14

Noncompact Manifolds and Ends

14.1 Introduction

The previous chapter showed that the notion of the ends of a manifold provides an important distinction between Euclidean space (R^n, λ), where ergodicity is typical for volume preserving homeomorphisms, and the strip $(R^1 \times [0,1], \lambda)$, where ergodicity is not typical for volume preserving homeomorphisms. The distinction between the spaces is that R^n (for $n \geq 2$) has only one end, whereas the strip has two. In this chapter we present a formal description of the ends of the manifold X, and related topological results. This will enable us to obtain in the following chapter conditions on the ends of a measured manifold (X, μ) under which ergodicity is typical in certain closed subspaces of $\mathcal{M}[X, \mu]$.

We now provide the formal definitions and results mentioned in this informal discussion. Most of these results come from [20] and [22].

14.2 End Compactification

The notion of an end of a manifold has already been discussed informally. We now give a formal treatment.

Definition 14.1 *An* end e *of the manifold* X *is a function which assigns to each compact subset* K *of* X *a nonempty unbounded component* $e(K)$ *of* $X - K$ *in such a way that*

$$K_1 \subset K_2 \textit{ implies } e(K_2) \subset e(K_1). \tag{14.1}$$

The set of all ends of X *is denoted* $E[X]$.

Note that if X is itself compact then it has no unbounded subsets. The notion of an end only has significance for us when the manifold X is not compact.

Observe that because X is a manifold, $X-K$ has only a finite number of unbounded components (see Lemma A2.10 from Appendix 2) and for each end e only one of those components of $X-K$ (namely $e(K)$) 'leads to' e. By adjoining to a compact set K the union of all the bounded components of $X-K$, we obtain a larger compact set $\hat{K}$ whose complement has no unbounded components. Furthermore the components of $X-\hat{K}$ are the same as the unbounded components of $X-K$.

The manifold X is compactified by adjoining the set of ends $E[X]$, and defining for each compact set $K \subset X$ a basic neighborhood $N_K(e_0)$ of an end $e_0 \in E[X]$, as the set

$$N_K(e_0) = e_0(K) \cup \{e \in E[X] : e(K) = e_0(K)\}.$$

With this topology, $X \cup E[X]$ is a compact Hausdorff space containing $E[X]$ as a closed subset. Again because for each compact set $K \subset X$ there are only finitely many unbounded components of $X-K$, these neighborhoods $N_K(e_0) \cap E[X]$ form a basis of closed and open sets for e_0 in $E[X]$; thus, with the relative topology on $E[X]$, the ends form a totally disconnected set.

14.3 Examples of End Compactifications

We now reconsider some of the examples given informally in the previous chapters. For $n \geq 2$, we have noted that R^n has a single end. The 'end compactification' topology of $R^n \cup E[R^n]$ given by the above definition is the usual one-point compactification which makes it into a topological sphere (homeomorphic to S^n). The cylinder $R^1 \times S^1$ however has two ends since compact sets such as $[a,b] \times S^1$ divide the space into two unbounded components, one of them leading to point at '$-\infty$' on the cylinder (the left end) and the other leading to the point at '$+\infty$' (the right end). Here again the compactification of the cylinder is the sphere S^2 and here the end set $E \subset S^2$ consists of two points. In general one can start with a compact manifold like S^2 and let $E \subset S^2$ be any totally disconnected set. Then E is the end set for the noncompact manifold $S^2 - E$. Although we will not need to deal with such problems we do note that the end sets may be embedded in S^2 in a very complicated fashion. For example when we take E to be the wild Cantor set as in Alexander's horned sphere, then the components $e(K)$ are intertwined

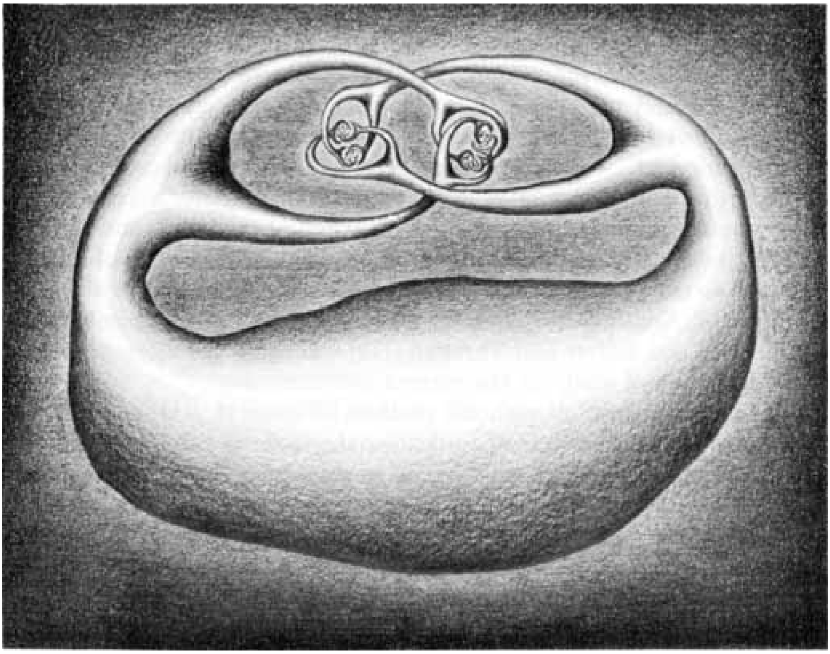

Fig. 14.1. Alexander's horned sphere (with permission from [73])

in an intricate manner. A picture of Alexander's horned sphere from Hocking and Young's *Topology* textbook [73, p. 176] is reproduced here in Figure 14.1.

14.4 Algebra $\mathcal{Q}$ of Clopen Sets

We return to the general setting of a sigma compact manifold X with end set $E[X]$. For any subset $Q \subset E[X]$, and compact set $K \subset X$, we define

$$Q(K) = \bigcup_{e \in Q} e(K).$$

Each compact set $K \subset X$ determines the following equivalence relation $\sim_K$ on $E[X]$, namely all the ends 'contained in' the component $e(K)$:

$$e \sim_K e' \text{ if and only if } e(K) = e'(K). \tag{14.2}$$

For each compact set $K \subset X$ there are only finitely many unbounded components in $X - K$. Let $\mathcal{P}_K$ denote the finite partition of $E[X]$ into equivalence classes modulo $\sim_K$ and let $\mathcal{Q}_K$ denote the finite algebra generated by $\mathcal{P}_K$. That is, the elements of $\mathcal{P}_K$ are the atoms of the algebra $\mathcal{Q}_K$. Observe that the algebra $\mathcal{Q} = \bigcup_K \mathcal{Q}_K$ (where the union is taken as K ranges through the family of compact subsets of X) is identical with the family of closed-open (clopen) subsets of $E[X]$, in the relative topology on $E[X]$, of the end compactification of X. It is on this algebra that we will define our measure induced on the ends $E[X]$. This analysis shows more clearly that $E[X]$ is totally disconnected.

14.5 Measures on Ends

An OU measure μ on X induces a two-valued (0 and ∞) measure μ^* on $(E[X], \mathcal{Q})$ as follows. If $Q \in \mathcal{Q}$, then Q belongs to $\mathcal{Q}_K$ for some compact set K. Define $\mu^*(Q) = 0$ if $\mu(Q(K)) < \infty$ and $\mu^*(Q) = \infty$ if $\mu(Q(K)) = \infty$.

Remark First we note that this definition of μ^* is independent of the compact set K once K is large enough so that $Q \in \mathcal{Q}_K$. So suppose Q is also in $\mathcal{Q}_{K'}$ for some compact set K' and for now we first make the special assumption that $K \subset K'$. *We need to show that* $\mu(Q(K)) = \infty$ *if and only if* $\mu(Q(K')) = \infty$ (the general situation will then follow from this special case). Now note that $Q(K)$ is the finite union of connected components of $X - K$ and $Q(K) = Q(K') \cup (K' \cap Q(K))$. Thus since the OU measure μ is finite on the compact set $K' \cap Q(K)$, it follows that either both $Q(K)$ and $Q(K')$ are finite μ-measured sets or they are both infinite μ-measured sets. For two arbitrary compact sets K_1 and K_2, if $Q \in \mathcal{Q}_{K_1} \cap \mathcal{Q}_{K_2}$, then let $K' = K_1 \cup K_2$. Then by the above argument $Q(K_1)$ has infinite measure if and only if $Q(K')$ does and has infinite measure if and only if $Q(K_2)$ does. This shows that the set function μ^* is well defined.

Lemma 14.2 *The set function* μ^* *taking the two values 0 and* ∞ *is a measure on the algebra of clopen sets* $\mathcal{Q}$ *on* $E[X]$.

Proof First we note μ^* is trivially a finitely additive set function. Indeed let $Q_1, Q_2, \ldots, Q_k \in \mathcal{Q}$. Let K be a large enough compact set so that $Q_1, Q_2, \ldots, Q_k \in \mathcal{Q}_K$. Then $\bigcup_{i=1}^k Q_i(K)$ has infinite μ-measure if and only if one of the $Q_i(K)$ does. Thus $\mu^*(\bigcup_{i=1}^k Q_i) = \sum_{i=1}^k \mu^*(Q_i)$. The proof is completed by noting that the algebra of clopen sets on the totally disconnected set $E[X]$ has the property that any countable union of disjoint clopen sets can be written as a finite union of these clopen sets. □

Thus μ^* is a measure on $(E[X], \mathcal{Q})$ taking on only the two values, 0 and ∞. The measure is nontrivial ($\mu^*(E[X]) \neq 0$) as long as $\mu(X) = \infty$. We again remind the reader that the measurable sets (the clopen sets) $\mathcal{Q}$ constitute only an algebra, and *not* a sigma algebra. Finally we say that the end $e \in E[X]$ is an end of infinite measure if and only if $\mu(e(K)) = \infty$ for all compact sets K. Let $E^\infty[X]$ denote the set of *ends of infinite measure.*

Lemma 14.3 *The set of infinite measured ends $E^\infty[X]$ is a closed subset of $E[X]$.*

Proof Suppose that the end e_0 is a limit point of infinite measured ends. For any compact set $K \subset X$, $\mu(e_0(K)) = \infty$ because the neighborhood of e_0, $N_K(e_0)$, contains an infinite measured end $e \in E^\infty[X]$ and $e(K) = e_0(K)$. Since $\mu(e(K))$ is infinite, so too is $\mu(e_0(K))$. □

Note that the set $E^\infty[X]$ may not be in the algebra $\mathcal{Q}$ of clopen sets. An example may be obtained on the Manhattan manifold $\hat{X}$ of Example 13.1. Take as the measure μ a measure equal to area measure for $y \leq 1$ and such that μ is finite on each of the vertical strips. In this case $E^\infty[\hat{X}]$ consists only of the 'horizontal' ends called $+\infty$ and $-\infty$, which are limits of the finite measured ends $\{\ldots, -1, 0, 1, \ldots\}$. So $E^\infty[\hat{X}]$ is not open and hence not clopen.

We end this section by giving some more examples of noncompact manifolds X with OU measures μ and the measures μ^* on the ends $E[X]$.

Example 14.4 (See [67]) *Let $X = R^2 - \{\bar{0}\}$. Then this manifold has two ends, one of them identifiable with the origin $\bar{0} = (0, 0)$ and the other with 'the point at infinity' in R^2 – this is easily seen by taking K to be a Jordan curve with the origin inside. Consider the following two different OU measures obtained by integrating the following 2-forms*

$$\omega = dx \times dy, \text{ and } \tau = (dx \times dy)/F(x^2 + y^2)$$

where $F : (0, \infty) \to (0, \infty)$ is any smooth function with the property

$$F(r) = 1 \quad \text{if} \quad r \geq 1 \quad \text{and} \quad F(r) = r^2 \quad \text{if} \quad 0 < r < 1/2.$$

This means (abusing notation) that $\tau(A) = \int_A \tau$ and similarly $\omega(A) =$

$\int_A \omega$. *Note that* ω *is just our area measure on the plane. The point at infinity has infinite measure for each of these measures but (the end at) the origin is infinite for only one of these measures:* $\omega^*(\bar{0}) = 0$, $\tau^*(\bar{0}) = \infty$.

Other noncompact manifolds and measures may be obtained similarly. Let $G(x, y)$ be a smooth function on $R^2 - E$ which is positive at all points except at some set E of 'singular points for G', where by a singularity at $p \in E$ we mean that $G(p_k) \to \infty$ for any sequence of points p_k in $R^2 - E$, converging to the point $p \in E$. The singularities will correspond to the ends of the manifold. We may define an OU measure ν_G on $R^2 - E$ by the equation $\nu_G(A) = \int_A G(x, y)dxdy$, for any Borel set A. By choosing the behavior at the 'singularities' of G appropriately (i.e., so that the improper integral around a singular point is finite or infinite), we can obtain ends of finite or infinite measure.

Example 14.5 *A noncompact manifold can be obtained from a compact manifold* D *by removing some closed totally disconnected subset* C *from it. So consider the manifold* $X = D - C$ *where* $D = \{(x, y) : x^2 + y^2 \leq 1\}$ *is the closed unit disk in the plane and* C *is the standard Cantor ternary set lying on the line* $[-1/2, 1/2]$ *along the* x*-axis. The Cantor set* C *may be identified with the set* $E[X]$ *of ends of* X*. We now give an example of an OU measure on* X*. Let* $C(0)$ *and* $C(1)$ *denote the left and right thirds of the interval* $[-1/2, 1/2]$ *on the* x*-axis. For* $i_k = 0, 1$ *and* $m \geq 1$*, let* $C(i_1, \ldots, i_m, 0)$ *and* $C(i_1, \ldots, i_m, 1)$ *be the left and right thirds of* $C(i_1, \ldots, i_m)$ *respectively. Let* λ_1 *and* λ_2 *denote respectively 1- and 2-dimensional Lebesgue measure (i.e., length and area measures respectively). For each Borel subset* A *of* X *define* $\mu(A)$ *by the formula*

$$\mu(A) = \lambda_2(A) + \sum_{m=1}^{\infty} \sum_{i_1, \ldots, i_m} 3^m \lambda_1(A \cap C(i_1, \ldots, i_m)).$$

Then note that all of the ends of X *(namely the points in the Cantor set* C*) have infinite* μ^**-measure.*

Another OU measure on the same manifold $X = D - C$ which gives all ends infinite measure is the following. Let $R(i_1, \ldots, i_m)$ be the closed rectangular $3^{-(m+1)}$-neighborhood of $C(i_1, \ldots, i_m)$ in D and let

$$K_m = D - \bigcup_{i_1, \ldots, i_m} \text{Int } \; R(i_1, \ldots, i_m).$$

Then the sets $L_m = K_{m+1} - K_m$ consist of 2^m congruent components each with volume

$$a_m = \lambda_2(L_m)/2^m.$$

For each Borel subset A of X define $\nu(A)$ by the formula

$$\nu(A) = \lambda_2(A) + \sum_m (1/a_m)\lambda_2(A \cap L_m).$$

Then ν is an infinite OU measure on $D - C$ which gives every end in C infinite measure.

Finally we note that there is nothing special about dimension 2 in any of these examples. We could just as easily have done all of this in dimensions $n > 2$.

14.6 Compact Separating Sets

Results from topology show that a general sigma compact manifold X can be written as an increasing sequence of compact connected manifolds with special properties. We describe these compact connected submanifolds and these special properties. The proofs of some of these results may be found in our Appendix 2.

We say that $K \subset X$ is an *n-cell* if it is homeomorphic to the closed unit n-cube I^n. A set $K \subset X$ is called a *relative n-cell* if there exists a continuous function $\phi : I^n \to K$ such that

(i) ϕ is onto
(ii) ϕ restricted to Int I^n is a homeomorphism onto its image
(iii) $\phi^{-1}\phi(\partial I^n) = \partial I^n$.

In other words, a relative n-cell is a subset of X which can be obtained from the n-cube by making boundary identifications, namely a compact connected n-manifold. Recall that in Chapter 9 we stated Brown's Theorem (Theorem 9.6) which said that every compact connected n-manifold is a relative n-cell.

An important fact about a sigma compact connected n-manifold is that it can be written as the countable union of an increasing family of compact connected manifolds. In [20], it is shown how this fact follows from the deep results of Kirby, Siebenmann and Quinn ([78] and [99]). However, the weaker result that the manifold is the increasing union of relative n-cells follows from the work of Berlanga and Epstein [38] (see Lemma A2.9 in Appendix 2). Furthermore, let μ be an OU measure on X (i.e., μ is a sigma finite nonatomic Borel measure which is positive on open sets, and zero on the boundary of X – we note that the fact that μ is a Borel measure implies that μ is also finite on compact sets). When the manifold X has nonempty boundary ∂X and K is a subset of X, we denote by Bdry K the union of the topological frontier of K and $\partial X \cap K$. Not only can we choose a sequence of relative n-cells increasing to X, but it follows from Berlanga and Epstein's work that the relative n-cells can be chosen so that their boundaries have measure zero. This will follow from the theorem below.

Theorem 14.6 *Let μ be an OU measure on a sigma compact connected n-manifold X. Then any compact subset C of X is contained in the interior of a relative n-cell K such that $X - K$ has no bounded components and $\mu(\text{Bdry } K) = 0$.*

Proof First apply Lemma A2.9 with $A = \emptyset$ and $B = C$ the given compact set. Then since $X - A$ is connected (the 'furthermore' part of the Lemma states) there is a single relative n-cell which we call L_1 containing C. Letting $\hat{L}_1$ be the union of L_1 and all of the bounded (compact) components of $X - L_1$, Lemma A2.10 implies that $\hat{L}_1$ is a compact set with only unbounded components in its complement. Another application of Lemma A2.9 but this time with $A = \emptyset$ and $B = \hat{L}_1$ gives the required relative n-cell K. □

Definition 14.7 *A relative n-cell K such that $X-K$ has only unbounded components, and $\mu(\text{Bdry } K) = 0$, is called a separating set.*

Lemma 14.8 *Let μ be an OU measure on a sigma compact connected manifold X, and μ^* the measure induced on $(E[X], \mathcal{Q})$. Let $Q \in \mathcal{Q}$ be a clopen set of ends in $E[X]$. Then there is a separating set $K \subset X$ such that $Q \in \mathcal{Q}_K$.*

Proof Since $Q \in \mathcal{Q}$, there is a compact set K_1 such that $Q \in \mathcal{Q}_{K_1}$. By the above theorem, there is a separating set K containing K_1 in its interior. □

14.7 End Preserving Lusin Theorem

In the later portions of the book, when proving genericity results for homeomorphisms of noncompact manifolds, we will need an extension of our Lusin Theorem 10.2 which preserves the end structure of the manifold. This extension is given below as Theorem 14.9. To motivate this extension, we refer briefly to the proof of Theorem 12.6, which gave a genericity result related to the manifold R^n. In that proof we used the Lusin Theorem 10.2 (actually, we used it twice) to approximate an automorphism $\bar{g}$ which left an n-cube K invariant and had small norm on K (that is, $|\bar{g}(x) - x| < \epsilon$ for λ-a.e. x in K). The approximating homeomorphism $h_1 \in \mathcal{M}[R^n, \lambda]$ obtained via Theorem 10.2 also had small norm on K, left K invariant, and *consequently* left $e_\infty(K) = R^n - K$ invariant. Here we use the notation e_∞ to denote the single end at infinity of R^n. The invariance of $e_\infty(K)$ is a trivial consequence of the fact that there is a single end, since whenever a set is invariant under a bijection, so is its complement. However, when there is more than one end, the invariance of K does not ensure the invariance of the sets $e(K)$, so this has to be part of the conclusion of any Lusin theorem that we use. Of course, there are only finitely many sets of the form $e(K)$. These are the sets $P(K)$, for $P \in \mathcal{P}_K$. So we give the needed Lusin theorem in the following form.

Theorem 14.9 (End Preserving Lusin Theorem) *Let μ be an OU measure on the sigma compact metric manifold (X, d) and let K be any separating subset of X. Let $g \in \mathcal{G}[X, \mu]$ be any automorphism of X satisfying*

(i) $d(g(x), x) < \epsilon$ *for μ-a.e.* $x \in K$
(ii) $g(K) = K$
(iii) $g(P(K)) = P(K)$ *for every* $P \in \mathcal{P}_K$.

Then any weak topology neighborhood of g contains a homeomorphism $h \in \mathcal{M}[X, \mu]$ with compact support which also satisfies properties (i)–(iii) *(with h replacing g). We note that condition* (iii) *is equivalent to the condition*

(iii′) $g(Q(K)) = Q(K)$ *for every* $Q \in \mathcal{Q}_K$.

Proof The proof is similar to the part of the proof of Theorem 12.6 (generic properties for R^n) where the compact form of the Lusin Theorem 10.2 was used twice: on the n-cube K with a norm bound, and on the annulus $C - K$ without a norm bound. Here we will use Theorem 10.2 $j + 1$ times, where j is the cardinality of $\mathcal{P}_K$: once on the separating set K of the theorem, and once for each component $P(K)$ where $P \in \mathcal{P}_K$.

The application of Theorem 10.2 to the separating set K so that the resulting homeomorphism h^K of K satisfies (i) and (ii) is immediate. Now fix some $P \in \mathcal{P}_K$. If B_i, $i = 1, \ldots, m$, are the finite measured sets in the given weak neighborhood of g (see Section 11.2), choose sets $C_i^P \equiv C_i \subset B_i \cap P(K)$, $i = 1, \ldots, m$, so that $\mu((B_i \cap P(K)) - C_i)$ is very small and $\bigcup_i (C_i \cup g(C_i))$ is a relatively compact subset of $P(K)$. Hypothesis (iii) makes this possible. Let R_P be a relative n-cell in $P(K)$ which contains $\bigcup_i (C_i \cup g(C_i))$ in its interior and has boundary measure zero (using Theorem 14.6). Apply the Lusin Theorem 10.2 to any automorphism $\hat{g}$ of R_P which agrees with g on $\bigcup_i C_i$, and extend the resulting homeomorphism h^P to all of $P(K)$ by setting it equal to the identity on $P(K) - R_P$. If we piece together the homeomorphisms h^K on K, and h^P on $P(K)$ for all $P \in \mathcal{P}_K$, we obtain the required homeomorphism h. □

14.8 Induced Homeomorphism h^*

Every homeomorphism of the manifold induces a homeomorphism on the ends. We have seen in Chapter 13 two examples of these end homeomorphisms (recall the Manhattan dynamical system and the end action induced by the translation to the right homeomorphism). We now give a formal description of the homeomorphism of the ends.

Definition 14.10 *Every homeomorphism $h : X \to X$ induces a homeomorphism h^* on the ends, $h^* : E[X] \to E[X]$ defined by*

$$[h^*(e)](K) = h(e(h^{-1}(K))) \tag{14.3}$$

for all $e \in E[X]$ and compact $K \subset X$. We say that $h \in \mathcal{H}[X]$ is end preserving if h^* is the identity on $E[X]$. If h is μ-preserving (i.e., $h \in \mathcal{M}[X, \mu]$) then h^* preserves the measure μ^*. Thus every $h \in \mathcal{M}[X, \mu]$ induces a measure preserving system $(E[X], \mathcal{Q}, \mu^*, h^*)$.

Keeping in mind that the measurable sets $\mathcal{Q}$ constitute only an algebra (the algebra of clopen sets), and *not* a sigma algebra, the following definitions deserve more than the usual scrutiny.

Definition 14.11 *Let σ be a μ^*-preserving homeomorphism of $E[X]$. Then the system $(E[X], \mathcal{Q}, \mu^*, \sigma)$ is called* compressible *if there is a clopen set $Q \in \mathcal{Q}$ such that $\mu^*(\sigma Q - Q) = 0$ and $\mu^*(Q - \sigma Q) > 0$ (note since μ^* is a two-valued measure, this means that $\mu^*(Q - \sigma Q) = \infty$).*

Otherwise it is called incompressible. *The system is called* ergodic *if for every (invariant) set* $I \in \mathcal{Q}$ *with* $\mu^*(I \triangle \sigma I) = 0$, *either* $\mu^*(I) = 0$ *or* $\mu^*(E[X] - I) = 0$.

We observed earlier that if $\mu(X) < \infty$, then $\mu^*(E[X]) = 0$. Hence in this case *any* $h \in \mathcal{M}[X, \mu]$ induces an incompressible and ergodic system $(E[X], \mathcal{Q}, \mu^*, h^*)$.

In contrast to the usual case in ergodic theory where an ergodic measure preserving automorphism must necessarily be incompressible, the following example presents an ergodic and compressible end homeomorphism.

Example Let E be the totally disconnected space consisting of $-1, \ldots,$ $-1 + \frac{1}{3}$, $-1 + \frac{1}{2}$, 0, $1 - \frac{1}{2}$, $1 - \frac{1}{3}$, $\ldots, 1$ with its usual topology. Suppose σ fixes -1 and 1 and moves all other points to the next larger number. Assume the measure μ^* is infinite for each nonempty set. The clopen set $Q = \{0, 1 - \frac{1}{2}, 1 - \frac{1}{3}, \ldots, 1\}$ satisfies $\sigma Q - Q = \emptyset$ and $Q - \sigma Q = \{0\}$, so $(E, \mathcal{Q}, \mu^*, \sigma)$ is compressible. The only nonempty invariant clopen set is E, and hence σ is ergodic and compressible.

The reader should recognize that the translation to the right homeomorphism $\hat{h}$ on the Manhattan manifold $\hat{X}$ from the previous chapter (see Example 13.1) has an induced end homeomorphism which is topologically conjugate to the end action $(E, \mathcal{Q}, \mu^*, \sigma)$ of the previous example.

The next lemma relates the action of the induced end homeomorphism $h^* = \sigma$ on the clopen sets of ends, to the action of the homeomorphism h on the components of $X - K$ containing those ends (for sufficiently large compact sets K).

Lemma 14.12 *Let* $h \in \mathcal{M}[X, \mu]$ *and write* $\sigma = h^*$. *Let* Q_1 *and* Q_2 *belong to* $\mathcal{Q}_K$ *for some separating set* K. *Then* $\mu^*(\sigma Q_1 \cap Q_2) = \infty$ *if and only if* $\mu(h(Q_1(K)) \cap Q_2(K)) = \infty$.

Proof First suppose $\mu^*(\sigma Q_1 \cap Q_2) = \infty$. By the definition of μ^* it follows that in particular $\mu((\sigma Q_1 \cap Q_2)(B)) = \infty$ for $B = K \cup hK$.

$$
\begin{aligned}
(\sigma Q_1 \cap Q_2)(B) &= (h^* Q_1)(B) \cap Q_2(B) && \text{since } \sigma = h^* \\
&= (h Q_1 h^{-1})(B) \cap Q_2(B) && \text{by (14.3)} \\
&\subseteq h Q_1 h^{-1}(hK) \cap Q_2(K) \\
&= h(Q_1(K)) \cap Q_2(K)
\end{aligned}
$$

where the inclusion above follows from (14.1), and the fact that B contains hK and K as subsets. Therefore $\mu(h(Q_1(K)) \cap Q_2(K)) = \infty$.

Now assume $\mu^*(\sigma Q_1 \cap Q_2) < \infty$, or equivalently that for a separating set $R \supset K \cup hK$ we have $\mu((\sigma Q_1 \cap Q_2)R) < \infty$. Observe that

$$\begin{aligned} h(Q_1(K)) &= hQ_1h^{-1}(hK) && (14.4) \\ &= \sigma Q_1(hK). && (14.5) \end{aligned}$$

From (14.1) in the definition of ends, it follows that

$$\begin{aligned} (E[X] - \sigma Q_1)(hK) &= \bigcup_{e \in E[X]-\sigma Q_1} e(hK) \\ &\supset \bigcup_{e \in E[X]-\sigma Q_1} e(R) = (E[X] - \sigma Q_1)(R). \end{aligned}$$

Since the complement in X of $(E[X] - \sigma Q_1)(hK)$ is just $\sigma Q_1(hK) \cup hK$ and $(E[X] - \sigma Q_1)(R)$ has $\sigma Q_1(R) \cup R$ as its complement it follows that

$$\sigma Q_1(hK) \cup hK \subset \sigma Q_1(R) \cup R.$$

Consequently combining this with equation (14.5) from above we have

$$h(Q_1(K)) \subset \sigma Q_1(R) \cup R.$$

Since $K \subset R$, then $Q_2(K) \subset Q_2(R) \cup (R - K)$ and so

$$Q_2(K) \subset Q_2(R) \cup R.$$

Therefore

$$h(Q_1(K)) \cap Q_2(K) \subset R \cup [(\sigma Q_1)(R) \cap Q_2(R)],$$

and

$$\mu(h(Q_1(K)) \cap Q_2(K)) < \mu(R) + \mu[(\sigma Q_1 \cap Q_2)(R)] < \infty.$$

□

Definition 14.13 *Suppose that $\sigma : E[X] \to E[X]$ is an end homeomorphism induced by some homeomorphism in $\mathcal{M}[X, \mu]$. Then define*

$$\mathcal{M}_\sigma[X, \mu] = \{f \in \mathcal{M}[X, \mu] : f^* = \sigma\}.$$

We can apply Baire category arguments to the space of homeomorphisms $\mathcal{M}_\sigma[X, \mu]$ because of the following lemma.

Lemma 14.14 *$\mathcal{M}_\sigma[X, \mu]$ is a closed subset of $\mathcal{M}[X, \mu]$ with respect to the compact-open topology.*

Proof Denote by Ψ, the * map; i.e.,

$$\Psi : \mathcal{M}[X,\mu] \to \mathcal{M}[E[X],\mu^*]$$

defined by $\Psi(h) = h^*$. Since the * map Ψ is continuous, the inverse image of a point $\sigma \in \mathcal{M}[E[X],\mu^*]$ is closed in $\mathcal{M}[X,\mu]$. But this inverse image is nothing more than $\mathcal{M}_\sigma[X,\mu]$. □

Recall that when we considered the Manhattan dynamical system $(\hat{X},\hat{\mu},\hat{h})$, whose induced end homeomorphism $\hat{h}^*$ is compressible, we showed that $\hat{h}$ could not be $\hat{\mu}$-recurrent. We now generalize this to show for all (X,μ) that *any* homeomorphism $f \in \mathcal{M}[X,\mu]$ which induces a compressible end homeomorphism $f^* : E[X] \to E[X]$ cannot be μ-recurrent on (X,μ).

Lemma 14.15 *If $(E[X],\mathcal{Q},\mu^*,\sigma)$ is a compressible system then no homeomorphism in $\mathcal{M}_\sigma[X,\mu]$ is μ-recurrent, and therefore no homeomorphism in $\mathcal{M}_\sigma[X,\mu]$ can be ergodic.*

Proof Suppose that h belongs to $\mathcal{M}_\sigma[X,\mu]$ (i.e., the action of h on the ends is $h^* = \sigma$). Then because σ is compressible, there is some clopen set of ends $Q \in \mathcal{Q}$ which satisfies $\mu^*(\sigma Q - Q) = 0$ and $\mu^*(Q \cap \sigma(E[X] - Q))$ > 0. Choose a separating set K such that $Q \in \mathcal{Q}_K$. Then by Lemma 14.12 $\mu(h(Q(K)) - Q(K)) < \infty$ and $\mu(Q(K) - h(Q(K))) = \infty$. But taking $V = Q(K)$, this yields

$$\mu(h(V) - V) \neq \mu(V - h(V))$$

violating the condition in Lemma 13.3 necessary for h to be μ-recurrent. □

In order to state our next two theorems, we will need to define some properties of finite $m \times m$ matrices $\mathbf{T} = (t_{i,j})$ whose entries are all 0s and 1s. Such a matrix $\mathbf{T}$ is called *irreducible* if for any i and j there is a positive integer a such that $t^a_{i,j} = 1$. If moreover there is a single positive integer N such that $t^N_{i,j} = 1$ for all i and j, then $\mathbf{T}$ has the stronger property that we will call *mixing*. The matrix $\mathbf{T}$ is called *recurrent* if for every i there is a positive integer a such that $t^a_{i,i} = 1$. The greatest common divisor of such powers a is called the *period* of i. (If $\mathbf{T}$ is irreducible then every i has the same period.) If every i has period 1 then we say that $\mathbf{T}$ is *aperiodic*. It is easy to see that $\mathbf{T}$ is mixing if and only if it is both irreducible and aperiodic, and that if $\mathbf{T}$ is recurrent it can be decomposed into irreducible submatrices. Similar definitions are given in Appendix 1 for stochastic matrices with either finitely or infinitely many states.

The next theorem gives a kind of 'ergodic decomposition' for an incompressible end homeomorphism σ with respect to a given clopen partition of E^∞. It produces a minimal 'clumped' partition of E^∞ into σ-invariant clopen sets, of which the given partition is a refinement. This result will be used in the proof of Theorem 15.1 (in Lemma 16.3) to produce an ergodic μ-preserving manifold homeomorphism h with $h^* = \sigma$.

Theorem 14.16 *Let $\sigma : E \to E$ be an incompressible end homeomorphism and let $E_1, \ldots, E_m$ form a partition of E into clopen sets of infinite μ^* measure. Define an $m \times m$ 0–1 matrix $\boldsymbol{T}$ by $t_{ij} = 1$ if $\mu^* (\sigma E_i \cap E_j) = \infty$ and $t_{ij} = 0$ otherwise. Then the relation $E_i \sim E_j$ if $t_{ij}^a > 0$ for some integer $a \geq 1$ is an equivalence relation on the set $\{E_1, \ldots, E_m\}$, and hence $\boldsymbol{T}$ is recurrent. The corresponding equivalence classes determine a partition of E^∞ into minimal σ-invariant clopen sets $C_1, \ldots, C_p$ (which are each unions of the sets E_i) and an associated decomposition of $\boldsymbol{T}$ into square irreducible submatrices.*

Proof Consider the directed graph G with vertex set $\{1, \ldots, m\}$ and an arc from i to j if $t_{ij} = 1$. Since σ preserves the measure μ^* it follows that every vertex has at least one arc going out of it. First suppose that the relation $\sim$ is not symmetric. Then for some vertices i and j there is a path from i to j but no path from j to i. Relabel the vertices so that $i = 1$, $j = b$, and $1, 2, \ldots, b$ is such a path (from i to j). If there is no path from b to any of the vertices $1, 2, \ldots, b$, set $c = b$; otherwise let c be the least number (under the relabeling) such that there is a path from b to c. Note that in either case $c > 1$ by assumption (no path from j to i). Let F denote the set of vertices z such that there is a path from c to z. Observe that $c - 1$ does not belong to F since in this case the path from b to c followed by the path from c to $c - 1$ would be a path from b to $c - 1$, contradicting the minimality of c. Now define the clopen end set $Q = E_{c-1} \cup \left(\bigcup_{k \in F} E_k\right)$ and observe that $\sigma(Q) = \bigcup_{k \in F} E_k \subset Q$ and $\mu^*(Q - \sigma Q) = \mu^*(E_{c-1}) = \infty$. This would imply that σ is compressible, contrary to hypothesis, so our additional assumption that the relation $\sim$ is not symmetric was false. The transitivity of the relation $\sim$ is obvious. Reflexivity follows from the fact that for each vertex k there is an arc to some vertex k' (possibly equal to k) and consequently by symmetry there is a path from k' back to k. □

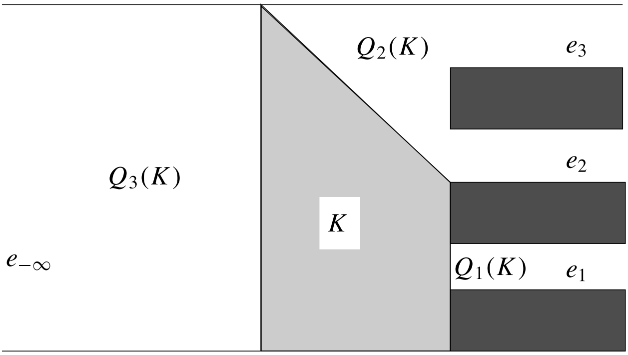

Fig. 14.2. The four-ended cylinder

We illustrate these concepts with the following example.

Example 14.17 (Four-ended cylinder) *Define a manifold by*

$$X = R \times [0,1] - \bigcup_{k=0}^{2} (1,\infty) \times \left(\frac{2k}{6}, \frac{2k+1}{6}\right),$$

with the top and bottom identified, that is, $(x,0)$ *with* $(x,1)$ *for all* $x \in R$. *This manifold has four ends, all of infinite measure, which we denote by* $E = \{e_{-\infty}, e_1, e_2, e_3\}$. *Roughly speaking* $e_{-\infty}$ *corresponds to going to infinity along the line* $(x, 1/2)$ *as* x *goes to* $-\infty$, *while each end* e_i, $i = 1,2,3$ *corresponds to going to infinity along the line* $\left(x, \frac{4i-1}{12}\right)$ *as* x *goes to* ∞. *The homeomorphism defined by* $h(x,y) = (x, y+1/3)$, *where addition in* $y + 1/3$ *is taken mod 1, preserves area. The induced end homeomorphism* h^* *fixes* $e_{-\infty}$ *and cyclically permutes the remaining ends* $e_1 \to e_2 \to e_3 \to e_1$. *It is incompressible. Let* K *denote the compact set bounded by the circle* $\{-1\} \times [0,1]$ *and the lines joining the points* $(-1,0), (+1,0), (+1,1/2)$ *and* $(-1,1) = (-1,0)$. *The set* $X - K$ *has three components, corresponding to the end partition* $\mathcal{P}_K = \{Q_1 = \{e_1\}, Q_2 = \{e_2, e_3\}, Q_3 = \{e_{-\infty}\}\}$. *The incidence matrix* $\mathbf{T} = \mathbf{T}(h,K)$ *defined by* $t_{ij} = 1$ *if* $h^*(Q_i) \cap Q_j \neq \emptyset$ *and otherwise* 0, *and the submatrix* $\mathbf{B}$ *corresponding to the* h^* *invariant set* $C = \{Q_1, Q_2\}$ *are* *given by*

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

If we want the matrix $\mathbf{T}$ in Theorem 14.16 to be mixing rather than merely irreducible, we must make stronger assumptions on σ. The required notion is a variation on Furstenberg's definition of topological weak mixing [64], which we call componentwise weak mixing.

Definition 14.18 *A homeomorphism σ of a compact topological space into itself is called* componentwise weak mixing *if for all clopen sets U, V the set $\{k : \sigma^k U \cap V \neq \emptyset\}$ contains consecutive integers.*

Theorem 14.19 *Let $h \in \mathcal{M}_\sigma[X, \mu]$ and assume $(E[X], \mathcal{Q}, \mu^*, \sigma)$ is incompressible and σ is componentwise weak mixing on the invariant subset $E^\infty[X]$ consisting of the ends of infinite measure. For any separating set K, let $E_1, \ldots, E_m$ be an enumeration of the elements of $\mathcal{P}_K$ having infinite μ^*-measure. Then the 0–1 matrix $\mathbf{T}$, defined by $t_{ij} = 1$ if and only if $\mu(h(E_i(K)) \cap E_j(K)) = \infty$, is mixing.*

Proof For $i = 1, \ldots, m$, define $E_i^\infty = E_i \cap E^\infty[X]$, and observe that these are clopen sets in the relative topology on $E^\infty[X]$. It follows from Lemma 14.12 that $\sigma^k(E_i^\infty) \cap E_j^\infty \neq \emptyset$ implies that $t_{ij}^k = 1$. The assumption of componentwise weak mixing implies that the matrix $\mathbf{T}$ is irreducible, so that all states $i = 1, \ldots, m$ have the same period p. (This means that $t_{ii}^k = 1$ only when p divides k.) Consequently $\{k : \sigma^k(E_i^\infty) \cap E_i^\infty \neq \emptyset\} \subset \{k : t_{ii}^k = 1\} \subset pZ$, the multiples of p. If $\mathbf{T}$ is not aperiodic, then $p > 1$ and the set pZ does not contain two consecutive integers. Thus as $\mathbf{T}$ is not aperiodic, then σ could not be componentwise weak mixing, contrary to assumption. It follows that $\mathbf{T}$ is irreducible and aperiodic and consequently mixing. □

14.9 The Charge Induced by a Homeomorphism

Every homeomorphism $h \in \mathcal{M}[X, \mu]$ induces on the ends $E = E[X]$ a charge (a finitely additive signed measure) $c = c_h$ defined on the subalgebra $\mathcal{I}_{h^*}$ of $\mathcal{Q}$ consisting of h^*-invariant clopen sets. This section is devoted to defining this end charge c and establishing some of its elementary properties.

In the two examples in Chapter 13, the unit translation on the strip dynamical system $(\check{X}, \check{\mu}, \check{h})$, and the Manhattan dynamical system $(\hat{X}, \hat{\mu}, \hat{h})$, we computed the quantity

$$\mu(V - h(V)) - \mu(h(V) - V)$$

where V was an unbounded component containing the end fixed at $+\infty$ by both of the systems. The fact the the above quantity was nonzero was used to prove that both of the dynamical systems $(\hat{X}, \hat{\mu}, \hat{h})$ and $(\check{X}, \check{\mu}, \check{h})$ are not μ-recurrent (see Lemma 13.3). These examples and the quantity $\mu(V-h(V))-\mu(h(V)-V)$ (which we encountered in the previous section as well) motivate the definition of the charge of a homeomorphism c_h for a general $h \in \mathcal{M}[X, \mu]$.

Fix a homeomorphism $h \in \mathcal{M}[X, \mu]$ and let σ denote the action of h^* on the ends. Denote by $\mathcal{I} = \mathcal{I}_\sigma$ the subalgebra of σ-invariant clopen sets in $\mathcal{Q}$ given by $\mathcal{I} = \{I \in \mathcal{Q} : \mu^*(I \triangle \sigma I) = 0\}$. For every $I \in \mathcal{I}$ there is a separating set $K \subset X$ such that $I \in \mathcal{Q}_K$. For such a set, define

$$c(I, K) = \mu(I(K) - h(I(K))) - \mu(h(I(K)) - I(K)). \tag{14.6}$$

This quantity measures the net flow of mass by h into the invariant set of ends in $I(K)$.

In the definition of $c(I, K)$ above, we first note that this difference is well defined. To see this we note that $I \in \mathcal{I}$ is an invariant clopen set, and so $\mu^*(I - \sigma I) = 0 = \mu^*(\sigma I - I)$; thus from Lemma 14.12, we have $\mu(I(K) - h(I(K))) < \infty$ and $\mu(h(I(K)) - I(K)) < \infty$. Consequently the difference in (14.6) is well defined. The fact that the end charge does not depend on the set K follows from

Lemma 14.20 *Suppose I is a clopen set of ends invariant under $\sigma = h^*$ belonging to both $\mathcal{Q}_K$ and $\mathcal{Q}_{K'}$. Then $c(I, K) = c(I, K')$, a number which will be simply written as $c(I)$.*

Proof Let R be any separating set containing $K \cup hK \cup h^{-1}K$. Such a separating set can be found using Theorem 14.6. We will show $c(I, R) = c(I, K)$. This will prove the lemma since by Theorem 14.6 we can always find an R which simultaneously has this relationship to both K and K'. Let $B = I(K) - I(R)$. Since $R \supset K$, then it follows from (14.1) that

$I(R) \subset I(K)$ and $I(K) = (R \cap I(K)) \cup I(R)$. Consequently

$$B = I(K) - I(R) = R \cap I(K)$$

is a subset of a compact set and therefore has finite measure. If we apply equation (13.2) from Lemma 13.4 to the finite measured set $V = B$, and the μ-preserving automorphism $g = h$, then we get

$$\mu(h(B) - B) - \mu(B - h(B)) = 0.$$

Now observe that

$$h(B) - B = [I(R) - h(I(R))] \bigcup_{\text{disj}} [h(I(K)) - I(K)]$$

$$B - h(B) = [I(K) - h(I(K))] \bigcup_{\text{disj}} [h(I(R)) - I(R)].$$

Therefore $c(I, R) - c(I, K) = \mu(h(B) - B) - \mu(B - h(B)) = 0$. □

As a consequence of this lemma we can define the *charge* $c = c_h$ *induced by a homeomorphism* $h \in \mathcal{M}[X, \mu]$ on the algebra $\mathcal{I}$ of h^*-invariant clopen sets of ends in $E = E[X]$, by equation (14.6). We note that if $\sigma = h^*$ is ergodic on (E, μ^*), then there are no nontrivial invariant sets of ends and $\mathcal{I}_\sigma = \{E, \emptyset\}$; in this case the charge is identically zero ($c_h(E) = 0$, and $c_h(\emptyset) = 0$). More generally the following lemma gives further properties of the charge.

Lemma 14.21 *Let $c = c_h$ be the end charge induced by some homeomorphism h in $\mathcal{M}[X, \mu]$ on the algebra $\mathcal{I}$ of h^*-invariant clopen sets of E. Then*

(i) $c(E) = 0$.

(ii) *If $\mu^*(I) = 0$ then $c(I) = 0$.*

(iii) *If $I_1, I_2 \in \mathcal{I}$ with $I_1 \cap I_2 = \emptyset$, then $c(I_1 \cup I_2) = c(I_1) + c(I_2)$.*

(iv) *Let σ be a fixed μ^*-preserving homeomorphism of the ends. The charge $c = c_h$ is continuous on each space $\mathcal{M}_\sigma[X, \mu]$ in the sense that for each $I \in \mathcal{I}_\sigma$, the function $h \to c_h(I)$ is continuous on $\mathcal{M}_\sigma[X, \mu]$.*

Proof (i) For any separating set K, we have $E \in \mathcal{Q}_K$ and $E(K) = X - K$. By Lemma 13.4 and equation (13.2) applied to the finite measured set $V = K$, and h

$$c(E) = c(E, K) = \mu(h(K) - K) - \mu(K - h(K)) = 0.$$

(ii) If $\mu^*(I) = 0$ then by definition there is a separating set K with $\mu(I(K)) < \infty$. Again applying Lemma 13.4, this time to the finite measured set $V = I(K)$, it follows that

$$c(I) = \mu(I(K) - h(I(K))) - \mu(h(I(K)) - I(K)) = 0.$$

(iii) For any K with $I_1, I_2 \in \mathcal{Q}_K$, because $I_1 \cap I_2 = \emptyset$, we have $I_1(K) \cap I_2(K) = \emptyset$. For such a K, $c(I_1 \cup I_2, K)$ equals $c(I_1, K) + c(I_2, K)$.

(iv) Fix $I \in \mathcal{I}_\sigma$ and let

$$\Phi : \mathcal{M}_\sigma[X, \mu] \to R^1$$

be the map $\Phi(h) = c_h(I)$. We prove the stronger assertion that $\Phi(\cdot)$ is in fact continuous in the weak topology on $\mathcal{M}_\sigma[X, \mu]$. Using Lemma 14.20, fix any compact set K such that I belongs to $\mathcal{I} \cap \mathcal{Q}_K$, so that $c(I) = c(I, K)$. If $h_j \to h$ in the weak topology on $\mathcal{M}_\sigma[X, \mu]$, then

$$\mu(I(K) - h_j(I(K))) \quad \to \quad \mu(I(K) - h(I(K))) \tag{14.7}$$

$$\mu(h_j(I(K)) - I(K)) \quad \to \quad \mu(h(I(K)) - I(K)) \tag{14.8}$$

so that $\Phi(h_j) = c_{h_j}(I, K) \to \Phi(h) = c_h(I, K)$ as required. □

Note The proof above actually shows that equations (14.7) and (14.8) are valid if $h_j \to h$ in the weak topology in $\mathcal{M}[X, \mu]$, which is a stronger result than that required by the lemma.

We observe that in a trivial sense the finite additivity of c on $\mathcal{I}$ can be extended to countable additivity since if a clopen set $I \in \mathcal{I}$ is the denumerable disjoint union of clopen sets $I_1, I_2, \ldots$, then the compactness of I implies all but a finite number of those sets must be the empty set. We now give simple computations of the charge induced by some homeomorphisms.

Examples

(i) *The charge $c_{\hat{h}}$ for the Manhattan dynamical system $(\hat{X}, \hat{\mu}, \hat{h})$:* Since the action $\hat{h}^*$ on $E[\hat{X}] = \{-\infty, \ldots, -1, 0, 1, \ldots, +\infty\}$ is ergodic, there are no nontrivial invariant clopen sets and so $c_{\hat{h}} \equiv 0$. We note however that the action of $\hat{h}^*$ is *compressible.*

(ii) *The charge $c_{\check{h}}$ for the unit translation on the strip dynamical system $(\check{X}, \check{\mu}, \check{h})$:* Here the end set is $E[\check{X}] = \{-\infty, +\infty\}$ and $\check{h}$ fixes both of these ends. Thus if we let $I_- = \{-\infty\}$ and $I_+ = \{+\infty\}$ be the two nontrivial $\check{h}^*$-invariant clopen sets then for $K = [3/8, 5/8] \times [0, 1]$ we computed in (13.4) with $V = I_+(K)$ that

$$c_{\check{h}}(I_+) = c_{\check{h}}(I_+, K) = 1.$$

Consequently $c_{\check{h}}(I_-) = -1$.

(iii) *If* (X, μ) *has no ends of infinite measure then for any* $h \in \mathcal{M}[X, \mu]$ *the charge* $c_h(I) = 0$ *for all invariant clopen sets* $I \in \mathcal{I}_{h^*}$. This follows easily from part (ii) of the above lemma because the finiteness of μ implies that $\mu^*(I) = 0$ for every I.

(iv) *If* (X, μ) *has exactly one end of infinite measure then for any* $h \in \mathcal{M}[X, \mu]$ *the charge* $c_h(I) = 0$ *for all invariant clopen sets* $I \in \mathcal{I}_{h^*}$. If I contains only ends of finite measure, then $\mu^*(I) = 0$ and so again (ii) of Lemma 14.21 implies $c_h(I) = 0$. If I contains the end of infinite measure, then $E[X] - I$ has only ends of finite measure and so $c_h(E[X] - I) = 0$. The finite additivity of c_h along with $c_h(E[X]) = 0$ implies that $c_h(I) = 0$.

A consequence of the next result shows that the end charge varies continuously on $\mathcal{M}_\sigma[X, \mu]$, a strengthened form of Lemma 14.21(iv).

Theorem 14.22 *For a fixed homeomorphism* σ *of the ends, let* $\mathcal{M}_\sigma[X, \mu]$ *be the homeomorphisms which induce the end homeomorphism* σ*, and let* $\mathcal{I} = \mathcal{I}_\sigma$ *denote the algebra of* σ*-invariant clopen subsets of* $E[X]$.

For any fixed end charge c *on* $\mathcal{I}_\sigma$*, the set* $\mathcal{M}_\sigma^c[X, \mu]$*, consisting of all homeomorphisms* $h \in \mathcal{M}[X, \mu]$ *with* $h^* = \sigma$ *and fixed end charge* $c_h = c$*, is closed in* $\mathcal{M}_\sigma[X, \mu]$*. In particular, the set* $\mathcal{M}_\sigma^0[X, \mu]$ *consisting of all homeomorphisms* $h \in \mathcal{M}[X, \mu]$ *with* $h^* = \sigma$ *and identically zero end charge is a closed subset of the topologically complete space* $\mathcal{M}[X, \mu]$ *with the compact-open topology.*

Furthermore, $\mathcal{M}_\sigma^c[X, \mu]$ *is invariant under right composition with any* $h \in \mathcal{M}[X, \mu]$ *of compact support.*

Proof First fix an end charge c on $\mathcal{I} = \mathcal{I}_\sigma$. Since

$$\mathcal{M}_\sigma^c[X, \mu] = \bigcap_{I \in \mathcal{I}} \Phi^{-1}\{c(I)\}$$

and the continuity of Φ (from Lemma 14.21) implies each set in the intersection is closed in $\mathcal{M}_\sigma[X, \mu]$ with the compact-open topology, it follows that $\mathcal{M}_\sigma^c[X, \mu]$ is also closed in $\mathcal{M}[X, \mu]$.

Next suppose that $h_1 \in \mathcal{M}_\sigma[X, \mu]$ has end charge $c_{h_1} = c$. If $h_2 \in \mathcal{M}[X, \mu]$ has compact support, then $h_2(R) = R$ and $h_2(I(R)) = I(R)$ for some compact R containing the support of h_2, with $I \in \mathcal{Q}_R$. Then

$c_{h_1h_2}(I) = c_{h_1h_2}(I,R)$, and

$$\begin{aligned} c_{h_1h_2}(I,R) &= \mu(I(R) - h_1h_2(I(R))) - \mu(h_1h_2(I(R)) - I(R)) \\ &= \mu(I(R) - h_1(I(R))) - \mu(h_1(I(R)) - I(R)) \\ &= c_{h_1}(I,R) \\ &= c_{h_1}(I). \end{aligned}$$

□

The first part of this theorem will enable us to carry out Baire category proofs in $\mathcal{M}^0_\sigma[X,\mu]$, the zero charge homeomorphisms with end action σ. The case of nonzero charge is dealt with in the following.

Theorem 14.23 *If $h \in \mathcal{M}[X,\mu]$ induces a nonzero charge c_h, then h is not the compact-open limit of ergodic or even μ-recurrent homeomorphisms in $\mathcal{M}[X,\mu]$.*

Proof Since c_h is not identically zero we have $c_h(I) \neq 0$ for some h^*-invariant clopen set of ends I. This set I belongs to $\mathcal{Q}_K$ for some compact separating set $K \subset X$. Setting $V = I(K)$ we have

$$c_f(I) = c_f(I,K) = \mu(f(V) - V) - \mu(V - f(V)) \neq 0$$

not only for $f = h$ but also for all f in some compact-open neighborhood of h, by the continuity of $c_f(I)$ in f (Theorem 14.22). It follows that for all f in this neighborhood, f is not recurrent (and hence not ergodic) by Lemma 13.3. □

14.10 h-moving Separating Sets

In the following chapters we will show how to approximate a given homeomorphism h by an ergodic automorphism f which is close on a compact separating set K. The construction we will give has the property that not only is f uniformly close to h on K but it also satisfies

$$f\,(Q(K)) = h\,(Q(K)) \text{ for every end set } Q \in \mathcal{Q}_K.$$

The exact statement of this approximation result is given in Chapter 16 as Lemmas 16.2 and 16.3. Note that the displayed property above implies that in addition we have $f(K) = h(K)$. Consider first the very special case where h is the identity homeomorphism, and observe that any f satisfying these conditions would have K and all the sets $Q(K)$,

$Q \in \mathcal{Q}_K$, as nontrivial invariant sets, and hence could not be ergodic. This observation holds more generally if the algebra $\mathcal{A}$ generated by the partition of X

$$X = K \cup E_1(K) \cup E_2(K) \cup \cdots \cup E_r(K),$$

where $\mathcal{P}_K = \{E_i(K)\}_{i=1}^r$, has an atom A which is periodic in $\mathcal{A}$ under h (this means that for some period $m \geq 1$, we have $h^j(A) \in \mathcal{A}$, $j = 0, \ldots, m$, with $h^m(A) = A$), since the set $\bigcup_{j=0}^m h^j(A)$ would be invariant. So in order to carry out the ergodic approximation we will need some assumption regarding the behavior of h on the sets $Q(K)$ which precludes the existence of an h-periodic atom in $\mathcal{A}$. This motivates the following definition.

Definition 14.24 *Let $h \in \mathcal{M}[X,\mu]$ be given. A separating set K is called h-moving if $\mu(h(K) \cap K) > 0$ and $\mu(h(I(K)) \cap K) > 0$ for every h^*-invariant clopen set of ends $I \in \mathcal{Q}_K$.*

This concept is applied in the following lemma.

Lemma 14.25 *Let K be an h-moving separating set. Then the algebra $\mathcal{A}$ generated by the partition of X*

$$X = K \cup E_1(K) \cup E_2(K) \cup \cdots \cup E_r(K),$$

where $\mathcal{P}_K = \{E_i(K)\}_{i=1}^r$, does not have an atom A which is periodic in $\mathcal{A}$ under h.

Proof Suppose that an atom $A \in \mathcal{A}$ has h-period m in $\mathcal{A}$. If $A = K$ then since $h(K)$ cannot be any of the sets $E_i(K)$, we must have $m = 1$ and $h(K) = K$, which violates the h-moving assumption.

If $A = E_i(K)$ for some i, then the h-invariant set $\bigcup_{j=0}^m h^j(A)$ equals $I(K)$ for the h^*-invariant set $I \in \mathcal{Q}_K$ containing E_i. This means that $I(K) = h(I(K))$ belongs to the algebra $\mathcal{A}$. However, it follows from the definition of h-moving that

$$0 < \mu(h(I(K)) \cap K) < \mu(K)$$

and consequently $h(I(K))$ cannot belong to the algebra $\mathcal{A}$. □

Of course the compact-open topology has been defined on $\mathcal{G}[X,\mu]$ so that the relative topology on $\mathcal{M}[X,\mu]$ is the usual compact-open topology (uniform convergence on compact sets). For this relative topology we can require that the compact set K, which defines the basic compact-open neighborhood $\mathcal{C}(h, K, \epsilon)$ of a homeomorphism $h \in \mathcal{M}[X, \mu]$ (see Section 11.2), be a separating set; i.e., K is a relative n-cell with $\mu(\text{Bdry } K) = 0$ and $X - K$ has no unbounded components. Furthermore, the following lemma shows that we can require K in the compact-open neighborhood above to be an h-moving set.

Lemma 14.26 *Let $h \in \mathcal{M}[X, \mu]$, $\epsilon > 0$, and a separating set K be given. Then there is an $h' \in \mathcal{M}[X, \mu]$ with compact support such that $\sup_{x \in X} d(x, h'(x)) < \epsilon$ and K is hh'-moving. Consequently there is a subbasic family of compact-open topology open sets of the form $\mathcal{C}(g, K, \epsilon)$, $g \in \mathcal{M}[X, \mu]$, where K is g-moving.*

Proof For each h^*-invariant set of ends I in $\mathcal{I}_{h^*} \cap \mathcal{Q}_K$ choose distinct topological n-balls B_I with diameter less than ϵ such that $\mu(B_I \cap I(K)) > 0$ and $\mu(B_I \cap K) > 0$. For each I choose a μ-preserving homeomorphism h_I in $\mathcal{M}[X, \mu]$ with support in B_I such that $B_I \cap I(K)$ is not invariant under h_I. There are several ways of constructing h_I. One way is to take h_I to be any μ-preserving ergodic homeomorphism of B_I which fixes the boundary of B_I – extend h_I to X by setting it to be the identity off B_I. Such an ergodic homeomorphism exists by the results of Part I. The homeomorphism h' is the composition of these finitely many h_I's. By our construction h' is small and K is $g = hh'$-moving. □

14.11 End Conditions for Homeomorphic Measures

For noncompact manifolds X, we may ask when two OU measures μ and ν on X are homeomorphic (i.e., $\mu = \nu h$ for some homeomorphism h of X). Complications can arise if the behavior of the measures at the ends of the manifold is not taken into account. For example area measure μ on the strip $\check{X}$, which is infinite on both ends, cannot be homeomorphic to any measure ν which is finite on one end of the strip but infinite on the other. R. Berlanga and D. Epstein showed [38] in 1981 that a sufficient condition for two sigma finite OU measures on a connected sigma compact manifold X to be homeomorphic is if the measures are infinite on the same set of ends.

Theorem A2.8 [Berlanga–Epstein] *Let X be a sigma compact, connected n-manifold ($n \geq 2$) and let μ and ν be two nonatomic Borel measures, positive on open sets, finite on compact sets and zero on the* *boundary of X (i.e., two OU measures). Then μ and ν are homeomorphic if $\mu(X) = \nu(X)$ and μ and ν are infinite on the same set of ends (i.e., the measures μ^* and ν^* induced on the ends are identical). The homeomorphism h of X such that $\nu = \mu h$ can be chosen to fix the boundary of X (i.e., $h \in \mathcal{H}[X, \partial X]$), and be end preserving.*

Further discussion and a proof of this theorem is given in the second Appendix.