The Mathematical Structure Of Finite Gauge Transformations in Closed String Field Theory

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PART I: A FLASH INTRODUCTION TO CSFT

From worldsheet amplitudes to a spacetime action

A closed string sweeps out a two-dimensional surface, the **worldsheet**. Quantum fields on this surface describe string motion in target spacetime.

A puncture carries a closed-string state. Integrating worldsheet correlation functions over inequivalent punctured surfaces gives **scattering amplitudes**: transition amplitudes between string states.

In **closed string field theory (CSFT)**, a string field Ψ has an infinite number of states whose interactions is governed by a non-polynomial spacetime action $S[\Psi]$ from which Feynman diagrams and thus scattering amplitudes can be calculated.

A string background: conformal symmetry without anomaly

A **conformal field theory (CFT)** on the worldsheet is a two-dimensional quantum field theory whose dimensionless couplings do not change with the renormalization scale:

$$\boxed{\beta^I := \mu \frac{d\lambda^I}{d\mu} = 0.}$$

Here λ^I are the couplings multiplying worldsheet operators, μ is the renormalization scale, and β^I are the **beta functions**.

A **consistent bosonic string background** requires the complete **matter-plus-ghost** worldsheet theory to have no **Weyl anomaly** (no quantum dependence on a local rescaling of the worldsheet metric):

$$\boxed{\beta^I = 0, \qquad c_{\text{total}} = c_{\text{matter}} + c_{\text{ghost}} = 26 - 26 = 0.}$$

The **central charge** c is the coefficient of the central term in the worldsheet conformal (Virasoro) algebra. The matter theory has $c_{\text{matter}} = 26$; the b, c ghost system has $c_{\text{ghost}} = -26$.

The BRST operator

The **matter fields** describe string coordinates and their excitations. The anticommuting b, c **ghosts** arise when fixing worldsheet gauge symmetry.

The complete matter-plus-ghost CFT specifies

$$\begin{aligned} \mathcal{H}^q &\quad \text{string states of ghost number } q, \\ Q_B : \mathcal{H}^q \longrightarrow \mathcal{H}^{q+1} &\quad \text{the BRST operator,} \qquad Q_B^2 = 0. \end{aligned}$$

Ghost number assigns $+1$ to c and -1 to b. The operator Q_B is **nilpotent** when the worldsheet BRST symmetry is anomaly-free.

The classical string field and its gauge parameter have ghost numbers two and one:

$$\boxed{\Psi \in \mathcal{H}^2, \qquad \Lambda \in \mathcal{H}^1.}$$

One string field, infinitely many component fields

Choose a basis of worldsheet states $|e_a; k\rangle$ with spacetime momentum k. The **string field** is

$$\Psi = \sum_a \int \frac{d^{26}k}{(2\pi)^{26}}\, \psi^a(k)|e_a; k\rangle, \qquad \psi^a(x) = \int \frac{d^{26}k}{(2\pi)^{26}} e^{ik\cdot x}\psi^a(k).$$

Here a labels oscillator and ghost states; $\psi^a(x)$ are the **spacetime component fields**; x is a target-spacetime coordinate.

level 0 — tachyon T

level 1 — graviton $h_{\mu\nu}$, $B_{\mu\nu}$, dilaton ϕ

level 2, 3, . . . — infinitely many massive component fields

Ψ includes the **full infinite** tower, including the ghost components required by gauge symmetry.

The quadratic action and BRST cohomology

The **BPZ pairing** $\omega(X, Y)$ is a bilinear form obtained from a worldsheet two-point function with the required ghost insertion. The free action is

$$\boxed{S_2[\Psi] = \frac{1}{2}\omega(\Psi, Q_B\Psi), \qquad Q_B\Psi = 0.}$$

The **gauge transformation** of $S_2[\Psi]$ is $\delta_\Lambda\Psi = Q_B\Lambda$. Because $Q_B^2 = 0$,

$$Q_B(\Psi + Q_B\Lambda) = Q_B\Psi.$$

$$\mathcal{H}^1 \xrightarrow{Q_B} \mathcal{H}^2 \xrightarrow{Q_B} \mathcal{H}^3$$

BRST cohomology is the space of solutions of $S_2[\Psi]$ modulo gauge transformations:

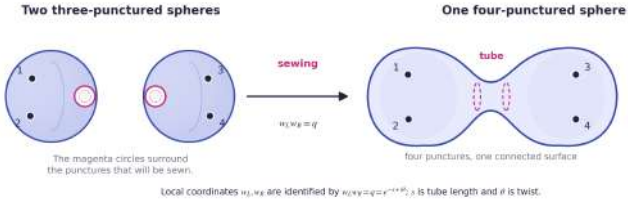
$$\boxed{H^2(Q_B) = \ker(Q_B : \mathcal{H}^2 \to \mathcal{H}^3)/\,\mathrm{im}(Q_B : \mathcal{H}^1 \to \mathcal{H}^2).}$$

The quotient identifies Ψ and $\Psi + Q_B\Lambda$.

Worldsheet vertices and the four-string interaction

A **string vertex** is an integral of a CFT correlation function over a chosen region in the **moduli space** of punctured surfaces. Moduli specify the shape of a punctured surface up to changes of coordinates.

A **propagator** is constructed by **sewing**: identifying neighborhoods of two punctures to form a tube. Sewing two three-string vertices gives three distinct four-string diagrams (s, t, u channels).



The sewn regions do not cover the full four-punctured-sphere moduli space. The remaining region defines the **fundamental four-string vertex** $\mathcal{V}_{0,4}$.

CSFT has elementary vertices with four, five, and arbitrarily many string inputs. These vertices define the **string products (bracket)** in the action.

The classical CSFT action and gauge symmetry

Let ℓ_n be the n-input string product obtained from the corresponding sphere vertex, with $\ell_1 = Q_B$.

$$S[\Psi] = \frac{1}{2}\omega(\Psi, Q_B\Psi) + \sum_{n\geq 2} \frac{1}{(n+1)!}\omega\big(\Psi, \ell_n(\Psi, \dots, \Psi)\big).$$

Varying S gives the equation of motion $F(\Psi) = 0$ and gauge transformation

$$F(\Psi) := Q_B\Psi + \sum_{n\geq 2} \frac{1}{n!}\ell_n(\Psi, \dots, \Psi),$$
$$\delta_\Lambda\Psi = Q_B\Lambda + \ell_2(\Psi, \Lambda) + \frac{1}{2}\ell_3(\Psi, \Psi, \Lambda) + \cdots =: Q_\Psi\Lambda.$$

The L_∞ **identities** are relations among $Q_B, \ell_2, \ell_3, \dots$ generalizing the Jacobi identity of a Lie algebra. They follow from sewing boundaries of the worldsheet vertex regions with n-punctures and ensure gauge invariance of the spacetime action $S[\Psi]$.

PART II: FINITE GAUGE TRANSFORMATIONS

Gauge transformations

At a fixed CFT, the infinitesimal gauge variation is

$$\delta_\Lambda \Psi = Q_\Psi \Lambda.$$

How do we integrate the L_∞ algebra to describe classical solutions, finite gauge transformations, and their higher relations? Does the result form a group, or a higher groupoid?

Finite gauge transformations

Gauge equivalence of classical solutions.
A finite gauge transformation is required to determine whether two solutions Ψ_i and Ψ_j are connected by gauge symmetry.

Composition of finite gauge transformations.
Given transformations $\Psi_i \to \Psi_j$ and $\Psi_j \to \Psi_k$, we would like to determine how their composition compares with a direct transformation $\Psi_i \to \Psi_k$.

Higher gauge redundancy.
In CSFT, a shift of the gauge parameter can leave $\delta_\Lambda \Psi$ unchanged. This redundancy of the gauge parameter is a **gauge-for-gauge transformation**.

Solutions and higher morphisms in one object

We want to organize **classical solutions, finite gauge transformations, and their higher homotopies** into a **simplicial set**: vertices, edges, triangles and higher simplices, with maps that remove or repeat vertices.

1-morphism.
A finite gauge transformation $G_{ij} : \Psi_i \to \Psi_j$ between two classical solutions.

2-morphism.
A **homotopy** is a family of gauge paths deforming one path into another while keeping its endpoints fixed. A 2-simplex H_{ijk} compares the two-step path with a direct path.

3-morphism and higher morphisms.
Transformations between these homotopies. A 3-simplex (tetrahedron) represents the next level.

The total field

On an m-dimensional parameter simplex Δ^m, define $A_0 := \Psi$ and

$$A_p \in \Omega^p(\Delta^m) \otimes \mathfrak{L}^{-p}.$$

where $\Omega^\bullet(\Delta^m)$ denotes smooth differential forms of all degrees on the m-simplex Δ^m. Thus each A_p has **total degree zero**: $p + (-p) = 0$.

Define the total field $\mathbf{\Psi}$

$$\boxed{\mathbf{\Psi} := \sum_{p=0}^{m} (-1)^{p(p-1)/2} A_p = \Psi + A_1 - A_2 - A_3 + A_4 + \cdots .}$$

We will construct these components explicitly for the following simplices:

(Δ^1) interval: $\quad A_1 = dt\,\Lambda,$

(Δ^2) triangle: $\quad A_1 = du\,\lambda_u + dv\,\lambda_v, \quad A_2 = du \wedge dv\,\xi_{uv},$

(Δ^3) tetrahedron: $\quad A_3 = du \wedge dv \wedge dw\,\upsilon_{uvw}.$

Formal series versus a convergent CSFT gauge flow

Simple example. Let $f(z) = 1 + z + z^2 + \cdots$. The identity $(1 - z)f(z) = 1$ holds *formally*: we compare coefficients of each power of z without summing the series.

$$|z| < 1 \quad \Longrightarrow \quad \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z} \qquad \text{(the partial sums converge).}$$

In CSFT. Solve $\dot{\Psi} = Q_\Psi(\varepsilon\Lambda)$, with $\Psi(0) = 0$, in powers of a small parameter ε:

$$\boxed{\Psi(1;\varepsilon) = \varepsilon Q_B\Lambda + \frac{\varepsilon^2}{2}\ell_2(Q_B\Lambda, \Lambda) + \cdots = \sum_{n\geq 1} \varepsilon^n \Psi_n.}$$

Every Ψ_n is a **formal** solution. To obtain a **convergent** solution, we will have to specify a norm $\|\cdot\|$ on the string-field space and compute estimates. For constants $M, C > 0$, suppose

$$\|\Psi_n\| \leq MC^n \quad \Longrightarrow \quad \sum_{n\geq 1} |\varepsilon|^n \|\Psi_n\| < \infty \quad \text{if } |\varepsilon| C < 1.$$

Solutions with all morphisms form a Kan complex

A total field Ψ on a **k**-simplex: a string-state-valued differential form on the **simplex** Δ^k, of total degree zero, that satisfies

$$d_\Delta \Psi + \sum_{n\geq 1} \frac{1}{n!}\, \ell_n(\Psi, \ldots, \Psi) = 0.$$

This is the **total Maurer–Cartan (MC) equation**. Its solutions form $\mathrm{Sol}_\bullet(L)$: vertices are classical solutions, edges are finite gauge transformations, and higher simplices compare those transformations.

The Kan condition: every compatible set of all but one faces of a simplex (a horn) can be extended to a full simplex. We claim that $\mathrm{Sol}_\bullet(L)$ has this property, hence an ∞-groupoid (morphisms and higher morphisms, invertible up to higher homotopy). This statement holds order by order in a **formal** expansion. **Analytically**, establishing the corresponding structure for string fields in full CSFT remains a challenge. A few attempts have been made in the literature. We will review them briefly towards the end, time permitting.

The de-Rham simplex differential d_Δ

Let $d_\Delta : \Omega^p(\Delta^m) \to \Omega^{p+1}(\Delta^m)$ denote the de-Rham differential, where $\Omega^p(\Delta^m)$ is the **space of smooth differential p-forms** on the m-simplex Δ^m

$$\Delta^m = \left\{ \left(t^1, \ldots, t^m\right) \in \mathbb{R}^m : t^a \geq 0, \quad \sum_{a=1}^m t^a \leq 1 \right\}.$$

The operator d_Δ is the degree-one derivation determined by

$$d_\Delta f = \sum_{a=1}^m \frac{\partial f}{\partial t^a} dt^a, \quad d_\Delta\left(dt^a\right) = 0$$

and the graded Leibniz rule

$$d_\Delta(\alpha \wedge \beta) = d_\Delta \alpha \wedge \beta + (-1)^p \alpha \wedge d_\Delta \beta, \quad \alpha \in \Omega^p(\Delta^m).$$

In particular, $d_\Delta^2 = 0$.

The Hilbert Space

Let $\mathcal{H}$ be the **graded string-state space, with shifted degree equal to ghost number minus two**. We work on

$$\Omega^\bullet(\Delta^m) \otimes \mathcal{H}, \qquad |\alpha \otimes X|_{\text{tot}} = \deg \alpha + |X|,$$

where $|X|$ denotes the shifted degree of the string state X.

Each string product ℓ_n has degree $+1$. For forms $\alpha_i \in \Omega^{p_i}(\Delta^m)$ and string states $X_i \in \mathcal{H}^{q_i}$, each of fixed degree, define

$$\begin{aligned}&\ell_n^\Omega(\alpha_1 \otimes X_1, \ldots, \alpha_n \otimes X_n)\\ &\quad := (-1)^{\sum_{i=1}^n p_i + \sum_{1\le i<j\le n} q_i p_j}(\alpha_1 \wedge \cdots \wedge \alpha_n) \otimes \ell_n(X_1, \ldots, X_n), \qquad n \ge 1,\end{aligned}$$

For example, the unary and binary extensions are

$$\ell_1^\Omega(\alpha \otimes X) = (-1)^p\, \alpha \otimes Q_B X,$$
$$\ell_2^\Omega(\alpha \otimes X, \beta \otimes Y) = (-1)^{p+r+qr}(\alpha \wedge \beta) \otimes \ell_2(X, Y),$$

One MC equation for each Δ^k

The MC equation for the total field $\mathbf{\Psi}$ is With $\mathbf{\Psi} = \Psi + A_1 - A_2 - A_3 + \cdots$, the first four equations give

$$
\begin{aligned}
F(\Psi) &= 0, \\
d_\Delta \Psi &= -Q^\Omega_\Psi A_1, \\
d_\Delta A_1 &= Q^\Omega_\Psi A_2 - \tfrac{1}{2}\ell_2^{\Psi,\Omega}(A_1, A_1), \\
d_\Delta A_2 &= -Q^\Omega_\Psi A_3 - \ell_2^{\Psi,\Omega}(A_1, A_2) + \tfrac{1}{3!}\ell_3^{\Psi,\Omega}(A_1, A_1, A_1).
\end{aligned}
$$

Differential forms on a simplex

The parameter simplex Δ^m is an interval for $m = 1$, a triangle for $m = 2$, and a tetrahedron for $m = 3$. Its differential forms,

$$\Omega^\bullet(\Delta^m) = \Omega^0(\Delta^m) \oplus \Omega^1(\Delta^m) \oplus \Omega^2(\Delta^m) \oplus \cdots ,$$

are the functions, one-forms, two-forms, and higher forms on Δ^m. For example,

$$\Delta^1 = \text{interval}, \qquad \Delta^2 = \text{triangle}, \qquad \Delta^3 = \text{tetrahedron}.$$

They form a **differential graded algebra**: forms multiply by the wedge product, and the exterior derivative d_Δ raises form degree by one, with $d_\Delta^2 = 0$.

A total field Ψ is a differential form on Δ^m whose coefficients are string states.

BACKGROUND AND LITERATURE REVIEW

The closed-string state space is graded

The closed-string states obeying the usual constraints $b_0^- X = L_0^- X = 0$ form a graded vector space

$$\boxed{L = \bigoplus_{q \in \mathbb{Z}} L^q.}$$

A **graded vector space** is a direct sum of subspaces L^q, one for each integer q. Here q is the **shifted degree**:

$$\boxed{|X| = q = \mathrm{gh}(X) - 2 \quad \text{for } X \in L^q.}$$

Thus $|\Psi| = 0$, a gauge parameter has $|\Lambda| = -1$, a gauge-for-gauge parameter has degree -2, and the equation of motion has degree $+1$.

The L_∞ algebra of CSFT

An **ordinary Lie algebra** is a vector space $\mathfrak{h}$ with an antisymmetric bilinear bracket $[\,,\,]$ satisfying the Jacobi identity

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0.$$

In an L_∞ algebra, we have instead graded-symmetric multilinear maps

$$\boxed{\ell_n\colon S^nL\longrightarrow L,\qquad |\ell_n|=+1,\qquad n\geq 1,}$$

satisfying generalized Jacobi identities.

Here S^nL is the **graded-symmetric** n-fold product: exchanging homogeneous X and Y gives $(-1)^{|X||Y|}$. $\ell_1=Q_B$ and $Q_B^2=0$.

Background-dressed Q

For $\Psi \in L^0$, define

$$\boxed{F(\Psi) := \sum_{n \geq 1} \frac{1}{n!} \ell_n(\Psi, \ldots, \Psi).}$$

The classical equation of motion is $F(\Psi) = 0$. A solution of this equation is a **Maurer–Cartan element.** For $q \geq 1$, define the background-dressed products

$$\ell_q^{\Psi}(X_1, \ldots, X_q) := \sum_{r \geq 0} \frac{1}{r!} \ell_{q+r}(\Psi^{\odot r}, X_1, \ldots, X_q),$$

and define

$$\boxed{Q_\Psi X := \ell_1^{\Psi}(X) = Q_B X + \ell_2(\Psi, X) + \frac{1}{2}\ell_3(\Psi, \Psi, X) + \cdots .}$$

Expanding around a solution

Start with products $\ell_1, \ell_2, \ell_3, \ldots$ and the Maurer–Cartan equation

$$F(\Phi) = \sum_{n \geq 1} \frac{1}{n!} \ell_n(\Phi, \ldots, \Phi) = 0.$$

Given a solution Ψ with $F(\Psi) = 0$, write a nearby field as

$$\Phi = \Psi + X,$$

where X is the fluctuation about Ψ.

We now substitute $\Phi = \Psi + X$ into $F(\Phi)$.

The terms linear in X determine the linearized operator about Ψ.

Expanding around a solution

Substituting $\Phi = \Psi + X$ gives

$$F(\Psi + X) = \sum_{n \geq 1} \frac{1}{n!} \ell_n(\Psi + X, \ldots, \Psi + X).$$

The term with no X is $F(\Psi)$. The terms linear in X are

$$\ell_1(X) + \ell_2(\Psi, X) + \frac{1}{2!}\ell_3(\Psi, \Psi, X) + \frac{1}{3!}\ell_4(\Psi, \Psi, \Psi, X) + \cdots .$$

Therefore

$$F(\Psi + X) = F(\Psi) + \left[\ell_1(X) + \ell_2(\Psi, X) + \frac{1}{2}\ell_3(\Psi, \Psi, X) + \cdots\right] + O(X^2).$$

Since $F(\Psi) = 0$, the bracketed expression is the linearized equation around Ψ.

Background-dressed ℓ_n with X

Define

$$\boxed{\ell_1^{\Psi}(X) := \sum_{r\geq 0} \frac{1}{r!}\ell_{r+1}(\Psi^{\otimes r}, X).}$$

Then

$$F(\Psi + X) = \ell_1^{\Psi}(X) + O(X^2).$$

With $\ell_1 = Q_B$, we define

$$\boxed{Q_{\Psi} := \ell_1^{\Psi}.}$$

Hence

$$\boxed{Q_{\Psi}X = Q_B X + \ell_2(\Psi, X) + \frac{1}{2}\ell_3(\Psi, \Psi, X) + \frac{1}{3!}\ell_4(\Psi, \Psi, \Psi, X) + \cdots .}$$

Background-dressed ℓ_n with X

For k free inputs, define

$$\boxed{\ell_k^{\Psi}(X_1, \ldots, X_k) := \sum_{r \geq 0} \frac{1}{r!} \ell_{k+r}(\Psi^{\otimes r}, X_1, \ldots, X_k).}$$

For $k = 1$,

$$\ell_1^{\Psi} = Q_{\Psi}.$$

For $k = 2$,

$$\ell_2^{\Psi}(X, Y) = \ell_2(X, Y) + \ell_3(\Psi, X, Y) + \frac{1}{2}\ell_4(\Psi, \Psi, X, Y) + \cdots .$$

Flat connections in ordinary Lie algebra

Let $\mathfrak{g}$ be an ordinary Lie algebra, such as $\mathfrak{su}(2)$, with $[e_b, e_c] = f^a{}_{bc}\, e_a$. A $\mathfrak{g}$-valued one-form $A = A^a e_a$ on Δ^m solves the Maurer–Cartan equation

$$\boxed{d_\Delta A + \tfrac{1}{2}[A, A] = 0,} \qquad \text{that is,} \qquad d_\Delta A^a + \tfrac{1}{2} f^a{}_{bc}\, A^b \wedge A^c = 0.$$

This says that A is a **flat connection** on the simplex Δ.

An ordinary Lie algebra sits entirely in degree -1, the gauge-parameter degree. Since the MC equation has only gauge parameters, only one-forms can appear.

Holonomy composes strictly, and there is one vertex

Along the edge from vertex i to vertex j, the holonomy

$$g_{ij} = \mathcal{P} \exp\Big(- \int_i^j A \Big),$$

is an element of the group G integrating $\mathfrak{g}$: a **finite gauge transformation**.

On a 2-simplex, flatness of A makes the transport around the boundary trivial:

$$\boxed{g_{jk}\, g_{ij} = g_{ik}.}$$

Composition is strict.

A point has no one-forms, so the only solution there is $A = 0$. **The Lie group has exactly one vertex**.

What changes for the L_∞ algebra of CSFT

The CSFT algebra L has a field at degree 0 as well as the degrees $-1, -2, \ldots$. So the same equation,

$$\boxed{d_\Delta \mathbf{\Psi} + \sum_{n\geq 1} \frac{1}{n!}\, \ell_n(\mathbf{\Psi}, \ldots, \mathbf{\Psi}) = 0,}$$

now has components of every form degree, $\mathbf{\Psi} = \Psi + A_1 - A_2 - A_3 + \cdots$. Their coefficients are the string field Ψ, the gauge parameters λ, the gauge-for-gauge parameters ξ, and υ, with ghost numbers $2, 1, 0, -1$.

Two things change. The vertices are now classical solutions Ψ. And where in the case the Lie algebra the 2-simplex closes strictly, **in CSFT it closes only up to** $Q_\Psi \xi_{uv}$, as the examples will show.

CONSTRUCTION OF THE KAN COMPLEX

Integrating an ordinary Lie algebra

Let $\mathfrak{h}$ be an ordinary Lie algebra and let H be a Lie group with Lie algebra $\mathfrak{h}$. Given $\lambda(t) \in \mathfrak{h}$, define $g(t) \in H$ by

$$g^{-1}(t)\dot{g}(t) = \lambda(t), \qquad g(0) = e.$$

The endpoint is

$$\boxed{g(1) = \mathcal{P} \exp\left(\int_0^1 \lambda(t)\, dt \right).}$$

Path ordering $\mathcal{P}$ is required when $\lambda(t_1)$ and $\lambda(t_2)$ do not commute. For an ordinary Lie algebra, the infinitesimal algebra integrates to finite group.

What is the corresponding construction for an L_∞ algebra?

Dimension zero: a vertex is one solution

The zero-simplex is just one point. Therefore

$$\mathrm{Sol}_0(U_i) = \mathrm{Sol}_0(L).$$

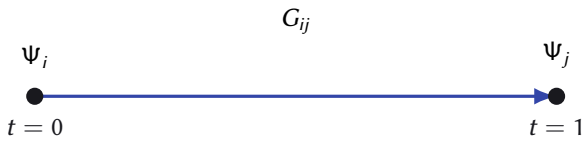
An element $\Psi_i \in \mathrm{Sol}_0(L)$ satisfies

$$\mathcal{F}(\Psi_i) = 0.$$



one point in parameter space
labels one complete string-field solution

Dimension one: interval



G_{ij} is a gauge transformation joining two solutions.

$$\Delta^1 = [0,1]$$

Define

$$\boxed{\mathbf{\Psi}_{ij} = \Psi(t) - dt\,\Lambda(t) \in \mathrm{Sol}(\Omega^\bullet([0,1]) \otimes L)\,.} \tag{4}$$

Endpoints:

$$\Psi(0) = \Psi_i, \qquad \Psi(1) = \Psi_j. \tag{5}$$

Expanding the total Maurer–Cartan equation gives

$$\mathcal{F}(\Psi(t)) = 0, \tag{6}$$

$$\dot{\Psi}(t) = Q_{\Psi(t)}\Lambda(t). \tag{7}$$

Every point in Δ^1 is an on-shell background, and motion along Δ^1 is a gauge transformation.

Gauge flow preserves the field equation

Assume $F(\Psi_i) = 0$. Then

$$Q_\Psi^2 X = -\ell_2^\Psi(F(\Psi), X).$$

Differentiating the equation-of-motion state along the gauge flow gives

$$\begin{aligned}\frac{d}{dt}F(\Psi(t)) &= Q_{\Psi(t)}\dot{\Psi}(t) \\ &= Q_{\Psi(t)}^2\Lambda(t) \\ &= -\ell_2^{\Psi(t)}(F(\Psi(t)), \Lambda(t)).\end{aligned}$$

The equation is homogeneous in $F(\Psi(t))$. Since $F(\Psi_i) = 0$, formal recursion gives $F(\Psi(t)) = 0$ **along the flow**.

THE 2-SIMPLEX

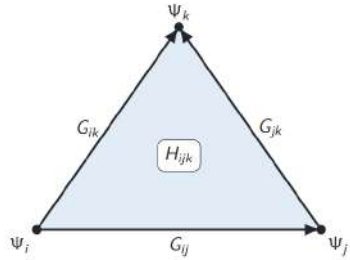
The 2-simplex Δ^2

Define

$$\Delta^2 := \{(u, v) \mid u \geq 0,\ v \geq 0,\ u + v \leq 1\}.$$

Let $\Psi(u, v)$ be an on-shell string field, and let $\lambda_u(u, v)$ and $\lambda_v(u, v)$ be ghost-number-one gauge parameters satisfying

$$\partial_u \Psi = Q_\Psi \lambda_u, \qquad \partial_v \Psi = Q_\Psi \lambda_v$$



Every vertex of the triangle is a **solution**. The three solutions are connected by finite gauge transformations.

There are **two different** paths from Ψ_i to Ψ_k:

$$G_{ik} \quad \text{and} \quad G_{jk} \circ G_{ij}$$

How do we compare these two transformations?

The difference between the two paths

We start from Q_Ψ

$$Q_\Psi X = Q_B X + \ell_2(\Psi, X) + \frac{1}{2}\ell_3(\Psi, \Psi, X) + \cdots .$$

Differentiating with respect to u gives

$$\begin{aligned}\partial_u(Q_\Psi X) = {} & Q_B(\partial_u X) + \ell_2(\partial_u \Psi, X) + \ell_2(\Psi, \partial_u X) \\ & + \frac{1}{2}\ell_3(\partial_u \Psi, \Psi, X) + \frac{1}{2}\ell_3(\Psi, \partial_u \Psi, X) + \frac{1}{2}\ell_3(\Psi, \Psi, \partial_u X) + \cdots .\end{aligned}$$

Collecting the terms containing $\partial_u X$ and the terms containing $\partial_u \Psi$, we get

$$\boxed{\partial_u(Q_\Psi X) = Q_\Psi(\partial_u X) + \ell_2^\Psi(\partial_u \Psi, X).}$$

The path $G_{jk} \circ G_{ij}$ implies $Q_\Psi C_{uv} = 0$

Partial derivatives commute: $0 = (\partial_u \partial_v - \partial_v \partial_u)\Psi$. Using the previous slide and substituting $\partial_u \Psi = Q_\Psi \lambda_u$, $\partial_v \Psi = Q_\Psi \lambda_v$ gives

$$0 = Q_\Psi(\partial_u \lambda_v - \partial_v \lambda_u) + \ell_2^\Psi(Q_\Psi \lambda_u, \lambda_v) - \ell_2^\Psi(Q_\Psi \lambda_v, \lambda_u).$$

For the two odd parameters, the two-input L_∞ identity is

$$Q_\Psi \ell_2^\Psi(\lambda_u, \lambda_v) + \ell_2^\Psi(Q_\Psi \lambda_u, \lambda_v) - \ell_2^\Psi(\lambda_u, Q_\Psi \lambda_v) = 0.$$

Since $Q_\Psi \lambda_v$ is even, $\ell_2^\Psi(\lambda_u, Q_\Psi \lambda_v) = \ell_2^\Psi(Q_\Psi \lambda_v, \lambda_u)$. Substituting gives

$$\boxed{Q_\Psi\big(\partial_u \lambda_v - \partial_v \lambda_u - \ell_2^\Psi(\lambda_u, \lambda_v)\big) = 0.}$$

Define $\mathbf{C_{uv} := \partial_u \lambda_v - \partial_v \lambda_u - \ell_2^\Psi(\lambda_u, \lambda_v)}$. **Then** $Q_\Psi C_{uv} = 0$ **(so closed) using only the path** $G_{jk} \circ G_{ij}$ **with its two gauge flow equations.**

Now include the entire 2-simplex

On the 2-simplex, $\mathbf{\Psi}$ is

$$\mathbf{\Psi} = \Psi + du\lambda_u + dv\lambda_v - du \wedge dv\xi_{uv}.$$

There are exactly three types of contribution to the coefficient of $du \wedge dv$.

Acting with d_Δ gives

$$d_\Delta \left(du\lambda_u + dv\lambda_v \right) = du \wedge dv \left(\partial_u \lambda_v - \partial_v \lambda_u \right).$$

Terms containing one ξ_{uv} and any number of Ψ 's are

$$-du \wedge dv \left[Q_B \xi_{uv} + \ell_2 \left(\Psi, \xi_{uv} \right) + \frac{1}{2} \ell_3 \left(\Psi, \Psi, \xi_{uv} \right) + \cdots \right] = -du \wedge dv Q_\Psi \xi_{uv}.$$

Terms containing both one-form components and any number of Ψ 's are

$$-du \wedge dv \left[\ell_2 \left(\lambda_u, \lambda_v \right) + \ell_3 \left(\Psi, \lambda_u, \lambda_v \right) + \frac{1}{2} \ell_4 \left(\Psi, \Psi, \lambda_u, \lambda_v \right) + \cdots \right] = -du \wedge dv \ell_2^\Psi \left(\lambda_u, \lambda_v \right).$$

C_{uv} is exact

Adding the three contributions, the Maurer-Cartan equation becomes

$$0 = du \wedge dv \left[\partial_u \lambda_v - \partial_v \lambda_u - \ell_2^{\Psi}(\lambda_u, \lambda_v) - Q_{\Psi} \xi_{uv}\right]$$

Therefore,

$$\underbrace{\partial_u \lambda_v - \partial_v \lambda_u - \ell_2^{\Psi}(\lambda_u, \lambda_v)}_{C_{uv}} = Q_{\Psi} \xi_{uv}.$$

$$\boxed{Q_{\Psi} \xi uv = C_{uv}}$$

A 2-simplex is determined by the following data: $\Psi, \lambda_u, \lambda_v, \xi_{uv}$ on Δ^2 satisfying the on-shell equation, two gauge-flow equations, and $C_{uv} = Q_{\Psi} \xi_{uv}$.

The $2 \to 3$ edge

For $s \in [0,1]$, set $(u(s), v(s)) = (1-s, s)$.

Then $du/ds = -1$ and $dv/ds = 1$.

Define $\Psi_{23}(s) := \Psi(1-s, s)$.

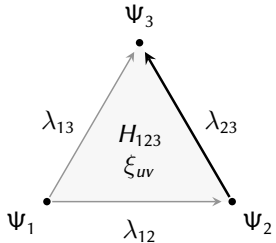
The chain rule gives

$$\begin{aligned}\frac{d\Psi_{23}(s)}{ds} &= -\partial_u \Psi + \partial_v \Psi \\ &= Q_{\Psi_{23}(s)}(-\lambda_u + \lambda_v).\end{aligned}$$

The right-hand side is evaluated at $(1-s, s)$.

$$\boxed{\lambda_{23}(s) := -\lambda_u(1-s, s) + \lambda_v(1-s, s).}$$

Thus $d\Psi_{23}/ds = Q_{\Psi_{23}} \lambda_{23}$. **The third edge is a 1-simplex.**



Along $2 \to 3$, u decreases and v increases.

The 2-simplex homotopy

The boundary gauge flows G_{12}, G_{23}, G_{13} are generated by $\lambda_{12}, \lambda_{23}, \lambda_{13}$.

The complete 2-simplex specifies a homotopy:

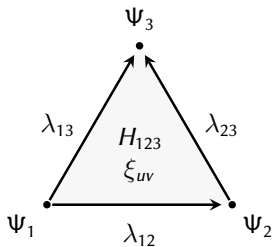
$$\boxed{H_{123} : G_{23} \circ G_{12} \Rightarrow G_{13}.}$$

It compares the paths $1 \to 2 \to 3$ and $1 \to 3$.

H_{123} **is the complete solution** 2-simplex.

The **two-form Maurer–Cartan equation** gives

$$\boxed{C_{uv} = \partial_u \lambda_v - \partial_v \lambda_u - \ell_2^{\Psi}(\lambda_u, \lambda_v) = Q_{\Psi} \xi_{uv}.}$$



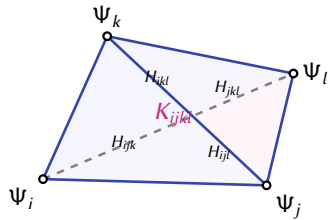
THE 3-SIMPLEX

The 3-simplex compares two ways of gluing 2-simplices

Define

$$\Delta^3 := \{(u, v, w) \mid u, v, w \geq 0,\ u + v + w \leq 1\}.$$

The functions $\lambda_u, \lambda_v, \lambda_w$ generate transformations along u, v and w. The ghost-number-zero functions $\xi_{uv}, \xi_{vw}, \xi_{wu}$ describe the three pairwise 2-simplex relations. Introduce one ghost-number-minus-one function υ_{uvw} for the three-dimensional relation. The four triangular faces are $H_{ijk}, H_{ijl}, H_{ikl}, H_{jkl}$.



Starting from $i \to j \to k \to l$, one can combine H_{ijk} with H_{ikl}, or combine H_{jkl} with H_{ijl}. The 3-simplex K_{ijkl} relates these two ways of combining the triangular relations.

The 3-form equation with ℓ_3^Ψ

On a 3-simplex, the three-form coefficient of the MC equation has four types of terms:

$$\underbrace{d_\Delta(\text{2-form part})}_{\partial\xi} - \underbrace{Q_\Psi(du \wedge dv \wedge dw\, v_{uvw})}_{Q_\Psi v} + \underbrace{\ell_2^\Psi(\text{1-form}, \text{2-form})}_{\ell_2^\Psi(\lambda,\xi)} + \underbrace{\frac{1}{3!}\ell_3^\Psi(\text{1-form})^3}_{\ell_3^\Psi(\lambda,\lambda,\lambda)}.$$

Collecting the coefficient of $du \wedge dv \wedge dw$ gives

$$\boxed{\begin{aligned}\partial_u\xi_{vw} + \partial_v\xi_{wu} + \partial_w\xi_{uv} = {} & Q_\Psi v_{uvw} + \ell_2^\Psi(\lambda_u, \xi_{vw}) + \ell_2^\Psi(\lambda_v, \xi_{wu}) \\ & + \ell_2^\Psi(\lambda_w, \xi_{uv}) + \ell_3^\Psi(\lambda_u, \lambda_v, \lambda_w).\end{aligned}}$$

The ℓ_3^Ψ term appears because three gauge directions are present simultaneously. A complete sign derivation is given in the Additional Slides.

Simplicial data so far

Each simplex contains data from lower-dimension simplexes **and** a filler data.

Dimension	Coefficient	Solutions and morphisms
0	Ψ (gh 2)	classical solution Ψ_i
1	Λ (gh 1)	G_{ij}: finite gauge transformation
2	ξ (gh 0)	H_{ijk}: homotopy between different paths
3	υ (gh -1)	K_{ijkl}: homotopy between compositions of 2-simplices

FROM EXPLICIT SIMPLICES TO THE GENERAL STRUCTURE

A simplicial set organizes simplices and their faces

A **simplicial set** $X_\bullet$ is a sequence of sets $X_0, X_1, X_2, \ldots$ with maps that remove or repeat vertices. A **face map** $d_a \colon X_m \to X_{m-1}$ removes vertex a; a **degeneracy map** $s_a \colon X_m \to X_{m+1}$ repeats it, producing an identity-type simplex. Removing vertices in different orders gives the same face:

$$\boxed{d_a d_b = d_{b-1} d_a, \qquad a < b.}$$

For solutions, d_a is restriction to face a. Restriction commutes with d_Δ ($d_a\,(d_\Delta \Psi) = d_\Delta\,(d_a \Psi)$) and the string products, so faces of solutions are solutions, and $\mathrm{MC}_\bullet(L)$ is a simplicial set. For the triangle already constructed,

$$\boxed{d_0 H_{ijk} = G_{jk}, \qquad d_1 H_{ijk} = G_{ik}, \qquad d_2 H_{ijk} = G_{ij}.}$$

The face maps recover the transformations we already built.

Horns

An m-**horn** is a family of solutions

$$\Psi_a \in \mathsf{MC}_{m-1}(L), \qquad a \neq r,$$

one on every face of Δ^m except face r.

Two faces of Δ^m meet on a $(m-2)$-dimensional face, and there the given solutions **must agree**:

$$\boxed{d_a \Psi_b = d_{b-1} \Psi_a, \qquad a < b, \quad a, b \neq r.}$$

Without this **compatibility condition**, the faces could not come from a single simplex.

We ask: *is there a solution* $\Psi \in \mathsf{Sol}_m(L)$ *with*

$$\boxed{d_a \Psi = \Psi_a, \qquad a \neq r\,?}$$

The Kan condition

If such an Ψ exists, it is a **horn filler**: a solution on the whole m-simplex whose restriction to every given facen is the solution already specified there. It also supplies the missing face, $d_r \Psi$.

The **Kan condition** says:

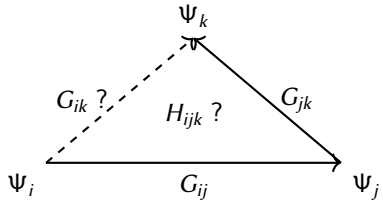
Every compatible horn has a filler.

A simplicial set with this property is a **Kan complex**, a model of an ∞-groupoid. Every horn must fill, including the outer ones, $r = 0$ and $r = m$; the outer horns give inverses.

Horn filling example

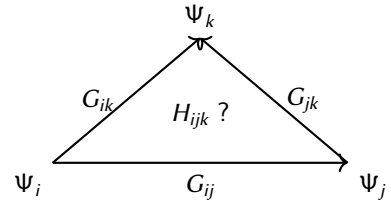
Take Δ^2 for example. Given two compatible edges, a **horn filler** specifies the third edge and the interior.

Horn: two edges given



G_{ik} is the composite edge.

Boundary: three edges given



These particular edges need not fill.

The groupoid structure

A **groupoid** has **objects** $x, y, z, \ldots$ and **1-morphisms** $f : x \to y$ between them. If $f : x \to y$ and $g : y \to z$, there is a composite $g \circ f : x \to z$, and composition is associative. Each object has an identity id_x, and each 1-morphism has an inverse $f^{-1} : y \to x$. A **group** is the special case with one object.

A group is the special case with **one** object.

The ∞-groupoid

In CSFT, the **classical solutions Ψ_i are the objects**. The finite gauge transformations G_{ij} are of 1-morphisms: the identities are the constant flows, and the inverses are inverse flows. But the G_{ij} compose only up to a 2-simplex horn filling, and similarly for an m-simplex.

Composition is **horn filling**. A composite of G_{ij} and G_{jk} exists, but it is unique only up to a **2-morphism**: the triangle H_{ijk}, relating the route through j, $G_{jk} \circ G_{ij}$, to the direct route G_{ik}. Associativity holds only up to a 2-morphism, which filling a tetrahedron supplies.

These higher morphisms carry **independent data**, So

$$\boxed{\Psi_i, \quad G_{ij}, \quad H_{ijk}, \quad K_{ijkl}, \ \ldots}$$

are the objects, 1-morphisms, 2-morphisms, 3-morphisms of a **weak ∞-groupoid** where composites exist but are **unique only up to homotopy, and associativity holds up to coherent homotopy**, modeled by the Kan complex $\mathrm{MC}_\bullet(L)$.

PART III: THE WESS–ZUMINO TERM

What Part III calculates

Here we apply the formalism in Part II to calculate the WZ term in an $SU(2)$ Chern–Simons theory on the unit three-sphere S^3.

Let A be an $\mathfrak{su}(2)$-valued one-form, where $\mathfrak{su}(2)$ consists of traceless anti-Hermitian 2×2 matrices. Let $g : S^3 \to SU(2)$, where $SU(2)$ is the group of unitary 2×2 matrices with determinant one. Define

$$A^g := g^{-1} A g + g^{-1} dg, \qquad \Delta S_{\mathrm{CS}} := S_{\mathrm{CS}}[A^g] - S_{\mathrm{CS}}[A].$$

The calculation shows

$$\boxed{\Delta S_{\mathrm{CS}} = S_{\mathrm{CS}}[g^{-1} dg] = 2\pi k N, \qquad N \in \mathbb{Z}.}$$

Here $k \in \mathbb{Z}$ is the Chern–Simons level and N is the winding number of g.

If a **smooth family** $g_s : S^3 \to SU(2)$ satisfies $g_0 = I_2$ and $g_1 = g$, where I_2 is the identity matrix, then $N = 0$ and $\Delta S_{\mathrm{CS}} = 0$. For $\mathbf{N \neq 0}$, **no such family exists on all of** S^3. We instead solve the MC equation on **two contractible spacetime regions** $U_1, U_2 \subset S^3$ and derive $2\pi k N$ from the two local gauge flows.

Define A, g, and $\theta = g^{-1}dg$

Take spacetime $\mathcal{M} = S^3$, the oriented unit sphere in $\mathbb{R}^4$. The group $SU(2)$ consists of complex 2×2 matrices g with

$$g^\dagger g = I_2, \qquad \det g = 1.$$

The Lie algebra $\mathfrak{su}(2)$ is represented by matrices X with $X^\dagger = -X$ and $\mathrm{tr}(X) = 0$.

A **connection** is a matrix-valued one-form

$$A = \sum_{\mu=1}^{3} A_\mu(x)\, dx^\mu, \qquad A_\mu(x) \in \mathfrak{su}(2),$$

where x^μ are local spacetime coordinates. Let d denote the spacetime exterior derivative. Products combine matrix multiplication with the exterior product $\wedge$; write $A^2 := A \wedge A$.

A smooth map $g : S^3 \to SU(2)$ acts on A by

$$\boxed{A^g = g^{-1}Ag + \theta, \qquad \theta := 0^g = g^{-1}dg.}$$

A **pure-gauge connection**, such as θ, is a gauge transform of the zero connection.

Vary the Chern–Simons action

Fix $k \in \mathbb{Z} \setminus \{0\}$, the **Chern–Simons level**. With tr the trace of a 2×2 matrix, the Chern-Simons action is given by

$$S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int_{S^3} \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A^3 \right).$$

For matrix-valued forms X, Y of degrees p_X, p_Y, **graded cyclicity** of the trace means $\mathrm{tr}(X \wedge Y) = (-1)^{p_X p_Y} \mathrm{tr}(Y \wedge X)$.

Varying $S_{\mathrm{CS}}[A]$ gives

$$\boxed{\delta S_{\mathrm{CS}} = \frac{k}{2\pi} \int_{S^3} \mathrm{tr}(\delta A \wedge F(A)), \qquad F(A) := dA + A^2 = 0.}$$

The MC equation $F(A) = 0$ defines a **flat connection**.

The cyclic L_∞ action

Let $\Omega^p(S^3, \mathfrak{su}(2))$ denote smooth matrix-valued p-forms and $\mathcal{H}_{\mathrm{CS}} := \Omega^\bullet(S^3, \mathfrak{su}(2))$ the space of forms on S^3.

$$[X, Y] := X \wedge Y - (-1)^{p_X p_Y} Y \wedge X,$$
$$\ell_1 X = dX, \qquad \ell_2(X, Y) = (-1)^{|X|}[X, Y], \qquad \ell_{n\geq 3} = 0,$$
$$\omega(X, Y) = (-1)^{|X|} \frac{k}{2\pi} \int_{S^3} \mathrm{tr}(X \wedge Y).$$

For the one-form A, $|A| = 0$ and $\ell_2(A, A) = 2A^2$, the CS action reads

$$\frac{1}{2}\omega(A, \ell_1 A) + \frac{1}{3!}\omega(A, \ell_2(A, A)) = S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int_{S^3} \mathrm{tr}(A \wedge dA) + \frac{k}{6\pi} \int_{S^3} \mathrm{tr}(A^3).$$

The Wess-Zumino term

The Chern-Simons action for a flat connection θ gives the **Wess-Zumino** term and ΔS_{CS}

$$S_{\mathrm{CS}}[\theta] = \frac{k}{4\pi}\int_{S^3} \mathrm{tr}\left(\theta \wedge d\theta + \frac{2}{3}\theta^3\right)$$
$$= \boxed{-\frac{k}{12\pi}\int_{S^3} \mathrm{tr}(\theta^3).}$$

With $A^g = g^{-1}Ag + \theta$, the Chern-Simons action shifts by

$$\Delta S_{\mathrm{CS}} = -\frac{k}{4\pi}\int_{S^3} d\,\mathrm{tr}(\theta \wedge g^{-1}Ag) - \frac{k}{12\pi}\int_{S^3}\mathrm{tr}(\theta^3)$$
$$= -\frac{k}{4\pi}\underbrace{\int_{\partial S^3}\mathrm{tr}(\theta \wedge g^{-1}Ag)}_{0} + S_{\mathrm{CS}}[\theta] = \boxed{S_{\mathrm{CS}}[\theta].}$$

A global gauge flow gives $\Delta S_{\mathrm{CS}} = 0$

A transformation g is **connected to the identity** if there is a smooth map $g(s, x)$ on $[0, 1] \times S^3$ with $g(0, x) = I_2$ and $g(1, x) = g(x)$. A **large gauge transformation** admits no such interpolation.

Suppose a **global** interpolation exists. Define $\theta(s) := g(s)^{-1} dg(s)$. For every s, $F(\theta(s)) = 0$. Then $\Delta S_{\mathrm{CS}} = 0$

$$\frac{d}{ds} S_{\mathrm{CS}}[\theta(s)] = \frac{k}{2\pi} \int_{S^3} \mathrm{tr}\big(\partial_s \theta(s) \wedge F(\theta(s))\big) = 0,$$

$$S_{\mathrm{CS}}[\theta] - S_{\mathrm{CS}}[0] = \int_0^1 ds \, \frac{d}{ds} S_{\mathrm{CS}}[\theta(s)] = 0.$$

Note that a large transformation g remains a smooth map on spacetime. The missing object here is a smooth interpolation $g(s, x)$ on **all** of $[0, 1] \times S^3$. We now choose interpolations on separate spacetime regions.

Patch-by-patch construction of spacetime

Let $\mathcal{M} = \bigcup_i U_i$ be spacetime covered by patches. In this section, i, j, k label patches. Define $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$.

Let Ψ_i be a local classical solution on U_i. One should first construct G_{ij} such that in U_{ij}, it is a **gauge transformation**

$$G_{ij} : \Psi_i|_{U_{ij}} \longrightarrow \Psi_j|_{U_{ij}} .$$

Likewise in a triple overlap

$$\boxed{H_{ijk} : G_{jk} \circ G_{ij} \Rightarrow G_{ik} \qquad \text{on } U_{ijk}.}$$

The same applies for higher homotopies

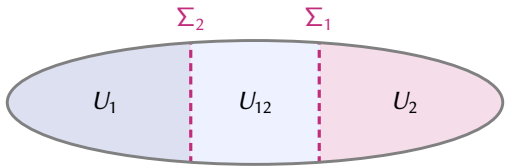
Two patches of S^3

Let z be a coordinate on S^3 and fix $0 < \varepsilon < 1$. Let

$$U_1 = \{z < \varepsilon\}, \quad U_2 = \{z > -\varepsilon\}, \quad U_{12} := U_1 \cap U_2 = \{-\varepsilon < z < \varepsilon\}.$$

Write $\Sigma_1 := \partial U_1$ at $z = \varepsilon$ and $\Sigma_2 := \partial U_2$ at $z = -\varepsilon$. We orient each surface by the outward normal from the indicated region. These orientations give $\partial U_{12} = \Sigma_1 + \Sigma_2$.

Dashed lines represent two-spheres. All fields extend smoothly to Σ_1 and Σ_2.



Both U_1 and U_2 are **contractible**. In contrast, U_{12} is $S^2 \times (-\varepsilon, \varepsilon)$: shrinking the interval leaves S^2.

Deriving the local gauge-flow equation

For $i = 1, 2$, choose a smooth interpolation on U_i with the boundary conditions

$$g_i(0, x) = I_2, \qquad g_i(1, x) = g(x) \quad (x \in U_i).$$

Define

$$\theta_i(s) := g_i(s)^{-1} dg_i(s), \qquad \lambda_i(s) := g_i(s)^{-1} \partial_s g_i(s).$$

λ_i is a zero-form, so a matrix-valued function. From $\partial_s g_i = g_i \lambda_i$, then

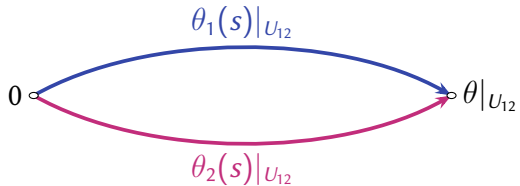
$$\begin{aligned} \partial_s \theta_i &= -\lambda_i \theta_i + g_i^{-1} d(g_i \lambda_i) \\ &= -\lambda_i \theta_i + \theta_i \lambda_i + d\lambda_i = d\lambda_i + [\theta_i, \lambda_i]. \end{aligned}$$

Here $[\theta_i, \lambda_i] = \theta_i \lambda_i - \lambda_i \theta_i$. Each $\theta_i(s)$ also satisfies $d\theta_i(s) + \theta_i(s)^2 = 0$.
Write $\theta_i := \theta_i(1) = \theta|_{U_i}$ to denote restriction to a specified spacetime patch U_i.

Boundary matching

On U_{12}, g_i and θ_i for $i = 1, 2$ have to match

$$g_1(1, x) = g_2(1, x) = g(x), \qquad \theta_1(0) = \theta_2(0) = 0, \qquad \theta_1(1) = \theta_2(1) = \theta|_{U_{12}}.$$



The arrows lie in the space of flat connections on U_{12}.

Express S_{CS} as $S_1 + S_2 - S_{12}$

The index $I \in \{1, 2, 12\}$ labels the regions U_1, U_2, U_{12}.

Define the **local** action on a_i

$$S_I[A_I] := \frac{k}{4\pi} \int_{U_I} \mathrm{tr}\left(A_I \wedge dA_I + \frac{2}{3} A_I^3 \right).$$

For $A_I := A|_{U_I}$, the **inclusion–exclusion identity** is

$$\boxed{S_{\mathrm{CS}}[A] = S_1[A_1] + S_2[A_2] - S_{12}[A_{12}].}$$

The pointwise weights explain the signs:

location of x	weight in $S_1 + S_2 - S_{12}$
$U_1 \setminus U_2$ or $U_2 \setminus U_1$	1
U_{12}	$1 + 1 - 1 = 1.$

The finite boundary correction cancels out

Now let us evaluate $S_I[A_I^{g_I}] - S_I[A_I]$ on U_I where the total derivative contributes on ∂U_I:

$$S_I[A_I^{g_I}] - S_I[A_I] = S_I[\theta_I] + \mathcal{A}_I, \qquad \mathcal{A}_I := -\frac{k}{4\pi}\int_{\partial U_I} \operatorname{tr}(dg_I\, g_I^{-1} \wedge A_I).$$

The scalar $\mathcal{A}_I$ is a **finite boundary correction**, not a connection. Computing $\mathcal{A}_{12}$ on U_{12} gives boundary terms

$$\begin{aligned}\mathcal{A}_{12} &= -\frac{k}{4\pi}\left(\int_{\Sigma_1} \operatorname{tr}(dg_1 g_1^{-1} \wedge A_1) + \int_{\Sigma_2} \operatorname{tr}(dg_2 g_2^{-1} \wedge A_2)\right)\\ &= \mathcal{A}_1 + \mathcal{A}_2.\end{aligned}$$

Using the inclusion–exclusion identity, we get ΔS_{CS}

$$\boxed{\begin{aligned}\Delta S_{\mathrm{CS}} &= S_1[\theta_1] + S_2[\theta_2] - S_{12}[\theta_{12}] + \underbrace{\mathcal{A}_1 + \mathcal{A}_2 - \mathcal{A}_{12}}_{0}\\ &= S_1[\theta_1] + S_2[\theta_2] - S_{12}[\theta_{12}].\end{aligned}}$$

The cancellation uses the finite endpoint maps g_I. No equality of $g_1(s,x)$ and $g_2(s,x)$ at intermediate s enters.

The primitive of the WZ three-form

For $i = 1, 2$, define the primitive of the WZ three-form, as the **spacetime two-form**

$$\boxed{b_i := -\frac{k}{4\pi}\int_0^1 ds\,\mathrm{tr}\big(\theta_i(s)\wedge\partial_s\theta_i(s)\big).}$$

Write $\vartheta = \theta_i(s)$ and $\dot\vartheta = \partial_s\theta_i(s)$. Differentiating $d\vartheta = -\vartheta^2$ gives $d\dot\vartheta = -\dot\vartheta\vartheta - \vartheta\dot\vartheta$. Consequently,

$$\begin{aligned} d\,\mathrm{tr}(\vartheta\wedge\dot\vartheta) &= \mathrm{tr}(d\vartheta\wedge\dot\vartheta - \vartheta\wedge d\dot\vartheta)\\ &= \mathrm{tr}(-\vartheta^2\dot\vartheta + \vartheta\dot\vartheta\vartheta + \vartheta^2\dot\vartheta)\\ &= \mathrm{tr}(\dot\vartheta\vartheta^2) = \tfrac{1}{3}\partial_s\mathrm{tr}(\vartheta^3). \end{aligned}$$

Integrating over s, using $\theta_i(0) = 0$ and $\theta_i(1) = \theta_i$ gives

$$db_i = -\frac{k}{12\pi}\big[\mathrm{tr}(\theta_i(s)^3)\big]_{s=0}^{s=1} = -\frac{k}{12\pi}\mathrm{tr}(\theta_i^3).$$

primitive of the WZ three-form

Use Stokes' theorem on U_1, U_2, and U_{12}

For a smooth two-form b on an oriented three-dimensional patch U, we can use Stokes' theorem to evaluate $S_i[\theta_i]$

$$S_i[\theta_i] = -\frac{k}{12\pi}\int_{U_i}\mathrm{tr}(\theta_i^3) = \int_{U_i} db_i = \int_{\Sigma_i} b_i, \qquad i = 1, 2.$$

On the overlap U_{12}

$$\theta_{12}(s) := \theta_2(s)|_{U_{12}}, \qquad b_{12} := -\frac{k}{4\pi}\int_0^1 ds\,\mathrm{tr}(\theta_{12}\wedge\partial_s\theta_{12}) = b_2|_{U_{12}}.$$

With $\partial U_{12} = \Sigma_1 + \Sigma_2$,

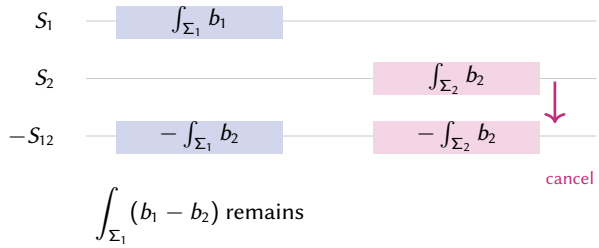
$$S_{12}[\theta_{12}] = \int_{\partial U_{12}} b_{12} = \int_{\Sigma_1} b_2 + \int_{\Sigma_2} b_2.$$

The connection $\theta_{12}|_{U_1}$ agrees with $\theta|_{U_{12}}$, even when $\theta_{12}(s)$ does not agree with $\theta_1(s)|_{U_{12}}$ for $0 < s < 1$.

Show that $\Delta S_{\mathrm{CS}} = \int_{\Sigma_1}(b_1 - b_2)$

Plugging the three boundary expressions into the local action sum yields

$$\Delta S_{\mathrm{CS}} = \int_{\Sigma_1} b_1 + \int_{\Sigma_2} b_2 - \int_{\Sigma_1} b_2 - \int_{\Sigma_2} b_2$$
$$= \boxed{\int_{\Sigma_1} (b_1 - b_2).}$$



Here S_I is $S_I[\theta_I]$. The two integrals over Σ_2 have opposite signs, so they cancel out The integral over Σ_1 however, maintains the difference between b_1 and b_2.

$d(b_1 - b_2) = 0$ does not imply $b_1 - b_2 = d\zeta$

On U_{12}, because the endpoints obey $\theta_1(1) = \theta_2(1)$, Then $d(b_1 - b_2)$ vanishes

$$d(b_1 - b_2) = -\frac{k}{12\pi}\mathrm{tr}\big(\theta_1(1)^3 - \theta_2(1)^3\big) = 0.$$

To distinguish closed from exact forms, let Ω_2 denote the area two-form of the oriented unit sphere S^2:

$$d\Omega_2 = 0, \qquad \int_{S^2} \Omega_2 = 4\pi.$$

A globally defined one-form ζ with $\Omega_2 = d\zeta$ would imply

$$4\pi = \int_{S^2} \Omega_2 = \int_{S^2} d\zeta = \int_{\partial S^2} \zeta = 0,$$

which is a contradiction. The overlap contains a closed two-sphere; the equation $d(b_1 - b_2) = 0$ alone does not force $\Delta S_{\mathrm{CS}} = 0$.

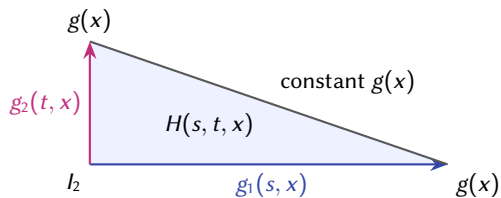
Define a homotopy with fixed endpoints

Define a smooth map on the 2-simplex

$$\Delta := \{(s,t) : s \geq 0,\ t \geq 0,\ s+t \leq 1\}, \qquad H : \Delta \times U_{12} \longrightarrow SU(2)$$

with boundary values

$$H(s,0,x) = g_1(s,x), \qquad H(0,t,x) = g_2(t,x),$$
$$H(v,1-v,x) = g(x), \qquad 0 \leq v \leq 1.$$



The paths $s \mapsto H(su, s(1-u), x)$ deform $g_2(s,x)$ at $u = 0$ into $g_1(s,x)$ at $u = 1$ with
$H(0,u,x) = I_2, \quad H(1,u,x) = g(x)$ for every $u \in [0,1]$. We demand that H be smooth on all of $\Delta \times U_{12}$.

Deriving the Maurer–Cartan equation on Δ^2

Define $\lambda_s := H^{-1}\partial_s H$ and $\lambda_t := H^{-1}\partial_t H$. The functions λ_s, λ_t take matrix values; the subscripts label parameter directions. Differentiation gives

$$\begin{aligned} \partial_s \lambda_t &= -\lambda_s \lambda_t + H^{-1}\partial_s \partial_t H, \\ \partial_t \lambda_s &= -\lambda_t \lambda_s + H^{-1}\partial_t \partial_s H, \\ \partial_s \lambda_t - \partial_t \lambda_s + [\lambda_s, \lambda_t] &= 0. \end{aligned}$$

This is **flatness in parameter space**, the compatibility condition for $\partial_s H = H\lambda_s$ and $\partial_t H = H\lambda_t$. On the diagonal,

$$0 = \frac{d}{dv} H(v, 1-v, x) = H(v, 1-v, x)\big(\lambda_s - \lambda_t\big)(v, 1-v, x).$$

Hence the constraint along the diagonal of Δ^2 is $\lambda_s - \lambda_t = 0$.

The group $SU(2)$ is **simply connected**: the group is path connected and every closed curve contracts to a point. The boundary map at each fixed x extends across Δ^2. This statement alone is not sufficient extend x smoothly throughout U_{12}.

Differentiate $b(u)$ along the homotopy

Assume H satisfies the boundary conditions and is **smooth** on $\Delta \times U_{12}$. Define

$$G(s,u,x) := H(su, s(1-u), x), \quad \vartheta(s,u) := G^{-1}dG, \quad \lambda_u := G^{-1}\partial_u G.$$

Here $s, u \in [0,1]$. The boundary values are $\vartheta(s,0) = \theta_2(s)$, $\vartheta(s,1) = \theta_1(s)$, $\vartheta(0,u) = 0$, and $\vartheta(1,u) = \theta|_{U_{12}}$.

Define $b(u) := -\frac{k}{4\pi}\int_0^1 ds\,\mathrm{tr}(\vartheta \wedge \partial_s \vartheta)$. Write $\vartheta_s := \partial_s \vartheta$ and $\vartheta_u := \partial_u \vartheta$. Integration by parts in s gives

$$\begin{aligned}
\partial_u b(u) &= -\frac{k}{4\pi}\int_0^1 ds\,\mathrm{tr}(\vartheta_u \wedge \vartheta_s + \vartheta \wedge \partial_s \vartheta_u) \\
&= -\frac{k}{4\pi}\left(2\int_0^1 ds\,\mathrm{tr}(\vartheta_u \wedge \vartheta_s) + \left[\mathrm{tr}(\vartheta \wedge \vartheta_u)\right]_{s=0}^{s=1}\right) \\
&= -\frac{k}{2\pi}\int_0^1 ds\,\mathrm{tr}(\vartheta_u \wedge \vartheta_s).
\end{aligned}$$

The endpoint boundary term vanishes because $\vartheta_u(0,u) = \vartheta_u(1,u) = 0$.

Showing $b_1 - b_2$ is exact

Differentiating $\vartheta = G^{-1}dG$ with $\lambda_u = G^{-1}\partial_u G$ gives $\vartheta_u = d\lambda_u + [\vartheta, \lambda_u]$. Differentiating $d\vartheta = -\vartheta^2$ in s gives $d\vartheta_s = -\vartheta_s\vartheta - \vartheta\vartheta_s$. In the trace,

$$\begin{aligned}\operatorname{tr}(\vartheta_u \wedge \vartheta_s) &= d\operatorname{tr}(\lambda_u\vartheta_s) - \operatorname{tr}(\lambda_u d\vartheta_s) + \operatorname{tr}([\vartheta, \lambda_u]\vartheta_s) \\ &= d\operatorname{tr}(\lambda_u\vartheta_s) + \operatorname{tr}\big(\lambda_u(\vartheta_s\vartheta + \vartheta\vartheta_s)\big) - \operatorname{tr}\big(\lambda_u(\vartheta_s\vartheta + \vartheta\vartheta_s)\big) \\ &= d\operatorname{tr}(\lambda_u\vartheta_s).\end{aligned}$$

Integrate $\partial_u b(u)$ from $u = 0$ to $u = 1$:

$$\boxed{b_1 - b_2 = d\zeta, \qquad \zeta := -\frac{k}{2\pi}\int_0^1 du \int_0^1 ds \operatorname{tr}(\lambda_u\vartheta_s).}$$

The one-form ζ is smooth on all of U_{12}. Since $\partial\Sigma_1 = \varnothing$,

$$\boxed{\Delta S_{\mathrm{CS}} = \int_{\Sigma_1}(b_1 - b_2) = \int_{\Sigma_1} d\zeta = \int_{\partial\Sigma_1}\zeta = 0.}$$

If $\Delta S_{\mathrm{CS}} \neq 0$, no smooth $H : \Delta \times U_{12} \to SU(2)$ satisfies the given boundary values.

To recap what just happened

A smooth 2-simplex over the entire spacetime overlap U_{12} implies $\Delta S_{\mathrm{CS}} = 0$.

Calculate the action shift.
For two local gauge paths g_1 and g_2, the two-patch calculation gives

$$\boxed{\Delta S_{\mathrm{CS}} = \int_{\Sigma_1} (b_1 - b_2).} \tag{8}$$

Here b_i are the boundary two-forms determined by the gauge paths, and Σ_1 is the closed oriented surface on which the two-patch calculation leaves a boundary integral.
At the path endpoints, connections are equal which implies

$$\boxed{d(b_1 - b_2) = 0.} \tag{9}$$

Equation (9) alone does *not* imply that the integral in (8) vanishes.

A globally smooth 2-simplex implies $\Delta S_{\mathrm{CS}} = 0$

Assume that the edge data (of the gauge paths) extend over Δ^2.
Let $\Delta^2 = \{(s,t) : s \geq 0,\ t \geq 0,\ s+t \leq 1\}$. Suppose a smooth map exists,

$$H : \Delta^2 \times U_{12} \longrightarrow SU(2), \tag{10}$$

whose restrictions to the three edges satisfy

$$\begin{aligned} H(s,0,x) &= g_1(s,x), \\ H(0,t,x) &= g_2(t,x), \\ H(v,1-v,x) &= g(x), \qquad s,t,v \in [0,1], \quad x \in U_{12}. \end{aligned} \tag{11}$$

Under this additional assumption, the homotopy calculation yields a *globally defined* one-form $\zeta \in \Omega^1(U_{12})$ such that

$$\boxed{b_1 - b_2 = d\zeta.} \tag{12}$$

Since $\partial\Sigma_1 = \varnothing$, Stokes' theorem gives

$$\boxed{\Delta S_{\mathrm{CS}} = \int_{\Sigma_1} d\zeta = \int_{\partial\Sigma_1} \zeta = 0.} \tag{13}$$

A nonzero Wess–Zumino term obstructs the 2-simplex

Now consider a large gauge transformation.
For a map $g : S^3 \to SU(2)$ of winding number $N \neq 0$, the Chern–Simons action changes by

$$\boxed{\Delta S_{\mathrm{CS}} = 2\pi k N \neq 0, \qquad k \neq 0.} \tag{14}$$

Here k is the Chern–Simons level and $N = \deg(g) \in \mathbb{Z}$. The integrality of N follows from the classical degree theorem.
Equations (13) and (14) cannot both hold. Consequently,

$$\boxed{N \neq 0 \quad \Longrightarrow \quad \text{no smooth } H \text{ satisfying (11) exists on } \Delta^2 \times U_{12}.} \tag{15}$$

At an individual point $x \in U_{12}$, the two gauge paths may admit an endpoint-fixed homotopy. Equation (15) excludes a choice that depends smoothly on x throughout U_{12}.

The calculation in CSFT?

In this calculation, we used $\ell_1 = d$, the graded matrix bracket ℓ_2, and the normalized three-form ν on $SU(2)$. But the CSFT gauge-flow equations contain the complete string products ℓ_n.

Let χ_I be a smooth cutoff equal to one inside a region and zero outside a surrounding transition region. For string-state inputs $X_1, \ldots, X_n$, define

$$[\ell_n, \chi_I](X_1, \ldots, X_n) := \ell_n(\chi_I X_1, X_2, \ldots, X_n) - \chi_I \ell_n(X_1, \ldots, X_n).$$

A CSFT product is **nonlocal** when the output depends on fields at separated spacetime points. The terms $[\ell_n, \chi_I]$ need not be confined to a sharp boundary. The transition region of χ_I is a **smeared boundary**.

CSFT also includes **diffeomorphisms**, smooth invertible coordinate maps with smooth inverses. Finite transformations can thus move the boundaries of the spacetime patches.

In full CSFT, we should evaluate the cutoff terms, establish finite matching conditions, and determine whether a normalized integral with integer values follows from the string products.

A FEW RELATED WORK

Formal proofs: Getzler and Rogers:

Getzler E. Getzler, *Ann. Math.* 170 (2009), 271–301. Works with a **nilpotent** L_∞ algebra. works with **smooth forms**, whose coefficients are polynomials in the simplex coordinates, so no limits are ever taken. Nilpotence makes the recursion that fills a horn stop after finitely many steps:

$$\boxed{L \text{ nilpotent } \implies \mathrm{Sol}_\bullet(L) \text{ is a Kan complex.}}$$

Rogers C. L. Rogers, *J. Pure Appl. Algebra* 227 (2023), 107403; arXiv:2008.01706.

Replaces nilpotence by a **complete filtration**: a nested sequence $L = F^1L \supset F^2L \supset \cdots$ compatible with the products,

$$\ell_n\big(F^{p_1}L, \ldots, F^{p_n}L\big) \subset F^{p_1+\cdots+p_n}L,$$

in which every infinite sum of ever-deeper corrections converges. Then

$$\boxed{L \text{ complete filtered } \implies \mathrm{Sol}_\bullet(L) \text{ is a Kan complex.}}$$

Getzler and Rogers: extending solutions on simplices

Getzler (2009). E. Getzler, *Ann. Math.* 170 (2009) 271–301, arXiv:math/0404003;
An L_∞ algebra is **nilpotent** when nested brackets with sufficiently many inputs vanish. Getzler proves that solutions on the faces of a horn, agreeing on their common faces, extend to the simplex.

Rogers (2023). C. L. Rogers, *J. Pure Appl. Algebra* 227 (2023) 107403, arXiv:2008.01706.
Rogers allows corrections to the solution at every order. For CSFT, $\mathfrak{g}$ is the graded space of string states with the products ℓ_n. A **filtration** sorts series by their lowest power of a formal parameter τ. **Completeness** means that answers to every order, agreeing at all lower orders, determine one series. Then

$$\boxed{\mathrm{Sol}_\bullet(\tau\mathfrak{g}[[\tau]]) \text{ is a Kan complex.}}$$

The higher homotopies exist as formal series. Convergence at a nonzero numerical value of τ is not proved.

An analytic attempt: Henriques

For a **finite type** L_∞ algebra, **Henriques** replaces polynomial forms by C^r **forms**, whose coefficients have r continuous derivatives. These form a **Banach space** — a complete normed vector space — so calculus, such as the implicit function theorem, is available. He proves

$$\boxed{L \text{ of finite type} \implies \int L \text{ is a Kan simplicial Banach manifold with } \textbf{one} \text{ vertex.}}$$

His algebras have **no field degree**, so there is **one** vertex: a **Lie** ∞**-group**.

For an ordinary Lie algebra, his construction gives the **classifying space** of the simply connected group — the one-vertex simplicial space whose m-simplices are m-tuples of group elements. For the **string Lie 2-algebra** — a Lie algebra extended by a ternary bracket built from its invariant 3-form — it recovers a model of the **string group**, the higher version of the spin group that governs string structures in heterotic string theory.

For CSFT: L **is infinite with many vertices (solutions).**

Another analytic attempt: Ševera–Širaň

An L_∞**-algebroid** is an L_∞ algebra whose brackets vary over a **base manifold** M: a bundle of graded vector spaces over M, with brackets on its sections. Equivalently, it is a **differential graded manifold**: a manifold with extra graded coordinates and a vector field Q with $Q^2 = 0$. When M is a point, it is an ordinary L_∞ algebra.

Ševera and Širaň integrate L_∞-algebroids over a finite-dimensional base. The Maurer–Cartan forms on simplices form a

Kan simplicial Banach (or Fréchet) manifold, with many vertices.

A **Fréchet space** is a slightly more general kind of complete space. The vertices are the points of M. Their motivating examples include Poisson manifolds and Courant algebroids, whose bracket governs diffeomorphisms combined with B-field gauge transformations.

For CSFT: the vertices would have to be the classical solutions, an infinite-dimensional space that need not be smooth. The finite-dimensional base is exactly what fails.

Henriques: solutions on whole simplices

Henriques (2008). A. Henriques, *Integrating L_∞-algebras*, *Compos. Math.* 144 (2008) 1017–1045, arXiv:math/0603563.

Constructs differential forms solving the Maurer–Cartan equation on whole simplices, not just coefficients of a formal series. He proves the Kan property without assuming nilpotence.

The algebra is **finite type**: each graded subspace is finite-dimensional. In this talk's grading it has only negative degrees, so the total field has no zero-form component. On a point only the zero solution remains. **The integrated simplicial set has one vertex.**

L_∞-algebra connections and applications to String- and Chern-Simons n-transport

Hisham Sati, Urs Schreiber, and Jim Stashe (2008). *In: Fauser, B., Tolksdorf, J., Zeidler, E. (eds) Quantum Field Theory. Birkhäuser Basel. https://doi.org/10.1007/978-3-7643-8736-5-17.*

Gives a generalization of the notion of a Cartan-Ehresmann connection from Lie algebras to L_∞-algebras and use it to study the obstruction theory of lifts through higher String-like extensions of Lie algebras. We find (generalized) Chern-Simons and BF-theory functionals this way and describe aspects of their parallel transport and quantization.

This is especially relevant to our Chern-Simons work if we level-truncate to a some level N.

Other related work

Schubert (1992)

Lett. Math. Phys. 26 (1992) 259–264.

Schubert constructs finite gauge transformations of classical, nonpolynomial CSFT by iterating infinitesimal transformations, and gives a simple rule for the coefficients of the resulting series.

Hohm and Zwiebach (2017)

Fortsch. Phys. 65 (2017) 1700014, arXiv:1701.08824.

Hohm and Zwiebach show that a gauge theory carries an L_∞ algebra for the gauge structure and a larger L_∞ algebra for the full interacting theory. A gauge structure that is a strict Lie algebra can still require the full L_∞ structure once interactions are included.

Hohm and Zwiebach (2013)

JHEP 02 (2013) 075, arXiv:1207.4198.

In double field theory, Hohm and Zwiebach construct finite gauge transformations by exponentiating the infinitesimal ones, and describe the finite transformations by coordinate maps $X \mapsto X'$ of the doubled space. Naseer proves that the formula equals the exponential for a suitable choice of parameter. Two successive finite transformations combine through the Courant bracket and form a group.

Future directions and ongoing work

- Finite gauge structure and spacetime gluing: Application to Frenkel-Kim patch-by-patch description of spacetime?
- Classical background independence?
- An analytic proof of the Kan complex ∞ groupoid with an infinite number of vertices ?

ADDITIONAL SLIDES

Toward an analytic statement

The idea. Make the order-by-order construction converge in a **norm** instead rather perturbatively expanding in powers of a small parameter τ.

The input: a growth estimate. On a Banach completion of the state space, the string products are continuous, obey the L_∞ identities, and satisfy, with the same C, A for every $n \geq 1$,

$$\boxed{\|\ell_n(X_1, \ldots, X_n)\| \leq C\, n!\, A^{n-1}\, \|X_1\| \cdots \|X_n\|.}$$

For $n = 1$ this includes Q_B, we use the norm $\|X\| + \|Q_B X\|$ can handle; We don't know what norm to use for ℓ_n with $n \geq 2$.

What remains open

The estimate. The growth estimate $\|\ell_n(X_1, \ldots, X_n)\| \leq C\, n!\, A^{n-1} \|X_1\| \cdots \|X_n\|$ has not been proved for the string products of CSFT.

Smoothness. The space of solutions may be singular. Even in a small finite-dimensional example satisfying the estimate, the solutions form the crossing of two axes, $ab = 0$. So, unlike in Henriques and Ševera–Širaň, the Kan complex should not be expected to be a manifold.

Provided we can derive the growth estimate, the solutions within a radius R of a reference solution form an analytic Kan complex. Proving the estimate for CSFT remains an open problem.

PATCH-BY-PATCH DESCRIPTION OF SPACETIME

Finite gauge structure and spacetime gluing

Within one CSFT, how do finite gauge transformations between classical solutions compose, and what are their higher homotopies?

$$\Psi_i, \qquad G_{ij}, \qquad H_{ijk}, \qquad K_{ijkl}, \ldots$$

Alex Frenkel and Manki Kim construct a gauge-invariant patchwise action satisfying the BV master equation.

Their construction uses **interpolating string fields** on **spacetime** overlaps and **smeared-boundary conditions**.

A. Frenkel and M. Kim, *Non-linear sigma model in string field theory*, arXiv:2509.20527v1 (2025), §3.

Application to Frenkel-Kim patch-by-patch construction

Let $\mathcal{M} = \bigcup_i U_i$ be spacetime covered by patches. In this section, i, j, k label patches. Define $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$.

Let Ψ_i be a local classical solution on U_i. One should first construct G_{ij} such that in U_{ij}, it is a **gauge transformation**

$$G_{ij} : \Psi_i|_{U_{ij}} \longrightarrow \Psi_j|_{U_{ij}} .$$

Likewise in a triple overlap

$$\boxed{H_{ijk} : G_{jk} \circ G_{ij} \Rightarrow G_{ik} \qquad \text{on } U_{ijk}.}$$

The same applies for higher homotopies

The task now is to explicitly construct of these higher morphisms such that they all satisfy Frenkel-Kim's interpolation maps and boundary conditions.

Reconstruct a global configuration from local (patch) fields

For scalar fields.
Functions $f_i : U_i \to \mathbb{R}$ satisfying $f_i = f_j$ on U_{ij} combine into one function $f : \mathcal{M} \to \mathbb{R}$ with $f|_{U_i} = f_i$.

For gauge fields, it is more subtle because of gauge transformations.
In this generalization, the local data is

$$\begin{aligned}
&\Psi_i \quad \text{on } U_i && : \text{local classical solutions,}\\
&G_{ij} \quad \text{on } U_{ij} && : \text{finite identifications,}\\
&H_{ijk} \quad \text{on } U_{ijk} && : \text{homotopies of those identifications,}\\
&\vdots && \text{higher homotopies on higher overlaps.}
\end{aligned}$$

A **compatible** collection should describe **one field configuration**. Transformations on double overlaps **must be the boundaries of complete higher homotopies on triple and higher overlaps.** The **transition maps** are essential here.

Homotopy descent must recover the transformations

Assume that solutions in each patch and their restriction maps are given.

Define $\mathrm{Sol}_\bullet(\mathcal{M})$ to be the **solution ∞-groupoid** on $\mathcal{M}$.

Define $\mathrm{Desc}_\bullet(\{U_i\})$ to contain **all compatible** collection of data $(\Psi_i, G_{ij}, H_{ijk}, \ldots)$, **their gauge transformations, and higher homotopies**.

Descent requires restriction maps to satisfy an equivalence relation

$$\boxed{\mathrm{Sol}_\bullet(\mathcal{M}) \xrightarrow{\simeq} \mathrm{Desc}_\bullet(\{U_i\}).}$$

Intuitively: the data on $\mathcal{M}$ and patches must be equivalent up to gauge transformation, together with equivalent gauge symmetries and higher homotopies.

V. Hinich, arXiv:alg-geom/9606010; R. Bandiera, arXiv:1705.02880.

Čech homotopy descent

Let $\mathcal{M} = \bigcup_i U_i$ be an open cover of spacetime. Suppose that every U_i has a solution ∞ -groupoid $\mathsf{Sol}_\bullet(U_i)$, and that restriction to smaller regions is defined and respects composition of restriction maps.

Write $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. The theory satisfies **Čech homotopy descent** for $U_{i_0 \cdots i_p}$ when the restriction map

$$\mathsf{Sol}_\bullet(\mathcal{M}) \longrightarrow \operatorname*{holim}_{[p] \in \Delta} \prod_{i_0, \ldots, i_p} \mathsf{Sol}_\bullet \left(U_{i_0 \cdots i_p} \right)$$

is a **weak equivalence**. The category Δ contains the ordered sets $[p] = \{0, \ldots, p\}$ as objects and order-preserving maps as morphisms.