Contravariant functoriality for twisted K-theory of stacks

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The purpose of this note is to clarify what is perhaps a small gap in the literature by explaining how the twisted K-theory of topological stacks as defined in [1] is contravariantly functorial. The content here is an exposition of a result from [4], an extension from Morita equivalences to proper generalized morphisms of the main theorem of [3].

Let $\mathfrak{X}$ be a topological stack and $\mathfrak{G} \to \mathfrak{X}$ an S^1-banded gerbe, so there is a twisted K-group $K^{\mathfrak{G},*}(\mathfrak{X})$. If

$$\begin{tikzcd}
\mathfrak{H} \arrow[r] \arrow[d] & \mathfrak{G} \arrow[d] \\
\mathfrak{Y} \arrow[r, "f"] & \mathfrak{X}
\end{tikzcd}$$

is a morphism of S^1-gerbes then one hopes that there is a well-defined homomorphism

$$f^* : K^{\mathfrak{G},*}(X) \to K^{\mathfrak{H},*}(\mathfrak{Y}).$$

and that these maps should respect composition. We show how to do this when f is *proper* in an appropriate sense. To keep the description simple we present the construction only in the untwisted case; it is entirely straightforward to extend to the twisted setting.

Proper generalized morphisms. By choosing an atlas we can represent a stack by a groupoid. A representable morphism of topological stacks can be given on the level of atlases by a homomorphism of groupoids if one is allowed to refine the atlas of the domain. The data of a homomorphism out of a refinement of the domain atlas is called a 'generalized morphism' of groupoids. See [2] for more details.

For our purposes the bibundle formulation of generalized morphisms is more convenient. Let $A_1 \rightrightarrows A_0$ and $B_1 \rightrightarrows B_0$ be locally compact Hausdorff groupoids.

Definition 0.1. *A generalized morphism* $Z : A_\bullet \to B_\bullet$ *is a diagram*

$$A_0 \xleftarrow{\alpha} Z \xrightarrow{\beta} B_0$$

where Z *has a left* $A_\bullet$*-action with respect to* α *and a commuting right* $B_\bullet$*-action with respect to* β *such that* $Z \xrightarrow{\alpha} A_0$ *is a principal* $B_\bullet$*-bundle.*

Definition 0.2. *Such a generalized morphism is said to be proper if the action of* $A_\bullet$ *on* Z *is proper and for every compact set* $K \subset B_0$ *there exists a compact set* $L \subset Z$ *such that* $\beta^{-1}(K)$ *is contained in the* $A_\bullet$*-orbit of* L*.*

Note that a homomorphism of groupoids $f : A_\bullet \to B_\bullet$ produces a generalized morphism $Z_f = A_0 \times_{f,B_0,\mathrm{trgt}} B_1$ which is proper if f is a proper homomorphism. Also, Morita equivalences of groupoids are proper.

Locally compact groupoids with *isomorphism classes* of generalized morphisms form a category. The composition of

$$A_0 \leftarrow Z \rightarrow B_0 \text{ and } B_0 \leftarrow W \rightarrow C_0$$

is $A_0 \leftarrow Z \times_{B_\bullet} W \rightarrow C_0$. The composition of two proper generalized morphisms is again proper. For homomorphisms f, g one has $Z_{f\circ g} \cong Z_f \circ Z_g$.

From groupoids to C^*-algebras. Let $(A_\bullet, \lambda)$ be a locally compact groupoid with Haar system, so $C_c(A_1)$ becomes an algebra by convolution. For each $x \in A_0$ there is a natural *-representation of $C_c(A_1)$ on $L^2(A_1^x, \lambda^x)$. The *reduced norm* of $f \in C_c(A_1)$ is the given by taking the supremum of the operator norms of these representations. The completion with respect to the reduced norm is the *reduced C^*-algebra* $C_r^*(A_\bullet, \lambda)$.

Let $(B_\bullet, \mu)$ be a second groupoid with Haar systems and consider a proper generalized morphism $A_\bullet \xrightarrow{Z} B_\bullet$. Associated to these two groupoids are reduced C^*-algebras $\mathscr{A} = C_r^*(A_\bullet, \lambda)$ and $\mathscr{B} = C_r^*(B_\bullet, \mu)$. We will now show how to construct a special kind of ($\mathscr{B}$,$\mathscr{A}$)-bimodule $\mathscr{M}_Z$ from the generalized morphism of groupoids Z.

Start with $C_c(Z)$. This has a left action of $C_c(B_1)$ given by

$$(f \cdot \zeta)(z) = \int_{B_1^{\beta(z)}} f(b)\zeta(zb) d\mu^{\beta(z)}(b),$$

and a right action of $C_c(A_1)$ given by

$$(\zeta \cdot g)(z) = \int_{A_1^{\alpha(z)}} \zeta(a^{-1}z) g(a^{-1}) d\lambda^{\alpha(z)}(a).$$

Furthermore, it has a $C_c(A_1)$-valued inner product given as follows: choose $z \in Z$ with $\beta(z) = \mathrm{src}(a)$ (note that by assumption $B_\bullet$ acts freely and transitively on the fibres of β), and set

$$\langle \zeta, \xi \rangle_{\mathscr{A}}(a) = \int_{B^{\beta(z)}} \overline{\zeta(a^{-1}zb)} \xi(zb) d\mu^{\beta(z)}(b)$$

(Note that we cannot also define a $C_c(B_1)$-valued inner product unless $Z \rightarrow B_0$ is a principal $A_\bullet$-bundle.) The bimodule $\mathscr{M}_Z$ is defined as the completion of $C_c(Z)$ with respect to the norm given by the $\mathscr{A}$-valued norm followed by the norm on $\mathscr{A}$. Clearly $\mathscr{M}_Z$ is a right $\mathscr{A}$-Hilbert module with a left action of $\mathscr{B}$.

Proposition 0.1. *$\mathscr{B}$ acts by $\mathscr{A}$-compact operators, i.e. operators contained in*

$$\mathbb{K}_{\mathscr{A}}(\mathscr{M}_Z) = \textit{operator norm closure of } \{|\zeta\rangle\langle\zeta'|_{\mathscr{A}} \textit{ for } \zeta, \zeta' \in \mathscr{M}_Z\} \subset \mathrm{End}(\mathscr{M}_Z).$$

Proof. Let ψ be an element of $C_c(B_1)$ and let $K = \operatorname{supp} \psi$. By the definition of properness for generalized morphisms there exists $L \subset Z$ compact with $\beta^{-1}(K)$ contained in the $A_\bullet$-orbit of L. One can choose functions $f_1, \ldots, f_n \in C_c(Z)$ with

$$\sum_{i=1}^{n} \int_{A_1^{\alpha(z)}} f_i(a^{-1}z) d\lambda^{\alpha(z)}(a) = 1 \text{ for all } z \in L.$$

Hence for any $z \in Z$ and $\zeta \in C_c(Z)$

$$(\psi \cdot \zeta)(z) = \sum_{i=1}^{n} \int_{A_1^{\alpha(z)}} f_i(a^{-1}z) \int_{B_1^{\beta(z)}} \psi(b)\zeta(zb)d\mu^{\beta(z)}(b)d\lambda^{\alpha(z)}(a). \tag{1}$$

Since $Z \xrightarrow{\alpha} A_0$ is a $B_\bullet$ principal bundle we can set $F_i(z, zb) = f_i(z)\psi(b)$, which gives a well-defined and compactly supported function on a closed subset of $Z \times Z$. Extend F_i to an element $\widetilde{F}_i$ of $C_c(Z \times Z)$. Now the algebraic tensor product $C_c(Z) \otimes C_c(z)$ is dense in $C_c(Z \times Z)$ with respect to the obvious norm, so one can write

$$\widetilde{F}_i = \sum_{k=1}^{\infty} g_{ik} \otimes \overline{h}_{ik}.$$

Then

$$f_i(a^{-1}z)\psi(b) = F_i(a^{-1}z, a^{-1}zb) = \sum_{k=1}^{\infty} g_{ik}(a^{-1}z)\overline{h_{ik}(a^{-1}zb)}. \tag{2}$$

Observe that for $\zeta, \zeta', \xi \in \mathscr{M}_Z$ we have

$$|\zeta\rangle\langle\zeta'|\xi\rangle_{\mathscr{A}}(z) = \int_{A_1^{\alpha(z)}} \zeta(a^{-1}z) \int_{B_1^{\beta(z)}} \overline{\zeta'(a^{-1}zb)}\xi(zb)d\mu^{\beta(z)}(b)d\lambda^{\alpha(z)}(a). \tag{3}$$

By inserting (2) into (1), we see from (3) that

$$\psi \cdot \zeta = \sum_{i=1}^{n} \sum_{k=1}^{\infty} |g_{ik}\rangle\langle h_{ik}|\zeta\rangle_{\mathscr{A}}.$$

(Here we use that convergence of functions $\widetilde{F}_i$ in $C_c(Z \times Z)$ implies convergence of the associated finite rank operators in the operator norm.) ☐

Note that the C^*-algebraic suspension $S\mathscr{M}_Z := \mathscr{M}_Z \otimes C_0(\mathbb{R})$ is naturally a right $S\mathscr{A}$-Hilbert module with an action of $S\mathscr{B}$ by $S\mathscr{A}$-compact operators.

Kasparov's bivariant KK-theory. For C^*-algebras $\mathscr{A}$, $\mathscr{B}$, Kasparov has defined a bivariant theory $KK(\mathscr{B}, \mathscr{A})$ such that

1. $K_*(\mathscr{A}) \cong KK(\mathbb{C}, S^*\mathscr{A})$.
2. There are associative Kasparov products
$$KK(\mathscr{C}, \mathscr{B}) \times KK(\mathscr{B}, \mathscr{A}) \xrightarrow{\otimes} KK(\mathscr{C}, \mathscr{A}).$$
3. A right $\mathscr{A}$-Hilbert module with a left action of $\mathscr{B}$ by $\mathscr{A}$-compact operators represents a class in $KK(\mathscr{B}, \mathscr{A})$. (But not all classes are represented by such bimodules.)

Now, suppose $Z : A_\bullet \to B_\bullet$ is a generalized morphism of groupoids. Pick Haar systems and form the reduced C^*-algebras $\mathscr{A} = C_r^*(A_\bullet, \lambda)$ and $\mathscr{B} = C_r^*(B_\bullet, \mu)$, and form the bimodule $\mathscr{M}_Z$. We get a morphism $K^*(B_\bullet) \to K^*(A_\bullet)$ from the diagram

$$K^{-i}(B_\bullet) = K_i(\mathscr{B}) \cong KK(\mathbb{C}, S^i\mathscr{B}) \xrightarrow{(-)\otimes S^i\mathscr{M}_Z} KK(\mathbb{C}, S^i\mathscr{A}) \cong K^{-i}(B_\bullet).$$

These morphisms behave correctly with respect to composition.

Proposition 0.2. *For generalized morphisms*

$$A_0 \leftarrow Z \rightarrow B_0 \text{ and } B_0 \leftarrow W \rightarrow C_0$$

one has the identity $[\mathscr{M}_{W \circ Z}] = [\mathscr{M}_W] \otimes [\mathscr{M}_Z]$ *in* $KK(\mathscr{C}, \mathscr{A})$.

Proof. This is because the algebraic tensor product $C_c(W) \otimes_{\mathscr{B}} C_c(Z)$ becomes isomorphic to $C_c(W \times_{B_\bullet} Z)$ after completing both of these into C^*-algebras. ☐

Proposition 0.3. *If* Z *is the identity generalized morphism then multiplication by* $\mathscr{M}_Z$ *is the identity on* KK*-theory.*

Proof. The identity generalized morphisms on $A_\bullet$ is $A_0 \overset{\text{trgt}}{\leftarrow} A_1 \overset{\text{trgt}}{\rightarrow} A_0$. So $\mathscr{M}_{\text{id}}$ is $\mathscr{A} = C^*_r(A_\bullet, \lambda)$, and tensoring with this is clearly the identity. ☐

Thus we have a functor from locally compact groupoids with Haar system and proper generalized morphisms to the category of $\mathbb{Z}/2$-graded abelian groups.

Theorem 0.1. *The above gives a well-defined functor from (locally compact Hausdorff) topological stacks and representable proper morphisms to* $\mathbb{Z}/2$*-graded abelian groups.*

Proof. Given any two atlases for a stack there is a canonical generalized isomorphism between them (i.e. a Morita equivalence). Furthermore, given any choice of Haar systems on these atlas groupoids the associated reduced C^*-algebras $\mathscr{A}$, $\mathscr{B}$ have a canonical bimodule (constructed from the canonical generalized isomorphism) representing an element in $KK(\mathscr{B}, \mathscr{A})$. Multiplication by this element induces a canonical isomorphism between the K-groups constructed from the two atlases and Haar systems. So the functor is well-defined on objects. It is clearly now also well-defined on arrows. ☐

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