isomorphism  $p^* : K_{\text{orb}}(\mathcal{X}) \rightarrow K_G(M)$ .*

**Corollary 3.7** *If  $\mathcal{X}$  is a effective orbifold, we can identify its orbifold K-theory with the equivariant K-theory of its frame bundle.<sup>2</sup>*

It is possible to extend this definition of orbifold K-theory in the usual way; indeed if  $\mathcal{X}$  is an orbifold, then  $\mathcal{X} \times \mathbb{S}^n$  is also an orbifold and, moreover, the inclusion  $i : \mathcal{X} \rightarrow \mathcal{X} \times \mathbb{S}^n$  is an orbifold morphism. Let  $i_n^* : K_{\text{orb}}(\mathcal{X} \times \mathbb{S}^n) \rightarrow K_{\text{orb}}(\mathcal{X})$ ; then we can define  $K_{\text{orb}}^{-n}(\mathcal{X}) = \ker(i_n^*)$ . However, the canonical identification outlined above shows that for a quotient orbifold this extension must agree with the usual extension of equivariant complex K-theory to a  $\mathbb{Z}/2\mathbb{Z}$ -graded theory (i.e., there will be Bott periodicity). Our approach here will be to study orbifold K-theory using equivariant K-theory, as it will enable us to make some meaningful computations. Note that if an orbifold  $\mathcal{X}$  is presented in two different ways as a quotient, say  $M/G \cong \mathcal{X} \cong M'/G'$ , then we have shown that  $K_{\text{orb}}^*(\mathcal{X}) \cong K_G^*(M) \cong K_{G'}^*(M')$ . Another point to make is that the homomorphism  $G \rightarrow G_{\text{eff}}$  will induce a ring map  $K_{\text{orb}}^*(\mathcal{X}_{\text{eff}}) \rightarrow K_{\text{orb}}^*(\mathcal{X})$ .

We also introduce the (K-theoretic) orbifold Euler characteristic.<sup>3</sup>

**Definition 3.8** Let  $\mathcal{X}$  be an orbifold. The *orbifold Euler characteristic* of  $\mathcal{X}$  is

$$\chi_{\text{orb}}(\mathcal{X}) = \dim_{\mathbb{Q}} K_{\text{orb}}^0(\mathcal{X}) \otimes \mathbb{Q} - \dim_{\mathbb{Q}} K_{\text{orb}}^1(\mathcal{X}) \otimes \mathbb{Q}$$

It remains to show that these invariants are tractable, or even well defined.

**Proposition 3.9** *If  $\mathcal{X} = M/G$  is a compact quotient orbifold for a compact Lie group  $G$ , then  $K_{\text{orb}}^*(\mathcal{X})$  is a finitely generated abelian group, and the orbifold Euler characteristic is well defined.*

<sup>2</sup> This has also been proposed by Morava [115], and also appears implicitly in [147].

<sup>3</sup> This definition extends the string-theoretic orbifold Euler characteristic which has been defined for global quotients.

*Proof* We know that  $M$  is a finite, almost free  $G$ -CW complex. It follows from [142] that there is a spectral sequence converging to  $K_{\text{orb}}(\mathcal{X}) = K_G(M)$ , with

$$E_1^{p,q} = \begin{cases} 0 & \text{if } q \text{ is odd,} \\ \bigoplus_{\bar{\sigma} \in X^{(p)}} R(G_\sigma) & \text{otherwise.} \end{cases}$$

Here,  $X^{(p)}$  denotes the collection of  $p$ -cells in the underlying space  $X$  of  $\mathcal{X}$ , and  $R(G_\sigma)$  denotes the complex representation ring of the stabilizer of  $\sigma$  in  $M$ . In fact, the  $E_2$  term is simply the homology of a chain complex assembled from these terms. By our hypotheses, each  $G_\sigma$  is finite, and there are finitely many such cells; hence each term is finitely generated as an abelian group, and there are only finitely many of them. We conclude that  $E_1$  satisfies the required finiteness conditions, and so must its subquotient  $E_\infty$ , whence the same holds for  $K_{\text{orb}}^*(\mathcal{X}) = K_G^*(M)$ .  $\square$

**Corollary 3.10** *With notation as before, we have*

$$\chi_{\text{orb}}(\mathcal{X}) = \sum_{\bar{\sigma} \in X} (-1)^{\dim \bar{\sigma}} \text{rank } R(G_\sigma).$$

The spectral sequence used above is in fact simply the equivariant analog of the Atiyah–Hirzebruch spectral sequence. We have described the  $E_1$ -term as a chain complex assembled from the complex representation rings of the isotropy subgroups. Actually, the  $E_2$ -term coincides with the equivariant Bredon cohomology  $H_G^*(M; R(-))$  of  $M$  described in the previous section, with coefficients in the representation ring functor. In fact this spectral sequence collapses rationally at the  $E_2$ -term (see [101, p. 28]). Consequently,  $H_{\text{orb}}^*(\mathcal{X}; \mathbb{R})$ ,  $K_{\text{orb}}^*(\mathcal{X}) \otimes \mathbb{R}$ , and  $H_G^*(M; R(-) \otimes \mathbb{R})$  are all additively isomorphic. What is more, the last two invariants have the same ring structure (provided that we take the  $\mathbb{Z}/2\mathbb{Z}$ -graded version of Bredon cohomology).

Computations for equivariant K-theory can be quite complicated. Our approach will be to study the case of global quotients arising from actions of finite and discrete groups. The key computational tool will be an equivariant Chern character, which we will define for almost free actions of compact Lie groups. This will be used to establish the additive rational equivalences outlined above. However, we note that  $K_{\text{orb}}(\mathcal{X})$  can contain important *torsion* classes, and so its rationalization is a rather crude approximation.

Let us review the special case of a global quotient, where the K-theoretic invariant above is more familiar.

**Example 3.11** Let  $G$  denote a finite group acting on a manifold  $Y$  and let  $\mathcal{X} = Y/G$ . In this case we know that there is an isomorphism  $K_{\text{orb}}(\mathcal{X}) \cong K_G(Y)$ .

Tensoring with the rationals, the equivariant K-theory decomposes as a direct sum, and we obtain the well-known formula

$$K_{orb}^*(\mathcal{X}) \otimes \mathbb{Q} \cong \bigoplus_{g \in G}^{(g)} K^*(Y^{(g)} / \mathbf{C}_G(g)) \otimes \mathbb{Q}, \tag{3.4}$$

where  $(g)$  is the conjugacy class of  $g \in G$  and  $\mathbf{C}_G(g)$  denotes the centralizer of  $g$  in  $G$ . Note that this decomposition appears in [11], but can be traced back (independently) to [144], [151], and [89].

One of the key elements in the theory of orbifolds is the inertia orbifold  $\wedge \mathcal{X}$  introduced in the previous chapter. In the case of a global quotient  $\mathcal{X} = Y/G$ , it can be shown (see [38]) that we have a homeomorphism

$$|\wedge \mathcal{X}| \cong \bigsqcup_{g \in G}^{(g)} Y^{(g)} / \mathbf{C}_G(g), \tag{3.5}$$

so we see that  $K_{orb}^*(\mathcal{X}) \cong_{\mathbb{Q}} K^*(|\wedge \mathcal{X}|)$ , where  $|\wedge \mathcal{X}|$  is the underlying space of the inertia orbifold  $\wedge \mathcal{X}$ . The conjugacy classes are used to index the so-called *twisted sectors* arising in this decomposition. We will use this as a model for our more general result in the following section.

### 3.4 A decomposition for orbifold K-theory

We will now prove a decomposition for orbifold K-theory using the methods developed by Lück and Oliver in [101]. The basic technical result we will use is the construction of an equivariant Chern character. Cohomology will be assumed  $\mathbb{Z}/2\mathbb{Z}$ -graded in the usual way. We have the following theorem of Adem and Ruan [5].

**Theorem 3.12** *Let  $\mathcal{X} = M/G$  be a compact quotient orbifold, where  $G$  is a compact Lie group. Then there is an equivariant Chern character which defines a rational isomorphism of rings*

$$K_{orb}^*(\mathcal{X}) \cong_{\mathbb{Q}} \prod_{C \subseteq G \text{ cyclic}}^{(C)} [H^*(M^C / \mathbf{C}_G(C)) \otimes \mathbb{Q}(\zeta_{|C|})]^{\mathbf{W}_G(C)},$$

where  $(C)$  ranges over conjugacy classes of cyclic subgroups,  $\zeta_{|C|}$  is a primitive root of unity, and  $\mathbf{W}_G(C) = \mathbf{N}_G(C) / \mathbf{C}_G(C)$ , a necessarily finite group.

*Proof* As has been remarked, we can assume that  $M$  is a finite, almost free  $G$ -CW complex. Now, as in [101] and [11], the main idea of the proof is to

construct a natural Chern character for any  $G$ -space as above, and then prove that it induces an isomorphism for orbits of the form  $G/H$ , where  $H \subset G$  is finite. Using induction on the number of orbit types and a Mayer–Vietoris sequence will then complete the proof.

To begin, we recall the existence (see [101], Prop. 3.4) of a ring homomorphism

$$\psi : K_{\mathbf{N}_G(C)}^*(M^C) \rightarrow K_{\mathbf{C}_G(C)}^*(M^C) \otimes R(C);$$

in this much more elementary setting, it can be defined by its value on vector bundles. Namely,

$$\psi([E]) = \sum_{V \in \text{Irr}(C)} [\text{Hom}_C(V, E)] \otimes [V]$$

for any  $\mathbf{N}_G(C)$ -vector bundle  $E \rightarrow M^C$ . We make use of the natural maps

$$\begin{aligned} K_{\mathbf{C}_G(C)}^*(M^C) \otimes R(C) &\rightarrow K_{\mathbf{C}_G(C)}^*(EG \times M^C) \otimes R(C) \\ &\rightarrow K^*(EG \times_{\mathbf{C}_G(C)} M^C) \otimes R(C), \end{aligned}$$

as well as the Chern map

$$\begin{aligned} K^*(EG \times_{\mathbf{C}_G(C)} M^C) \otimes R(C) &\rightarrow H^*(EG \times_{\mathbf{C}_G(C)} M^C; \mathbb{Q}) \otimes R(C) \\ &\cong H^*(M^C / \mathbf{C}_G(C); \mathbb{Q}) \otimes R(C). \end{aligned}$$

Note that the isomorphism above stems from the crucial fact that all the fibers of the projection map  $EG \times_{\mathbf{C}_G(C)} M^C \rightarrow M^C / \mathbf{C}_G(C)$  are rationally acyclic, as they are classifying spaces of finite groups. Finally, we will make use of the ring map  $R(C) \otimes \mathbb{Q} \rightarrow \mathbb{Q}(\zeta_{|C|})$ ; its kernel is the ideal of elements whose characters vanish on all generators of  $C$ . Putting all of this together and using the restriction map, we obtain a natural ring homomorphism

$$K_G^*(M) \otimes \mathbb{Q} \rightarrow H^*(M^C / \mathbf{C}_G(C); \mathbb{Q}(\zeta_{|C|}))^{\mathbf{N}_G(C) / \mathbf{C}_G(C)}. \quad (3.6)$$

Here we have taken invariants on the right hand side, as the image naturally lands there. Verification of the isomorphism on  $G/H$  is an elementary consequence of the isomorphism  $K_G^*(G/H) \cong R(H)$ , and the details are left to the reader.  $\square$

**Corollary 3.13** *Let  $\mathcal{X} = M/G$  be a compact quotient orbifold. Then there is an additive decomposition*

$$K_{orb}^*(\mathcal{X}) \otimes \mathbb{Q} = K_G^*(M) \otimes \mathbb{Q} \cong \bigoplus_{\substack{(g) \\ g \in G}} K^*(M^{(g)} / \mathbf{C}_G(g)) \otimes \mathbb{Q}.$$

Note that the (finite) indexing set consists of the  $G$ -conjugacy classes of elements in the isotropy subgroups – all of finite order. Thus, just as in the case of a global quotient, we see that the orbifold K-theory of  $\mathcal{X}$  is rationally isomorphic to the ordinary K-theory of the underlying space of the twisted sectors  $\wedge \mathcal{X}$ .

**Theorem 3.14** *Let  $\mathcal{X} = M/G$  denote a compact quotient orbifold. Then there is a homeomorphism*

$$\bigsqcup_{g \in G}^{(g)} M^{(g)} / \mathbf{C}_G(g) \cong |\wedge \mathcal{X}|,$$

and, in particular,  $K_{\text{orb}}^*(\mathcal{X}) \cong_{\mathbb{Q}} K^*(|\wedge \mathcal{X}|)$ .

*Proof* We begin by considering the situation locally. Suppose that we have a chart in  $M$  of the form  $V \times_H G$ , mapping onto  $V/H$  in  $X$ , where by assumption  $H \subset G$  is a finite group. Then

$$\begin{aligned} (V \times_H G)^{(a)} &= \{H(x, u) \mid H(x, ua) = H(x, u)\} \\ &= \{H(x, u) \mid uau^{-1} = h \in H, x \in V^{(h)}\}. \end{aligned}$$

Let us now define an  $H$  action on  $\bigsqcup_{t \in H} (V^{(t)}, t)$  by  $k(x, t) = (kx, kt k^{-1})$ . We define a map

$$\phi : (V \times_H G)^{(g)} \rightarrow \bigsqcup_{t \in H} (V^{(t)}, t) / H$$

by  $\phi(H(x, u)) = [x, ugu^{-1}]$ . We check that this is well defined: indeed, if  $H(x, u) = H(y, v)$  then there is a  $k \in H$  with  $(y, v) = k(x, u)$ , so  $y = kx, v = ku$ . This means that  $vgv^{-1} = kugv^{-1}k^{-1}$ , and so  $[y, \overline{v}gv^{-1}] = [kx, kugv^{-1}k^{-1}] = [x, ugu^{-1}]$  as  $k \in H$ . Now suppose that  $z \in \mathbf{C}_G(g)$ ; then  $\phi(H(x, uz)) = \phi(H(x, uz)) = [x, uzgz^{-1}u^{-1}] = [x, ugu^{-1}] = \phi(H(x, u))$ ; hence we have a well-defined map on the orbit space

$$\overline{\phi} : (V \times_H G)^{(g)} / \mathbf{C}_G(g) \rightarrow \bigsqcup_{t \in H} (V^{(t)}, t) / H.$$

This map turns out to be injective. Indeed, if  $(x, ugu^{-1}) = k(y, vgv^{-1})$  for some  $k \in H$ , then  $x = ky$  and  $g = u^{-1}kvgv^{-1}k^{-1}u$ , hence  $u^{-1}kv \in \mathbf{C}_G(g)$  and  $H(x, u)(u^{-1}kv) = H(x, kv) = H(ky, kv) = H(y, v)$ . The image of  $\overline{\phi}$  consists of the  $H$ -equivalence classes of pairs  $(x, t)$ , where  $x \in V^{(t)}$  and  $t$  is conjugate to  $g$  in  $G$ .

Putting this together and noting that  $(V \times_H G)^{(g)} = \emptyset$  unless  $g$  is conjugate to an element in  $H$ , we observe that we obtain a homeomorphism

$$\bigsqcup_{\substack{(g) \\ g \in G}} (V \times_H G)^{(g)} / C_G(g) \cong \bigsqcup_{t \in H} (V^{(t)}, t) / H \cong \bigsqcup_{\substack{(t) \\ t \in H}} V^{(t)} / C_H(t).$$

To complete the proof of the theorem it suffices to observe that by the compatibility of charts, the local homeomorphisms on fixed-point sets can be assembled to yield the desired global homeomorphism on  $M$ .  $\square$

**Remark 3.15** Alternatively, the theorem is an easy consequence of the fact that the translation groupoid  $\wedge G \ltimes M = G \ltimes \bigsqcup_{g \in G} M^{(g)}$  is Morita equivalent to the groupoid  $\bigsqcup_{(g)} C_G(g) \ltimes M^{(g)}$ . In fact, the inclusion of the latter into the former is an equivalence. Thus, their quotient spaces must be homeomorphic.

**Remark 3.16** We can compose the result above with the ordinary Chern character on  $|\wedge \mathcal{X}|$  to obtain a stringy Chern character

$$\text{ch} : K_{\text{orb}}^*(\mathcal{X}) \otimes \mathbb{C} \rightarrow H^*(|\wedge \mathcal{X}|; \mathbb{C}). \tag{3.7}$$

In fact, this is an isomorphism of graded abelian groups (where we take  $\mathbb{Z}/2\mathbb{Z}$ -graded cohomology on the right hand side). Note that  $H^*(|\wedge \mathcal{X}|; \mathbb{C})$  arises naturally as the target of the stringy Chern character. At this point, we only consider the additive structure of  $H^*(|\wedge \mathcal{X}|; \mathbb{C})$ ; in Chapter 4, we will endow it with a different grading and a stringy cup product. The resulting ring is often referred to as the *Chen–Ruan cohomology ring*.

**Corollary 3.17** We have  $\chi_{\text{orb}}(\mathcal{X}) = \chi(|\wedge \mathcal{X}|)$ .

**Example 3.18** We will now consider the case of the weighted projective space  $\mathbb{W}\mathbb{P}(p, q)$ , where  $p$  and  $q$  are assumed to be distinct prime numbers. Recall that  $\mathbb{W}\mathbb{P}(p, q) = \mathbb{S}^3 / \mathbb{S}^1$ , where  $\mathbb{S}^1$  acts on the unit sphere  $\mathbb{S}^3 \subset \mathbb{C}^2$  via  $\lambda(v, w) = (\lambda^p v, \lambda^q w)$ . There are two singular points,  $x = [1, 0]$  and  $y = [0, 1]$ , with corresponding isotropy subgroups  $\mathbb{Z}/p\mathbb{Z}$  and  $\mathbb{Z}/q\mathbb{Z}$ . The fixed-point sets are disjoint circles in  $\mathbb{S}^3$ , hence the formula for the orbifold  $K$ -theory yields

$$K_{\text{orb}}^*(\mathbb{W}\mathbb{P}(p, q)) \cong_{\mathbb{Q}} \mathbb{Q}(\zeta_p) \times \mathbb{Q}(\zeta_q) \times \Lambda(b_2), \tag{3.8}$$

where  $\zeta_p$  and  $\zeta_q$  are the corresponding primitive roots of unity (compare with Corollary 2.7.6 in [9]). More explicitly, we have an isomorphism

$$K_{\text{orb}}^*(\mathbb{W}\mathbb{P}(p, q)) \otimes \mathbb{Q} \cong \mathbb{Q}[x] / (x^{p-1} + x^{p-2} + \dots + x + 1) \\ \times (x^{q-1} + x^{q-2} + \dots + x + 1)(x^2),$$

from which we see that the orbifold Euler characteristic is  $\chi_{\text{orb}}(\mathbb{W}\mathbb{P}(p, q)) = p + q$ .

**Remark 3.19** The decomposition described above is based on entirely analogous results for proper actions of discrete groups (see [101]). In particular, this includes the case of *arithmetic orbifolds*, also discussed in [3] and [76]. Let  $G(\mathbb{R})$  denote a semi-simple  $\mathbb{Q}$ -group, and  $K$  a maximal compact subgroup. Let  $\Gamma \subset G(\mathbb{Q})$  denote an arithmetic subgroup. Then  $\Gamma$  acts on  $X = G(\mathbb{R})/K$ , a space diffeomorphic to a Euclidean space. Moreover, if  $H$  is any finite subgroup of  $\Gamma$ , then  $X^H$  is a totally geodesic submanifold, hence also diffeomorphic to a Euclidean space. We can make use of the Borel–Serre completion  $\bar{X}$  (see [25]). This is a contractible space with a proper  $\Gamma$ -action such that the  $\bar{X}^H$  are also contractible (we are indebted to Borel and Prasad for outlining a proof of this in [24]) but having a compact orbit space  $\Gamma \backslash \bar{X}$ . In this case, we obtain the multiplicative formula

$$K_{\Gamma}^*(X) \otimes \mathbb{Q} \cong K_{\Gamma}^*(\bar{X}) \otimes \mathbb{Q} \cong \prod_{\substack{(C) \\ C \subset \Gamma \text{ cyclic}}} H^*(B\mathbf{C}_{\Gamma}(C); \mathbb{Q}(\zeta_{|C|}))^{\mathbf{N}_{\Gamma}(C)}.$$

This allows us to express the orbifold Euler characteristic of  $\Gamma \backslash X$  in terms of group cohomology:

$$\chi_{\text{orb}}(\Gamma \backslash X) = \sum_{\substack{(\gamma) \\ \gamma \in \Gamma \text{ of finite order}}} \chi(B\mathbf{C}_{\Gamma}(\gamma)). \tag{3.9}$$

**Example 3.20** Another example of some interest is that of compact, two-dimensional, hyperbolic orbifolds. They are described as quotients of the form  $\Gamma \backslash PSL_2(\mathbb{R})/SO(2)$ , where  $\Gamma$  is a Fuchsian subgroup. The groups  $\Gamma$  can be expressed as extensions of the form

$$1 \rightarrow \Gamma' \rightarrow \Gamma \rightarrow G \rightarrow 1,$$

where  $\Gamma'$  is the fundamental group of a closed orientable Riemann surface, and  $G$  is a finite group (i.e., they are *virtual surface groups*). Geometrically, we have an action of  $G$  on a surface  $\Sigma$  with fundamental group  $\Gamma'$ ; this action has isolated singular points with cyclic isotropy. The group  $\Gamma$  is  $\pi_1(EG \times_G \Sigma)$ , which coincides with the orbifold fundamental group. Assume that  $G$  acts on  $\Sigma$  with  $n$  orbits of cells, having respective isotropy groups  $\mathbb{Z}/v_1\mathbb{Z}, \dots, \mathbb{Z}/v_n\mathbb{Z}$ , and with quotient a surface  $W$  of genus equal to  $g$ . The formula then yields (compare with the description in [105, p. 563])

$$K_{\text{orb}}^*(W) \otimes \mathbb{Q} \cong \tilde{R}(\mathbb{Z}/v_1\mathbb{Z}) \otimes \mathbb{Q} \times \dots \times \tilde{R}(\mathbb{Z}/v_n\mathbb{Z}) \otimes \mathbb{Q} \times K^*(W) \otimes \mathbb{Q}.$$

In this expression,  $\tilde{R}$  denotes the reduced representation ring, which arises because the trivial cyclic subgroup only appears once. From this we see that

$$\dim_{\mathbb{Q}} K_{\text{orb}}^0(W) \otimes \mathbb{Q} = \sum_{i=1}^n (v_i - 1) + 2, \quad \dim_{\mathbb{Q}} K_{\text{orb}}^1(W) \otimes \mathbb{Q} = 2g,$$

and so  $\chi_{\text{orb}}(W) = \sum_{i=1}^n (v_i - 1) + \chi(W)$ .

**Remark 3.21** This decomposition formula is analogous to the decomposition of equivariant algebraic K-theory which appears in work of Vezzosi and Vistoli [157, p. 5] and Toen (see [150, p. 29] and [149, p. 49]) in the context of algebraic Deligne–Mumford stacks. Under suitable conditions, Toen obtains rational isomorphisms between the G-theory of a Deligne–Mumford stack and that of its inertia stack. Vezzosi and Vistoli, on the other hand, express the equivariant algebraic K-theory  $K_*(X, G)$  of an affine group scheme of finite type over  $k$  acting on a Noetherian regular separated algebraic space  $X$  in terms of fixed-point data, again under suitable hypotheses (and after inverting some primes). A detailed comparison of these with the topological splitting above would seem worthwhile.

**Remark 3.22** It should also be observed that the decomposition above could equally well have been stated in terms of the computation of Bredon cohomology mentioned previously, i.e.,  $H_G^*(M, R_{\mathbb{Q}}) \cong \text{Hom}_{\text{Or}(G)}(\underline{H}_*(M); R_{\mathbb{Q}})$  and the collapse at  $E_2$  of the rationalized Atiyah–Hirzebruch spectral sequence:  $K_{\text{orb}}^*(\mathcal{X}) \otimes \mathbb{Q} \cong H_G^*(M; R_{\mathbb{Q}})$ . It had been previously shown that a Chern character with expected naturality properties inducing such an isomorphism cannot exist; in particular [63] contains an example where such an isomorphism is impossible. However, the example is for a circle action with stationary points, our result<sup>4</sup> shows that *almost free* actions of compact Lie groups do indeed give rise to appropriate equivariant Chern characters. A different equivariant Chern character for abelian Lie group actions was defined in [18], using a  $\mathbb{Z}/2\mathbb{Z}$ -indexed de Rham cohomology (called *delocalized equivariant cohomology*). Presumably it must agree with our decomposition in the case of almost free actions. Nistor [121] and Block and Getzler [22] have pointed out an alternative approach using cyclic cohomology.

**Remark 3.23** If  $\mathcal{X} = M/G$  is a quotient orbifold, then the K-theory of  $EG \times_G M$  and the orbifold K-theory are related by the Atiyah–Segal Completion Theorem in [10]. Considering the equivariant K-theory  $K_G^*(M)$  as a module

<sup>4</sup> Moerdijk has informed us that in unpublished work (1996), he and Svensson obtained essentially the same Chern character construction as that appearing in this chapter.

over  $R(G)$ , it states that  $K^*(EG \times_G M) \cong K_G^*(M)^\wedge$ , where the completion is taken at the augmentation ideal  $I \subset R(G)$ .

### 3.5 Projective representations, twisted group algebras, and extensions

We will now extend many of the constructions and concepts used previously to an appropriately *twisted* setting. This twisting occurs naturally in the framework of mathematical physics. In this section, we will always assume that we are dealing with finite groups, unless stated otherwise. Most of the background results which we list appear in [79, Chapt. III].

**Definition 3.24** Let  $V$  denote a finite-dimensional complex vector space. A mapping  $\rho : G \rightarrow GL(V)$  is called a *projective representation* of  $G$  if there exists a function  $\alpha : G \times G \rightarrow \mathbb{C}^*$  such that  $\rho(x)\rho(y) = \alpha(x, y)\rho(xy)$  for all  $x, y \in G$  and  $\rho(1) = \text{Id}_V$ .

Note that  $\alpha$  defines a  $\mathbb{C}^*$ -valued cocycle on  $G$ , i.e.,  $\alpha \in Z^2(G; \mathbb{C}^*)$ . Also, there is a one-to-one correspondence between projective representations of  $G$  as above and homomorphisms from  $G$  to  $PGL(V)$ . We will be interested in the notion of *linear equivalence* of projective representations.

**Definition 3.25** Two projective representations  $\rho_1 : G \rightarrow GL(V_1)$  and  $\rho_2 : G \rightarrow GL(V_2)$  are said to be *linearly equivalent* if there exists a vector space isomorphism  $f : V_1 \rightarrow V_2$  such that  $\rho_2(g) = f\rho_1(g)f^{-1}$  for all  $g \in G$ .

If  $\alpha$  is the cocycle attached to  $\rho$ , we say that  $\rho$  is an  $\alpha$ -representation on the space  $V$ . We list a few basic results regarding these structures.

**Lemma 3.26** Let  $\rho_i$  (for  $i = 1, 2$ ) be an  $\alpha_i$ -representation on the space  $V_i$ . If  $\rho_1$  is linearly equivalent to  $\rho_2$ , then  $\alpha_1$  is equal to  $\alpha_2$ .

It is easy to see that given a fixed cocycle  $\alpha$ , we can take the direct sum of any two  $\alpha$ -representations.

**Definition 3.27** We define  $M_\alpha(G)$  to be the monoid of linear isomorphism classes of  $\alpha$ -representations of  $G$ . Its associated Grothendieck group will be denoted  $R_\alpha(G)$ .

In order to use these constructions, we need to introduce the notion of a *twisted group algebra*. If  $\alpha : G \times G \rightarrow \mathbb{C}^*$  is a cocycle, we denote by  $\mathbb{C}^\alpha G$  the vector space over  $\mathbb{C}$  with basis  $\{\bar{g} \mid g \in G\}$  and product  $\bar{x} \cdot \bar{y} = \alpha(x, y)\bar{xy}$

extended distributively. One can check that  $\mathbb{C}^\alpha G$  is a  $\mathbb{C}$ -algebra with  $\bar{1}$  as the identity element. This algebra is called the  $\alpha$ -twisted group algebra of  $G$  over  $\mathbb{C}$ . Note that if  $\alpha(x, y) = 1$  for all  $x, y \in G$ , then  $\mathbb{C}^\alpha G = \mathbb{C}G$  is the usual group algebra.

**Definition 3.28** If  $\alpha$  and  $\beta$  are cocycles, then  $\mathbb{C}^\alpha G$  and  $\mathbb{C}^\beta G$  are equivalent if there exists a  $\mathbb{C}$ -algebra isomorphism  $\psi : \mathbb{C}^\alpha G \rightarrow \mathbb{C}^\beta G$  and a mapping  $t : G \rightarrow \mathbb{C}^*$  such that  $\psi(\bar{g}) = t(g)\bar{g}$  for all  $g \in G$ , where  $\{\bar{g}\}$  and  $\{\tilde{g}\}$  are bases for the two twisted algebras.

We have a basic result which classifies these twisted group algebras.

**Theorem 3.29** We have an isomorphism  $\mathbb{C}^\alpha G \cong \mathbb{C}^\beta G$  between twisted group algebras if and only if  $\alpha$  is cohomologous to  $\beta$ ; hence if  $\alpha$  is a coboundary,  $\mathbb{C}^\alpha G \cong \mathbb{C}G$ . Indeed,  $\alpha \mapsto \mathbb{C}^\alpha G$  induces a bijective correspondence between  $H^2(G; \mathbb{C}^*)$  and the set of equivalence classes of twisted group algebras of  $G$  over  $\mathbb{C}$ .

Next we recall how these twisted algebras play a role in determining  $R_\alpha(G)$ .

**Theorem 3.30** There is a bijective correspondence between  $\alpha$ -representations of  $G$  and  $\mathbb{C}^\alpha G$ -modules. This correspondence preserves sums and bijectively maps linearly equivalent (respectively irreducible, completely reducible) representations into isomorphic (respectively irreducible, completely reducible) modules.

**Definition 3.31** Let  $\alpha \in Z^2(G; \mathbb{C}^*)$ . An element  $g \in G$  is said to be  $\alpha$ -regular if  $\alpha(g, x) = \alpha(x, g)$  for all  $x \in \mathbf{C}_G(g)$ .

Note that the identity element is  $\alpha$ -regular for all  $\alpha$ . Also, one can see that  $g$  is  $\alpha$ -regular if and only if  $\bar{g} \cdot \bar{x} = \bar{x} \cdot \bar{g}$  for all  $x \in \mathbf{C}_G(g)$ .

If an element  $g \in G$  is  $\alpha$ -regular, then any conjugate of  $g$  is also  $\alpha$ -regular. Therefore, we can speak of  $\alpha$ -regular conjugacy classes in  $G$ . For technical purposes, we also want to introduce the notion of a *standard cocycle*. A cocycle  $\alpha$  is standard if (1)  $\alpha(x, x^{-1}) = 1$  for all  $x \in G$ , and (2)  $\alpha(x, g)\alpha(xg, x^{-1}) = 1$  for all  $\alpha$ -regular  $g \in G$  and all  $x \in G$ . In other words,  $\alpha$  is standard if and only if for all  $x \in G$  and for all  $\alpha$ -regular elements  $g \in G$ , we have  $\bar{x}^{-1} = \overline{x^{-1}}$  and  $\bar{x} \bar{g} \bar{x}^{-1} = \overline{ngx^{-1}}$ . It turns out that any cohomology class  $c \in H^2(G; \mathbb{C}^*)$  can be represented by a standard cocycle, so from now on we will make this assumption.

The next result is basic.

**Theorem 3.32** If  $r_\alpha$  is equal to the number of non-isomorphic irreducible  $\mathbb{C}^\alpha G$ -modules, then this number is equal to the number of distinct  $\alpha$ -regular

conjugacy classes of  $G$ . In particular,  $R_\alpha(G)$  is a free abelian group of rank equal to  $r_\alpha$ .

In what follows we will be using cohomology classes in  $H^2(G; \mathbb{S}^1)$ , where the  $G$ -action on the coefficients is assumed to be trivial. Note that  $H^2(G; \mathbb{S}^1) \cong H^2(G; \mathbb{C}^*) \cong H^2(G; \mathbb{Q}/\mathbb{Z}) \cong H^3(G; \mathbb{Z})$ . We will always use standard cocycles to represent any given cohomology class.

An element  $\alpha \in H^2(G; \mathbb{S}^1)$  corresponds to an equivalence class of group extensions

$$1 \rightarrow \mathbb{S}^1 \rightarrow \tilde{G}_\alpha \rightarrow G \rightarrow 1.$$

The group  $\tilde{G}_\alpha$  can be given the structure of a compact Lie group, where  $\mathbb{S}^1 \rightarrow \tilde{G}_\alpha$  is the inclusion of a closed subgroup. The elements in the extension group can be represented by pairs  $\{(g, a) \mid g \in G, a \in \mathbb{S}^1\}$  with the product  $(g_1, a_1)(g_2, a_2) = (g_1 g_2, \alpha(g_1, g_2) a_1 a_2)$ .

Consider the case when  $z \in \mathbf{C}_G(g)$ ; then we can compute the following commutator of lifts:

$$\begin{aligned} (z, 1)(g, 1)[(g, 1)(z, 1)]^{-1} &= (zg, \alpha(z, g))(z^{-1}g^{-1}, \alpha(g, z)^{-1}) \\ &= (1, \alpha(zg, (zg)^{-1})\alpha(z, g)\alpha(g, z)) \\ &= (1, \alpha(z, g)\alpha(g, z)^{-1}). \end{aligned}$$

This computation is independent of the choice of lifts. It can be seen that this defines a character  $\gamma_g^\alpha$  for the centralizer  $\mathbf{C}_G(g)$  via the correspondence  $z \mapsto \alpha(z, g)\alpha(g, z)^{-1}$ . This character is trivial if and only if  $g$  is  $\alpha$ -regular.

There is a one-to-one correspondence between isomorphism classes of representations of  $\tilde{G}_\alpha$  which restrict to scalar multiplication on the central  $\mathbb{S}^1$  and isomorphism classes of  $\alpha$ -representations of  $G$ . If  $\psi : \tilde{G}_\alpha \rightarrow GL(V)$  is such a representation, then we define an associated  $\alpha$ -representation via  $\rho(g) = \psi(g, 1)$ . Note that

$$\begin{aligned} \rho(gh) &= \psi(gh, 1) = \alpha(g, h)^{-1} \psi(gh, \alpha(g, h)) = \alpha(g, h)^{-1} \psi((g, 1)(h, 1)) \\ &= \alpha(g, h)^{-1} \rho(g)\rho(h). \end{aligned}$$

Conversely, given  $\rho : G \rightarrow GL(V)$ , we simply define  $\psi(g, a) = a\rho(g)$ ; note that

$$\begin{aligned} \psi((g, a)(h, b)) &= \psi(gh, \alpha(g, h)ab) = ab\rho(g)\rho(h) = a\rho(g)b\rho(h) \\ &= \psi(g, a)\psi(h, b). \end{aligned}$$

Therefore, we can identify  $R_\alpha(G)$  with the subgroup of  $R(\tilde{G}_\alpha)$  generated by representations that restrict to scalar multiplication on the central  $\mathbb{S}^1$ .

In the next section we will need an explicit understanding of the action of  $\widetilde{\mathbf{C}}_G(g)_\alpha$  on  $R_{\text{Res}(\alpha)}(\langle g \rangle)$ , where  $\text{Res}(\alpha)$  is the restriction of the cocycle to the subgroup  $\langle g \rangle$  (this restriction is cohomologous to zero). It is easiest to describe using the formulation above. Given a representation  $\phi$  for  $\widetilde{\langle g \rangle}_\alpha$ , an element  $(z, a) \in \widetilde{\mathbf{C}}_G(g)_\alpha$ , and  $(x, b) \in \widetilde{\langle g \rangle}_\alpha$ , we define  $(z, a)\phi(x, b) = \phi((z, a)(x, b)(z, a)^{-1})$ . Notice that this value is precisely  $\gamma_x^\alpha(z)\phi(x, b)$ ; this is independent of the choice of lifting and defines an action of  $\mathbf{C}_G(g)$ . For  $x, y \in \langle g \rangle$  we have  $\gamma_x^\alpha(z)\gamma_y^\alpha(z) = \gamma_{xy}^\alpha(z)$ . In particular, if  $g^n = 1$ , we have  $(\gamma_g^\alpha(z))^n = 1$ . The correspondence  $x \mapsto \gamma_x^\alpha(z)$  defines a character  $L^\alpha(z)$  for  $\langle g \rangle$ , whence the action is best described as sending an  $\alpha$ -representation  $\rho$  to  $L^\alpha(z)\rho$ . Note that the evaluation  $\phi \mapsto \text{tr}(\phi(g, 1))$  defines a  $\mathbb{C}\mathbf{C}_G(g)$ -homomorphism  $u : R_{\text{Res}(\alpha)}(\langle g \rangle) \otimes \mathbb{C} \rightarrow \gamma_g^\alpha$ .

### 3.6 Twisted equivariant K-theory

We are now ready to define a twisted version of equivariant K-theory for global quotients.<sup>5</sup> We assume as before that  $G$  is a finite group. Now suppose we are given a class  $\alpha \in Z^2(G; \mathbb{S}^1)$  and the compact Lie group extension which represents it,  $1 \rightarrow \mathbb{S}^1 \rightarrow \widetilde{G}_\alpha \rightarrow G \rightarrow 1$ ; finally, let  $X$  be a finite  $G$ -CW complex.

**Definition 3.33** An  $\alpha$ -twisted  $G$ -vector bundle on  $X$  is a complex vector bundle  $E \rightarrow X$  together with an action of  $\widetilde{G}_\alpha$  on  $E$  such that  $\mathbb{S}^1$  acts on the fibers through complex multiplication and the action covers the given  $G$ -action on  $X$ .

One may view such a bundle  $E \rightarrow X$  as a  $\widetilde{G}_\alpha$ -vector bundle, where the action on the base is via the projection onto  $G$  and the given  $G$ -action. Note that if we divide out by the action of  $\mathbb{S}^1$ , we obtain a *projective* bundle over  $X$ . These twisted bundles can be added, forming a monoid.

**Definition 3.34** The  $\alpha$ -twisted  $G$ -equivariant K-theory of  $X$ , denoted by  ${}^\alpha K_G(X)$ , is defined as the Grothendieck group of isomorphism classes of  $\alpha$ -twisted  $G$ -bundles over  $X$ .

As with  $\alpha$ -representations, we can describe this twisted group as the subgroup of  $K_{\widetilde{G}_\alpha}(X)$  generated by isomorphism classes of bundles that restrict to multiplication by scalars on the central  $\mathbb{S}^1$ . As the  $\mathbb{S}^1$ -action on  $X$  is trivial,

<sup>5</sup> By now there are many different versions of twisted K-theory; we refer the reader to [55] for a succinct survey, as well as connections to the Verlinde algebra.

we have a natural isomorphism  $K_{\mathbb{S}^1}(X) \cong K(X) \otimes R(\mathbb{S}^1)$ . Composing the restriction with the map  $K(X) \otimes R(\mathbb{S}^1) \rightarrow R(\mathbb{S}^1)$ , we obtain a homomorphism  $K_{\tilde{G}_\alpha}(X) \rightarrow R(\mathbb{S}^1)$ ; we can define  ${}^\alpha K_G(X)$  as the inverse image of the subgroup generated by the representations defined by scalar multiplication.

Just as in non-twisted equivariant K-theory, this definition extends to a  $\mathbb{Z}/2\mathbb{Z}$ -graded theory. In fact we can define  ${}^\alpha K_G^0(X) = {}^\alpha K_G(X)$  and  ${}^\alpha K_G^1(X) = \ker[{}^\alpha K_G(\mathbb{S}^1 \times X) \rightarrow {}^\alpha K_G(X)]$ . We can also extend the description given above to express  ${}^\alpha K_G^*(X)$  as a subgroup of  $K_{\tilde{G}_\alpha}^*(X)$ .

We begin by considering the case  $\alpha = 0$ ; this corresponds to the split extension  $G \times \mathbb{S}^1$ . Any ordinary  $G$ -vector bundle can be made into a  $G \times \mathbb{S}^1$ -bundle via scalar multiplication on the fibers; conversely, a  $G \times \mathbb{S}^1$ -bundle restricts to an ordinary  $G$ -bundle. Hence we readily see that  ${}^\alpha K_G^*(X) = K_G^*(X)$ .

**Theorem 3.35** *Let  $\alpha$  and  $\beta$  be cocycles. If  $\alpha = \beta\gamma$  for a coboundary  $\gamma$ , there is an isomorphism*

$$\kappa_\gamma : {}^\alpha K_G^*(X) \rightarrow {}^\beta K_G^*(X).$$

As a consequence,  $H^1(G; \mathbb{S}^1)$  acts as automorphisms of  ${}^\alpha K_G^*(X)$ .

Now we consider the case when  $X$  is a trivial  $G$ -space.

**Lemma 3.36** *Let  $X$  denote a CW-complex with a trivial  $G$ -action; then there is a natural isomorphism  $K(X) \otimes R_\alpha(G) \rightarrow {}^\alpha K_G(X)$ .*

*Proof* This result is the analog of the untwisted version (see [142, p. 133]). The natural map  $R(\tilde{G}_\alpha) \rightarrow K_{\tilde{G}_\alpha}(X)$  can be combined with the map  $K(X) \rightarrow K_{\tilde{G}_\alpha}(X)$  (which gives any vector bundle the trivial  $G$ -action) to yield a natural isomorphism  $K(X) \otimes R(\tilde{G}_\alpha) \rightarrow K_{\tilde{G}_\alpha}(X)$  which covers the restriction to the  $\mathbb{S}^1$ -action; the result follows from looking at inverse images of the subgroup generated by the scalar representation. □

The inverse of the map above is given by

$$[E] \mapsto \bigoplus_{\{[M] \in \text{Irr}(\tilde{G}_\alpha)\}} [\text{Hom}_{\tilde{G}_\alpha}(M, E)] \otimes [M].$$

Note that only the  $M$  which restrict to scalar multiplication on  $\mathbb{S}^1$  are relevant – these are precisely the irreducible  $\alpha$ -representations.

Let  $X$  be a  $G$ -space and  $Y$  a  $G'$ -space, and let  $h : G \rightarrow G'$  denote a group homomorphism. If  $f : X \rightarrow Y$  is a continuous map equivariant with respect to this homomorphism, we obtain a map  ${}^\alpha f^* : {}^\alpha K_{G'}(Y) \rightarrow {}^{h^*(\alpha)} K_G(X)$ , where  $h^* : H^2(G'; \mathbb{S}^1) \rightarrow H^2(G; \mathbb{S}^1)$  is the map induced by  $h$  in cohomology. Let  $H \subseteq G$  be a subgroup; the inclusion defines a restriction map  $H^2(G; \mathbb{S}^1) \rightarrow$

$H^2(H; \mathbb{S}^1)$ . In fact, if  $\tilde{G}_\alpha$  is the group extension defined by  $\alpha \in H^2(G; \mathbb{S}^1)$ , then  $\text{Res}_H^G(\alpha)$  defines the “restricted” group extension over  $H$ , denoted  $\tilde{H}_{\text{Res}(\alpha)}$ ; we have a restriction map  ${}^\alpha K_G(X) \rightarrow {}^{\text{Res}(\alpha)} K_H(X)$ .

Now consider the case of an orbit  $G/H$ . We claim that  ${}^\alpha K_G(G/H) = R_{\text{Res}_H^G(\alpha)}(H)$ . Indeed, we can identify  $K_{\tilde{G}_\alpha}(G/H) = K_{\tilde{G}_\alpha}(\tilde{G}_\alpha/\tilde{H}_\alpha) \cong R(\tilde{H}_\alpha)$ , and by restricting to the representations that induce scalar multiplication on  $\mathbb{S}^1$ , we obtain the result.

We are now ready to state a basic decomposition theorem for our twisted version of equivariant K-theory.

**Theorem 3.37** *Let  $G$  denote a finite group and  $X$  a finite  $G$ -CW complex. For any  $\alpha \in Z^2(G; \mathbb{S}^1)$ , we have a decomposition*

$${}^\alpha K_G^*(X) \otimes \mathbb{C} \cong \bigoplus_{\substack{(g) \\ g \in G}} (K^*(X^{(g)}) \otimes \gamma_g^\alpha)^{\mathbf{C}_G(g)}.$$

*Proof* Fix the class  $\alpha \in Z^2(G, \mathbb{S}^1)$ . To any subgroup  $H \subset G$ , we can associate  $H \mapsto R_{\text{Res}(\alpha)}(H)$ . Note the special case when  $H = \langle g \rangle$ , a cyclic subgroup. As  $H^2(\langle g \rangle; \mathbb{S}^1) = 0$ , the group  $R_{\text{Res}(\alpha)}(\langle g \rangle)$  is isomorphic to  $R(\langle g \rangle)$ .

Now consider  $E \rightarrow X$ , an  $\alpha$ -twisted bundle over  $X$ ; it restricts to a  $\text{Res}(\alpha)$ -twisted bundle over  $X^{(g)}$ . Recall that we have an isomorphism  $\text{Res}(\alpha) K_{\langle g \rangle}^*(X^{(g)}) \cong K^*(X^{(g)}) \otimes R_{\text{Res}(\alpha)}(\langle g \rangle)$ . Let  $u : R_{\text{Res}(\alpha)}(\langle g \rangle) \rightarrow \gamma_g^\alpha$  denote the  $\mathbb{C} \mathbf{C}_G(g)$ -map  $\chi \mapsto \chi(g)$  described previously, where the centralizer acts on the projective representations as described above. Then the composition

$$\begin{aligned} {}^\alpha K_G^*(X) \otimes \mathbb{C} &\xrightarrow{\text{Res}(\alpha)} K_{\langle g \rangle}^*(X^{(g)}) \otimes \mathbb{C} \rightarrow K^*(X^{(g)}) \otimes R_{\text{Res}(\alpha)}(\langle g \rangle) \otimes \mathbb{C} \\ &\rightarrow K^*(X^{(g)}) \otimes \gamma_g^\alpha \end{aligned}$$

has its image lying in the invariants under the  $\mathbf{C}_G(g)$ -action. Hence we can put these together to yield a map

$${}^\alpha K_G^*(X) \otimes \mathbb{C} \rightarrow \bigoplus_{(g)} (K^*(X^{(g)}) \otimes \gamma_g^\alpha)^{\mathbf{C}_G(g)}.$$

One checks that this induces an isomorphism on orbits  $G/H$ ; the desired isomorphism follows from using induction on the number of  $G$ -cells in  $X$  and a Mayer–Vietoris argument (as in [11]).  $\square$

Note that in the case when  $X$  is a point, we are saying that  $R_\alpha(G) \otimes \mathbb{C}$  has rank equal to the number of conjugacy classes of elements in  $G$  such that the associated character  $\gamma_g^\alpha$  is trivial. This of course agrees with the number of  $\alpha$ -regular conjugacy classes, as indeed  ${}^\alpha K_G(\text{pt}) = R_\alpha(G)$ .

**Remark 3.38** It is apparent that the constructions introduced in this section can be extended to the case of a proper action on  $X$  of a discrete group  $\Gamma$ . The group extensions and vector bundles used for the finite group case have natural analogs, and so we can define  ${}^\alpha K_\Gamma^*(X)$  for  $\alpha \in H^2(\Gamma; \mathbb{S}^1)$ . We will make use of this in the next section.

**Example 3.39** Consider the group  $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ; then  $H^2(G; \mathbb{S}^1) = \mathbb{Z}/2\mathbb{Z}$  (as can be seen from the Künneth formula). If  $a, b$  are generators for  $G$ , we have a projective representation  $\mu : G \rightarrow PGL_2(\mathbb{C})$  given by

$$a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that this gives rise to an extension  $\tilde{G} \rightarrow GL_2(\mathbb{C})$ . Restricted to  $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{S}^1$ , we get an extension of the form  $1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \tilde{D} \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \rightarrow 1$ ; however this is precisely the embedding of the dihedral group in  $GL_2(\mathbb{C})$ . Hence the extension  $\tilde{G}$  must also be non-split, and so represents the non-trivial element  $\alpha$  in  $H^2(G; \mathbb{S}^1)$ . One can easily verify that there is only one conjugacy class of  $\alpha$ -regular elements in  $G$ , comprising the trivial element. The representation  $\mu$  is clearly irreducible, hence up to isomorphism is the unique irreducible  $\alpha$ -twisted representation of  $G$ . In particular,  $R_\alpha(G) \cong \mathbb{Z}\langle \mu \rangle$ .

**Example 3.40** (Symmetric product) Let  $G = S_n$ , the symmetric group on  $n$  letters. Assume that  $n \geq 4$ ; it is well known that in this range  $H^2(G; \mathbb{S}^1) = \mathbb{Z}/2\mathbb{Z}$ . Denote the non-trivial class by  $\alpha$ . Using the decomposition formula, one can calculate (see Uribe’s thesis [154] for details)  ${}^\alpha K_{S_n}^*(M^n)$ , where the group acts on the  $n$ -fold product of a manifold  $M$  by permutation of coordinates. The quotient orbifold is the symmetric product considered in Example 1.13. From this one can recover a corrected version of a formula which appears in [43] for twisted symmetric products – the error was first observed and corrected by W. Wang in [160]:

$$\begin{aligned} \sum q^n \chi({}^\alpha K_{S_n}^*(M^n) \otimes \mathbb{C}) &= \prod_{n>0} (1 - q^{2n-1})^{-\chi(M)} + \prod_{n>0} (1 + q^{2n-1})^{\chi(M)} \\ &\quad \times \left[ 1 + \frac{1}{2} \prod_{n>0} (1 + q^{2n})^{\chi(M)} - \frac{1}{2} \prod_{n>0} (1 - q^{2n})^{\chi(M)} \right]. \end{aligned}$$

**Remark 3.41** There is a growing literature in twisted K-theory; in particular, a twisting of  $K_G(X)$  can be done using an element in  $H_G^1(X; \mathbb{Z}/2\mathbb{Z}) \times H_G^3(X; \mathbb{Z})$  (see [55, p. 422]). Given a  $G$ -space  $X$ , we can take the classifying map  $f_X : EG \times_G X \rightarrow BG$ ; hence given  $\alpha \in H^2(BG; \mathbb{S}^1) \cong H^3(G; \mathbb{Z})$  we obtain an element in  $H_G^3(X; \mathbb{Z})$  for any  $G$ -space  $X$ , and furthermore these elements

naturally correspond under equivariant maps. Our twisted version of K-theory  ${}^\alpha K_G$  specializes (for any  $X$ ) to the twisting by the element  $f_X^*(\alpha) \in H_G^3(X; \mathbb{Z})$ .

### 3.7 Twisted orbifold K-theory and twisted Bredon cohomology

Recall that a *discrete torsion*  $\alpha$  of an orbifold  $\mathcal{X}$  is defined to be a class  $\alpha \in H^2(\pi_1^{\text{orb}}(\mathcal{X}); \mathbb{S}^1)$ . As we saw in Section 2.2, the orbifold fundamental group  $\pi_1^{\text{orb}}(\mathcal{X})$  may be defined as the group of deck translations of the orbifold universal cover  $\mathcal{Y} \rightarrow \mathcal{X}$ .

For example, if  $\mathcal{X} = Z/G$  is a global quotient, the universal cover  $Y$  of  $Z$  is the orbifold universal cover of  $X$ . In fact, if  $EG \times_G Z$  is the Borel construction for  $Z$ , then we have a fibration  $Z \rightarrow EG \times_G Z \rightarrow BG$  which gives rise to the group extension  $1 \rightarrow \pi_1(Z) \rightarrow \pi_1^{\text{orb}}(\mathcal{X}) \rightarrow G \rightarrow 1$ ; here we are identifying  $\pi_1^{\text{orb}}(\mathcal{X})$  with  $\pi_1(EG \times_G Z)$ . Note that a class  $\alpha \in H^2(G; \mathbb{S}^1)$  induces a class  $f^*(\alpha)$  in  $H^2(\pi_1^{\text{orb}}(\mathcal{X}); \mathbb{S}^1)$ .

Now suppose that  $\mathcal{X} = M/G$  is a quotient manifold for a compact Lie group  $G$  and  $p : \mathcal{Y} \rightarrow \mathcal{X}$  is the orbifold universal cover. Note that  $p$  is an orbifold morphism. The same argument used in pulling back orbifold bundles implies that we can pull back the orbifold principal bundle  $M \rightarrow \mathcal{X}$  to obtain an orbifold principal  $G$ -bundle  $\tilde{M} \rightarrow \mathcal{Y}$ . Furthermore,  $\tilde{M}$  is smooth and has a free left  $\pi_1^{\text{orb}}(\mathcal{X})$ -action, as well as a right  $G$ -action. These can be combined to yield a left  $\pi = \pi_1^{\text{orb}}(\mathcal{X}) \times G$ -action. It follows that

$$K_\pi^*(\tilde{M}) \cong K_G^*(\tilde{M}/\pi_1^{\text{orb}}(\mathcal{X})) = K_{\text{orb}}^*(\mathcal{X}).$$

Consider a group  $\pi$  of the form  $\Gamma \times G$ , where  $\Gamma$  is a discrete group and  $G$  is a compact Lie group. Now let  $Z$  denote a proper  $\pi$ -complex such that the orbit space  $Z/\pi$  is a compact orbifold. We now fix a cohomology class  $\alpha \in H^2(\Gamma; \mathbb{S}^1)$ , corresponding to a central extension  $\Gamma_\alpha$ . From this we obtain an extension  $\tilde{\pi}_\alpha = \tilde{\Gamma}_\alpha \times G$ . We can define the  $\alpha$ -twisted  $\pi$ -equivariant K-theory of  $Z$ , denoted  ${}^\alpha K_\pi^*(Z)$  in a manner analogous to what we did before. Namely, we consider  $\tilde{\pi}_\alpha$ -bundles covering the  $\pi$  action on  $Z$ , such that the central circle acts by scalar multiplication on the fibers. Based on this we can introduce the following definition.<sup>6</sup>

**Definition 3.42** Let  $\mathcal{X} = M/G$  denote a compact quotient orbifold where  $G$  is a compact Lie group, and let  $\mathcal{Y} \rightarrow \mathcal{X}$  denote its orbifold universal

<sup>6</sup> Alternatively, we could have used an equivariant version of orbifold bundles and introduced the twisting geometrically. This works for general orbifolds, but we will not elaborate on this here.

cover, with deck transformation group  $\Gamma = \pi_1^{\text{orb}}(\mathcal{X})$ . Given an element  $\alpha \in Z^2(\pi_1^{\text{orb}}(\mathcal{X}); \mathbb{S}^1)$ , we define the  $\alpha$ -twisted orbifold  $K$ -theory of  $\mathcal{X}$  as  ${}^\alpha K_{\text{orb}}^*(\mathcal{X}) = {}^\alpha K_\pi^*(\tilde{M})$ , where  $\pi = \pi_1^{\text{orb}}(\mathcal{X}) \times G$ .

If  $\mathcal{Y}$ , the orbifold universal cover of  $\mathcal{X}$ , is actually a manifold, i.e., if  $\mathcal{X}$  is a good orbifold (see [105]), then the  $G$ -action on  $\tilde{M}$  is free, and in this case the  $\alpha$ -twisted orbifold  $K$ -theory will simply be  ${}^\alpha K_{\pi_1^{\text{orb}}(\mathcal{X})}^*(\mathcal{Y})$ . For the case of a global quotient  $X = Z/G$  and a class  $\alpha \in H^2(G; \mathbb{S}^1)$ , it is not hard to verify that in fact  ${}^{f^*(\alpha)} K_{\text{orb}}^*(\mathcal{X}) \cong {}^\alpha K_G^*(Z)$ , where  $f : \pi_1^{\text{orb}}(\mathcal{X}) \rightarrow G$  is defined as before.

In the general case, we note that  $\pi = \pi_1^{\text{orb}}(\mathcal{X}) \times G$  acts on  $\tilde{M}$  with finite isotropy. That being so, we can make use of “twisted Bredon cohomology” and a twisted version of the usual Atiyah–Hirzebruch spectral sequence. Fix  $\alpha \in Z^2(\pi_1^{\text{orb}}(\mathcal{X}); \mathbb{S}^1)$ , where  $\mathcal{X}$  is a compact orbifold. There is a spectral sequence of the form

$$E_2 = H_\pi^*(\tilde{M}; R_\alpha(-)) \Rightarrow {}^\alpha K_{\text{orb}}^*(\mathcal{X}).$$

The  $E_1$  term will be a chain complex built out of the twisted representation rings of the isotropy groups, all of which are finite. In many cases, this twisted Atiyah–Hirzebruch spectral sequence will also collapse at  $E_2$  after tensoring with the complex numbers. We believe that in fact this must always be the case – see Dwyer’s thesis [47] for more on this. In particular, we conjecture that if (1)  $\mathcal{X}$  is a compact good orbifold with orbifold universal cover the manifold  $Y$ , (2)  $\Gamma = \pi_1^{\text{orb}}(\mathcal{X})$ , and (3)  $\alpha \in H^2(\Gamma; \mathbb{S}^1)$ , then we have an additive decomposition

$${}^\alpha K_\Gamma^*(\mathcal{X}) \otimes \mathbb{C} \cong \bigoplus_{(g)} H^*(\text{Hom}_{\mathbf{C}_\Gamma(g)}(C_*(Y^{(g)}), \gamma_g^\alpha)) \cong H_{\text{CR}}^*(\mathcal{X}; \mathcal{L}_\alpha). \quad (3.10)$$

Here,  $(g)$  ranges over conjugacy classes of elements of finite order in  $\Gamma$ ,  $C_*(-)$  denotes the singular chains,  $\gamma_g^\alpha$  is the character for  $\mathbf{C}_\Gamma(g)$  associated to the twisting, and  $H_{\text{CR}}^*(\mathcal{X}; \mathcal{L}_\alpha)$  is the twisted Chen–Ruan cohomology defined in the next chapter.