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COHOMOLOGY AND FIELD THEORY[1,2]

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ABSTRACT

The relationship between cohomology and the quantization of certain coupling constants in physics is discussed. A brief discussion is given about the relationship between cohomology and the Schrödinger wavefunction in field theories with quantized coupling constants.

1 Introduction

The relationship between charge quantization and topology goes back to Dirac [1]. In recent years, a flurry of research has gone into elucidating the relationships among geometry, topology and quantum field theory. In particular, physicists have discovered that homotopy arguments are very useful in understanding quantization conditions. It is also possible to use cohomology arguments to obtain the same quantization conditions. These ideas have been discussed in detail in a paper [2]. In this talk I present a brief introduction to the subject. I will also discuss the classification of Schrödinger wavefunctions in a quantum field theory. This seems to be intimately related to the Cech cohomology ideas used in the topological quantization arguments. The bulk of the wavefunction study is still unpublished [3].

There are three very familiar quantization conditions in quantum field theory: magnetic charge quantization [1], quantization [4] of the Yang-Mills mass term [5] and the quantization [6] of the coupling of the Wess-Zumino Lagrangian [7]. Part of the Lagrangian for each of these theories can be interpreted as a differential form. The Lagrangian will be written as the sum of two terms $\mathcal{L} = \mathcal{L}_0 + T$ where T is the term of topological significance and

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magnetic monopole	$\int T$	$\sim \int A_\mu \dot{x}^\mu \, dt$ $\sim \int A_\mu \, dx^\mu$
Y-M mass	$\int T$	$\sim \int \epsilon^{\mu\nu\rho} \mathrm{Tr}(A_\mu \partial_\nu A_\rho + \cdots)\, d^3x$ $\sim \int \mathrm{Tr}(A \wedge dA + \cdots)$
W-Z (d=2)	$\int T$	$\sim \int \epsilon^{\mu\nu} \mathrm{Tr}(\pi \, \partial_\mu \pi \, \partial_\nu \pi + \cdots)\, d^2x$ $\sim \int \mathrm{Tr}(\pi \, d\pi \wedge d\pi + \cdots)$

$\mathcal{L}_0$ includes the kinetic energy terms and interactions which will not concern us. The three "topological" Lagrangians for each of the theories are schematically presented in the table. The Lagrangians have the following in common:

1. $\mathcal{L}_0$ is globally defined.
2. T has special properties:
   (a) T may be interpreted as a differential form.
   (b) Under an appropriate transformation T changes by a total derivative.[4]
   (c) T is not globally defined.

The common properties of the topological part of the Lagrangian, denoted by T, will lead to the topological quantization conditions.

2 Dirac's Quantization Condition Revisited

In this section, the familiar Dirac quantization condition is derived in a manner illustrating Cech cohomological concepts. We will use a generalization of some ideas of Wu and Yang [8]. The methods of this section extend to higher dimensional cases.

Consider the motion of a point particle on a two dimensional sphere with a magnetic monopole residing at the center of the sphere. The classical Lagrangian for this system is

$$\mathrm{L} = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + A_\mu \frac{dx^\mu}{dt}$$

The term of interest for us is the coupling of the vector potential to the velocity of the particle. This is the only term of topological interest and for the remaining part of this section we will completely disregard the kinetic energy terms. We would like to view this term as the line integral of the one form $A = A_\mu dx^\mu$ along the trajectory Γ of the particle:

$$\int_\Gamma A$$

This would be fine except for the fact that it is impossible to find an everywhere nonsingular vector potential over the entire sphere. Wu and Yang [8] pointed out how to

[4] This should remind the reader of the classical mechanics theorem about the equivalence of Lagrangians that differ by a total time derivative.

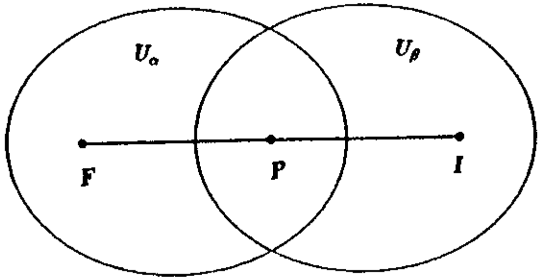

Figure 1: The worldline of the particle which begins at I and ends at F traverses two distinct coordinate patches. The point P is in the intersection of the two patches

modify the Lagrangian to take this into account. Cover the sphere with a collection of open sets $\mathcal{U} = \{U_\alpha\}$. On each open set choose a vector potential one form A_α. The subscript α on A_α is not a Lorentz index and refers to the open cover: $A_\alpha = A_{\alpha\mu}dx^\mu$. Consider the situation depicted in Figure 1 where one has a trajectory Γ that goes through a non-empty overlap $U_\alpha \cap U_\beta$. Let P be a point in the intersection. Naively one would write the vector potential contribution to the action as (remember that we are concentrating only on the term of possible topological interest)

$$I_P = \int_P^F A_\alpha + \int_I^P A_\beta$$

The problem with this definition is that it depends on the choice of the point P. To see this, consider another point Q in the overlap, construct I_Q and compute the difference $I_Q - I_P$:

$$I_Q - I_P = -\int_P^Q (A_\alpha - A_\beta)$$

We require knowledge of the gauge transformation on the overlap to evaluate the above. On each overlap it is necessary to specify a gauge transformation $\psi_{\alpha\beta}$ satisfying

$$d\psi_{\alpha\beta} = A_\alpha - A_\beta$$

Note that $-\psi_{\alpha\beta} = \psi_{\beta\alpha}$. Using the gauge transformation properties we see that:

$$I_Q - I_P = \psi_{\alpha\beta}(P) - \psi_{\alpha\beta}(Q)$$

In particular the quantity $I = I_Q + \psi_{\alpha\beta}(Q)$ is independent of the choice of point Q. More explicitly, I is given by

$$I = \int_Q^F A_\alpha + \psi_{\alpha\beta}(Q) + \int_I^Q A_\beta$$

This is the correct form of the action which was given by Wu and Yang. It seems to be a bit mysterious but its significance is more discernible by thinking about quantum mechanics.

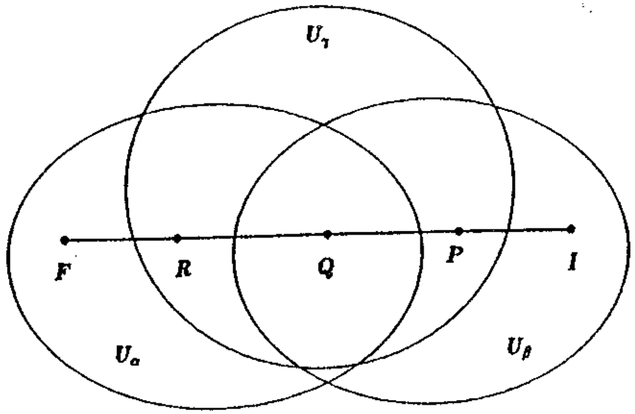

Figure 2: A third coordinate patch is introduced. The points P and R do not have to be in the triple intersection

According to the Feynman path integral formulation of quantum mechanics [9], the effect of a vector potential on propagation is to multiply the amplitudes by the exponential of the above equation. This is simply seen to be the amplitude for propagation in patch U_β, followed by a gauge transformation and terminating with the amplitude to propagate in the new gauge in patch U_α.

We now depart from the discussion of Wu and Yang and we ask the question, "What happens in a triple overlap?". The situation is depicted in Figure 2. Let us temporarily forget U_γ. The action is given by the Wu-Yang prescription. Remember that the value of the action is independent of the location of Q. Let us rewrite this term in such a way that contribution to the line integral from the part of the trajectory between P and R is expressed in terms of A_γ only. By using the gauge transformation law for the vector potential the action may be rewritten as:

$$\begin{aligned} I = \int_R^F A_\alpha + \psi_{\alpha\gamma}(R) + \int_P^R A_\gamma + \psi_{\gamma\beta}(P) + \int_I^P A_\beta \\ + \Big(\psi_{\alpha\beta}(Q) + \psi_{\beta\gamma}(Q) + \psi_{\gamma\alpha}(Q)\Big) \end{aligned} \qquad (1)$$

This equation is reminiscent of the Wu-Yang prescription. It is of the form line integral, gauge transformation, line integral, gauge transformation, line integral, and a left over piece. It is important to note that the left over piece contains the only reference to the point Q. The other pieces are just the Wu-Yang prescription for going from patch U_β to patch U_γ and ending in patch U_α. We will see that the left over piece contains all the information required to obtain Dirac's quantization condition.

The first piece of information we need is that the gauge transformations must satisfy a consistency condition on triple overlaps. Consider the following three equations:

$$A_\alpha - A_\beta = d\psi_{\alpha\beta}$$
$$A_\beta - A_\gamma = d\psi_{\beta\gamma}$$
$$A_\gamma - A_\alpha = d\psi_{\gamma\alpha}$$

Add all three equations to obtain the result

$$d(\psi_{\alpha\beta} + \psi_{\beta\gamma} + \psi_{\gamma\alpha}) = 0$$

To proceed further we need a special condition on the cover we chose for the sphere. It is possible to choose a cover such that each U_δ is diffeomorphic to an open ball, and each non-empty finite multiple intersection is also diffeomorphic to an open ball [10].This means that the Poincaré lemma is valid in each multiple intersection. In particular, we reach the conclusion that on $U_\alpha \cap U_\beta \cap U_\gamma$ one has

$$\psi_{\alpha\beta} + \psi_{\beta\gamma} + \psi_{\gamma\alpha} = c_{\alpha\beta\gamma}$$

where $c_{\alpha\beta\gamma}$ is a constant over the entire triple overlap. Therefore equation (1) is independent of Q as required.

There is an important lesson that this exercise teaches us. The classical action is ambiguous up to a constant. A priori, one could use the Wu-Yang prescription to write an expression involving patches U_α and U_β only, or write an expression involving patches U_α, U_β, U_γ. The difference between these two expressions is a constant which does not affect the classical equations of motion.

This classical ambiguity leads to quantum mechanical inconsistencies unless certain conditions are imposed on the collection of all $\{c_{\alpha\beta\gamma}\}$. The best way to see this is through path integral quantization. Consider the contribution of a trajectory Γ to the non-relativistic propagator:

$$\exp\left(i\int_\Gamma A\right) \cdot K_{\text{free}}(\Gamma)$$

The only ambiguity arises in how one decides to evaluate the vector potential line integral. There is an ambiguous phase factor of $\exp(ic_{\alpha\beta\gamma})$. Such a potential ambiguity exists at each non-empty triple intersection of patches on the sphere. The only way to avoid this mishap is to require that each phase factor be equal to one. In other words one has to choose all $c_{\alpha\beta\gamma}$ to be $2\pi \times$ (integer). Later we will see that this statement contains topological information about the manifold. It states that if the manifold's second cohomology class contains the integers then a consistent quantum theory requires an appropriate coupling constant to be quantized. In fact, the collection $\{c_{\alpha\beta\gamma}\}$ defines a two cocycle.

The Dirac flux quantization condition is related to the two cocycle $\{c_{\alpha\beta\gamma}\}$. With a little work one can see that the total magnetic flux is given by

$$\int_{S^2} F = \sum_{V_{\alpha\beta\gamma}} c_{\alpha\beta\gamma}$$

where the sum is over all triple overlaps with $V_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma \cap S^2$. We conclude that the total flux is given by

$$\int_{S^2} F = 2\pi \sum_{V_{\alpha\beta\gamma}} n_{\alpha\beta\gamma}$$

where the integer n is given by

$$c_{\alpha\beta\gamma} = 2\pi\, n_{\alpha\beta\gamma}$$

This is Dirac's quantization condition. We shall see that this condition generalizes in higher dimensions. Note that the quantization arose because of consistency conditions on triple overlaps of the different coordinate patches. There is a connection between the ambiguity in the classical action and the total flux through the sphere. In the case of a sphere, the construction given above is not necessary since one can cover the sphere with two coordinate patches, see Wu and Yang [8]. The above construction is valid for any manifold.

In this example, one finds that no further conditions are imposed by looking at quadruple or higher overlaps. This is not true in two dimensional field theories as we will see in the next section.

3 Cech Cohomology

Cech cohomology is the correct language for formulating the examples presented in the previous section. The machinery of Cech cohomology provides a means of cataloguing the necessary information required to extract the physics. In this section we will explain the relationship of Cech cohomology to the topology of the manifold, and we will also explain the cataloguing procedure. We will not present Cech cohomology in its most abstract setting. The general theory will be stripped down to a level sufficient to attack and solve the problems addressed in this talk. We assume the reader is familiar with the elementary aspects of simplicial homology [11], [12]. Namely, the concept of simplices, the existence of triangulations of a manifold, the notion of a chain (the formal sum of simplices), and the concept of the boundary of a chain. It is clear that the topology of the manifold will determine the allowed triangulations and that there are many possible triangulations. What is remarkable is that there are certain invariants which are independent of the triangulations. These invariants are the homology groups and their associated cohomology groups.

We will formulate Cech homology in a way that the connection to simplicial homology will be explicit. In all the manifolds we will consider it is always possible to choose an open cover $\mathcal{U} = \{U_\alpha\}$ such that each open set and each non-empty finite intersection is diffeomorphic to an open ball in $\mathbf{R}^n$ [10]. We will refer to these covers as *good covers*. At this stage we have already tailored Cech theory to some specifics we require. A major benefit of a good cover is that on each intersection the Poincaré lemma holds.

On each non-empty finite intersection define objects $U_{\alpha\beta}$, $U_{\alpha\beta\gamma}$, $U_{\alpha\beta\gamma\delta}$, etc. by

$$U_{\alpha\beta} = U_\alpha \cap U_\beta$$
$$U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$$
$$U_{\alpha\beta\gamma\delta} = U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$$

We define a formal orientation by requiring that $U_{\alpha\beta} = -U_{\beta\alpha}$, and likewise for the other objects. This good cover of the manifold defines a simplicial triangulation of the manifold. This is illustrated in Figure 3. In each open set U_α we choose a point in the interior, see Figure 3a . These points define the vertices of the triangulation. To each non-empty intersection we associate a one simplex, see Figure 3b . To each non-empty triple intersection

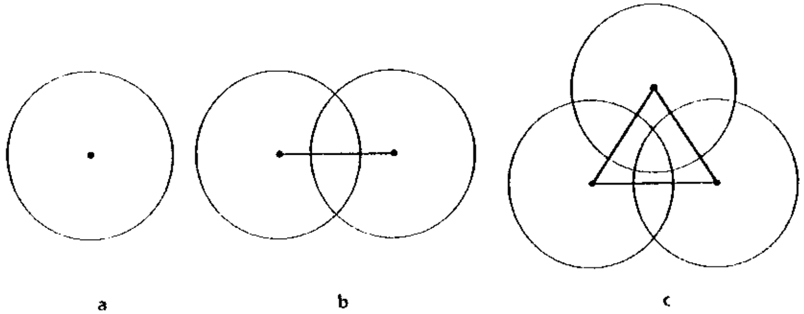

Figure 3: a: an open set and its associated vertex. b: two intersecting open sets and the associated segment. c: triply overlapping sets and the associated triangle.

we associate a two simplex, see Figure 3c . It is clear that the combinatorics of a good cover defines a simplicial triangulation of the manifold.

Our main interest is not Cech homology theory but Cech cohomology theory. For our purposes, Cech cohomology provides a systematic way of cataloguing information and a systematic way of dealing with singular fields configurations by avoiding the singularities. We will see how the ideas discussed in the previous sections may be discussed in the language of Cech cohomology.

A p-cochain with values in q-forms is an assignment of a nonsingular q-form to each p-chain. As an example consider the monopole. In that case we assigned to each open set U_α a vector potential A_α. The collection $\{A_\alpha\}$ is a zero cochain with values in one forms. We required that on U_α the vector potential A_α be nonsingular. A_α may be singular somewhere outside of U_α. This singularity is the famous Dirac string singularity. The collection of gauge transformation $\{\psi_{\alpha\beta}\}$ defines a one cochain with values in the zero forms.

Let us try to answer the following question, "When does a zero cochain define a global differential form?" Consider a zero cochain $\{\lambda_\alpha\}$. The zero cochain specifies a q-form on each open set in the cover. Assume $U_{\alpha\beta} \neq 0$, then on the overlap $U_{\alpha\beta}$ one must have $\lambda_\alpha - \lambda_\beta = 0$. If not then λ_α will not extend to λ_β. One can define a global differential q-form λ^{global} if and only if $\{\lambda_\alpha - \lambda_\beta\}$ vanishes identically. In other words, $\{\lambda_\alpha\}$ is a zero cocycle. This already gives us an inkling on what Cech theory will do for us. It will in certain situations allow us to piece local information into global information. We will see that Cech cohomology provides a systematic way for determining when local information can be pieced into less local information.

The coboundary operator δ is an operation between p-chains and (p+1)-chains. It is defined as follows for small values of p:

$$\delta\{A_\alpha\} = \{A_\alpha - A_\beta\}$$

$$\delta\{B_{\alpha\beta}\} = \{B_{\alpha\beta} + B_{\beta\gamma} + B_{\gamma\alpha}\}$$

$$\delta\{C_{\alpha\beta\gamma}\} = \{C_{\alpha\beta\gamma} + C_{\beta\gamma\delta} + C_{\gamma\delta\alpha} + C_{\delta\alpha\beta}\}$$

The generalization to larger values of p is straightforward. One can show that the coboundary operator satisfies $\delta^2 = 0$. The nilpotency of δ allows us to define a cohomology theory. We define the p-cocycles as those p-cochains that are annihilated by δ. A p-cocycle z is said to be *exact* if one can find a (p-1)-cochain y such that $z = \delta y$. The p-coboundaries are the image of the (p-1)-cochains under δ. Since δ is nilpotent, it follows that the p-coboundaries are a subset of the p-cochains. Therefore it is possible to define cohomology classes by taking the quotient of the cocycles by the coboundaries. Since a cocycle is defined by the condition $\delta z = 0$, the existence of non-trivial cohomology classes boils down to the question of whether every cocycle is exact. The answer is provided by the existence of a Poincaré lemma for the δ operator. This lemma states that if $p > 0$ then the equation $\delta z = 0$ for a p-chain z can always be "solved". Namely, there exists a (p-1)-cochain y such that $z = \delta y$. This seems to say that the cohomology classes are trivial. This is true except for a caveat which is related to the construction of y in the proof of the Poincaré lemma. This caveat will be used to constuct nontrivial cohomology classes called Cech cohomology classes. We will postpone the caveat until later. The p=0 case is the statement that a closed zero cochain defines a global differential form.

The final piece of formalism we need is the *tic-tac-toe* box [10]. We will be studying the so called *double chain complexes*. These ideas are best explained by looking at our magnetic monopole example. It will not be necessary to assume that the electrically charged particle moves on the surface of a sphere. The configuration space for the trajectory of the particle may be any compact manifold without boundary. In the box below we have included the vector potentials and the transition functions.

Ω^3					
Ω^2					
Ω^1		$\{A_\alpha\}$			
Ω^0			$\{\psi_{\alpha\beta}\}$		
d $\uparrow$					
δ	$\rightarrow$	C^0	C^1	C^2	C^3

The rows correspond to the degree of the differential form. The notation Ω^q stands for the q-forms. The columns correspond to the degree of the cochain. The notation C^p stands for the p-cochains. The d operator moves us vertically and the δ operator moves us horizontally. Let us apply the d and the δ operations to the elements in the above box. We can operate again with d and δ and get zero since these operators are nilpotent. Notice that one of the entries is the gauge transformation law. Also, the operators d and δ commute.

Ω^3		0			
Ω^2		$\{dA_\alpha\}$	0		
Ω^1		$\{A_\alpha\}$	$\delta\{A\}=\{d\psi\}$	0	
Ω^0			$\{\psi_{\alpha\beta}\}$	$\delta\{\psi_{\alpha\beta}\}$	0
d	$\uparrow$				
δ	$\rightarrow$	C^0	C^1	C^2	C^3

Define quantities F_α by $F_\alpha = dA_\alpha$, and $c_{\alpha\beta\gamma}$ by $\delta\{\psi_{\alpha\beta}\} = \{c_{\alpha\beta\gamma}\}$. The above box thus becomes:

Ω^3		0			
Ω^2		$\{F_\alpha\}$	0		
Ω^1		$\{A_\alpha\}$	$\delta\{A\}=\{d\psi\}$	0	
Ω^0			$\{\psi_{\alpha\beta}\}$	$\{c_{\alpha\beta\gamma}\}$	0
d	$\uparrow$				
δ	$\rightarrow$	C^0	C^1	C^2	C^3

We immediately learn that the F_α is d-closed and it is also a zero δ-cocycle. The means that F_α defines a closed global differential form F, the electromagnetic field strength two form.

The other piece of information we learn from the tic-tac-toe box involves the $\{c_{\alpha\beta\gamma}\}$. These objects are d-closed and they define a two cocycle. Since locally closed zero form is given by a constant, the $\{c_{\alpha\beta\gamma}\}$ must define a two cocycle. All this information is shown in the box below.

Ω^3	0	0			
Ω^2	F	$\{F_\alpha\}$	0		
Ω^1		$\{A_\alpha\}$	$\delta\{A\}=\{d\psi\}$	0	
Ω^0			$\{\psi_{\alpha\beta}\}$	$\{c_{\alpha\beta\gamma}\}$	0
d	$\uparrow$			$\{c_{\alpha\beta\gamma}\}$	0
δ	$\rightarrow$	C^0	C^1	C^2	C^3

The main conclusion is that given a collection of vector potentials $\{A_\alpha\}$ and transition functions $\{\psi_{\alpha\beta}\}$, the gauge transformation law $\delta A = \psi$, one can construct a closed global two form F and a locally constant two cocycle $\{c_{\alpha\beta\gamma}\}$. Since F is closed and global, it is a representative of the second DeRham cohomology class. Note that we wrote F "outside" the tic-tac-toe box. The reason is that even though F is closed, the Poincaré lemma is in general not valid globally. If something is outside the box then one has to be careful about applying the Poincaré lemma. We are not allowing singular vector potentials as acceptable solutions.

A similar thing happens with the Poincaré lemma for the δ operator. Note that the locally constant cocycle $\{c_{\alpha\beta\gamma}\}$ is exact. It is the δ of $\{\psi_{\alpha\beta}\}$. In general the ψ's will not be constant. The question is whether on can find a solution to the equation $\delta c = 0$ given by $c = \delta b$ where the $\{b_{\alpha\beta}\}$ are constants. In general, such a solution does not exist. This is the caveat we previously mentioned. There is no Poincaré lemma for the δ operator if one only uses locally constant cochains. This is analogous to not allowing singular differential forms in the Poincaré lemma for the d operator. In analogy to the previous case we write the c cocycle "outside" the box. One has to be careful in applying the Poincaré lemma outside the box. The Cech cohomology classes are defined by looking at locally constant cocycles and asking whether they are exact within the class of locally constant cocycles. The two cocycle c is a representative of the second Cech cohomology class of the manifold. There is an isomorphism between the DeRham classes and the Cech classes [10].

Remember that the total magnetic flux through the manifold was determined by the $\{c_{\alpha\beta\gamma}\}$. There several notes of interest. The total magnetic flux through the sphere was determined by conditions on triple overlaps. Quantum mechanics imposes a further condition on the cocycle $\{c_{\alpha\beta\gamma}\}$. The c's must be $2\pi \times$ (integer). This imposes a severe restriction on the cohomology classes. The integers $\mathbf{Z}$ are a subset of the real numbers $\mathbf{R}$. One can define objects $\{n_{\alpha\beta\gamma}\}$ to be integer valued cochains instead of real valued cochains. Since the δ Poincaré lemma does not apply to real valued cochains then it certainly does not apply to the integer valued cochains. Therefore, there will be non-trivial integer valued cohomology. These cohomology classes are called *Cech cohomology classes with integer coefficients* and they will be denoted by $H^p_C(M,\mathbf{Z})$. Quantum mechanics requires that the cocycle $\{c_{\alpha\beta\gamma}/(2\pi)\}$ must be integral. The existence of such a cocycle is determined by whether or not the manifold in question admits integral cocycles in its second Cech cohomology class, i.e., $\mathbf{Z} \subset H^2_C(M,\mathbf{Z})$. The existence of such integral cocycles is determined by the topology of the manifold. The magnetic flux will be quantized if the manifold admits a second cohomology class with integer coefficients that contains the integerc.

The situation becomes more interesting when one looks at a two dimensional example. Assume one has a two dimensional non-linear sigma model given as a map ϕ from a two dimensional spacetime S to a manifold M. For simplicity we take S to be $\mathbf{R} \times S^1$. Assume that part of the Lagrangian can be interpreted as the pull back of a two form on M. We will neglect completely the rest of the Lagrangian. Lagrangian will be taken to refer only to the term of possible topological significance. In analogy to the monopole example, the Lagrangian T might not be globally defined. Assume that there is a collection of locally defined two-forms $\{T_\alpha\}$, one two-form for each open set in a good cover of M. Assume that on a non-empty intersection $U_\alpha \cap U_\beta$ the respective Lagrangians differ by the differential of a one form $J_{\alpha\beta}$:

$$T_\alpha - T_\beta = dJ_{\alpha\beta}\ .$$

Note that the collection of Lagrangians defines a zero cocycle and the transition functions define a one cocycle. In the tic-tac-toe box below we have included the Lagrangian and its gauge transformation properties.

Ω^4						
Ω^3						
Ω^2		$\{T_\alpha\}$	$\delta\{T_\alpha\}=\{dJ_{\alpha\beta}\}$			
Ω^1			$\{J_{\alpha\beta}\}$			
Ω^0						
d	$\uparrow$					
δ	$\rightarrow$	C^0	C^1	C^2	C^3	C^4

First we record the consequence of multiple d and δ operations. This is shown in the box below:

Ω^4		0				
Ω^3		$\{dT_\alpha\}$	0			
Ω^2		$\{T_\alpha\}$	$\delta\{T_\alpha\} = \{dJ_{\alpha\beta}\}$	0		
Ω^1			$\{J_{\alpha\beta}\}$	$\delta\{J\}$	0	
Ω^0						
d	$\uparrow$					
δ	$\rightarrow$	C^0	C^1	C^2	C^3	C^4

One of the pieces of information we have is that δJ is closed, $d\delta J = 0$. This follows from the commutativity of the two operations. The tic-tac-toe box automatically takes this into account. Since δJ is closed and since the cohomology is trivial, there must exist a two cochain K such that J is its differential. This is illustrated in the box below

Ω^4		0				
Ω^3		$\{dT_\alpha\}$	0			
Ω^2		$\{T_\alpha\}$	$\delta\{T_\alpha\} = \{dJ_{\alpha\beta}\}$	0		
Ω^1			$\{J_{\alpha\beta}\}$	$\delta\{J\}$	0	
Ω^0				$\{K\}$		
d	$\uparrow$					
δ	$\rightarrow$	C^0	C^1	C^2	C^3	C^4

Applying the d and δ information to the box above we find:

Ω^4		0				
Ω^3		$\{dT_\alpha\}$	0			
Ω^2		$\{T_\alpha\}$	$\delta\{T_\alpha\} = \{dJ_{\alpha\beta}\}$	0		
Ω^1			$\{J_{\alpha\beta}\}$	$\delta\{J\}$	0	
Ω^0				$\{K\}$	$\delta\{K\}$	0
d	$\uparrow$					
δ	$\rightarrow$	C^0	C^1	C^2	C^3	C^4

We learn that δK is a closed three-cocycle. This cocycle is represented by constant cocycle $\{c_{\alpha\beta\gamma\delta}\} = \delta K$. The other piece of information we need to know is that $\{dT_\alpha\}$ defines a closed global differential form $\mathcal{G}$. This is all depicted in the box below.

Ω^4	0	0				
Ω^3	$\mathcal{G}$	$\{dT_\alpha\}$	0			
Ω^2		$\{T_\alpha\}$	$\delta\{T_\alpha\} = \{dJ_{\alpha\beta}\}$	0		
Ω^1			$\{J_{\alpha\beta}\}$	$\delta\{J\}$	0	
Ω^0				$\{K\}$	$\delta\{K\}$	0
d	$\uparrow$				$\{c_{\alpha\beta\gamma\delta}\}$	0
δ	$\rightarrow$	C^0	C^1	C^2	C^3	C^4

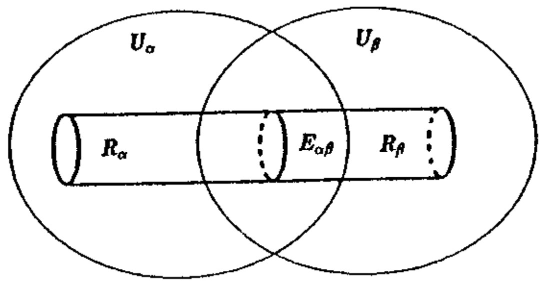

Figure 4: The evolution of a manifold with its spatial topology being a circle. The world sheet lies in two distinct patches. It is subdivided into regions R_α and R_β with the edge being $E_{\alpha\beta}$.

Just as in the electromagnetic case, the Lagrangian and its gauge transformation law determines a closed global differential form and a locally constant cocycle. Is there any significance to the K and c cochains in the above? What is the meaning of the global differential form $\mathcal{G}$? To see the meaning of these quantities on has to go back an see what is the analogue of the Wu-Yang prescription in the two dimensional case.

It is possible to generalize the Wu-Yang construction to this situation. For simplicity, let us assume that the image of spacetime $\phi(S)$ lies entirely in the patches U_α and U_β as depicted in Figure 4. By mimicking the Wu-Yang construction one can show that

$$\int_{R_\alpha} T_\alpha - \int_{E_{\alpha\beta}} J_{\alpha\beta} + \int_{R_\beta} T_\beta$$

is independent of where one chooses the boundary $E_{\alpha\beta}$. This prescription is actually incomplete. We will have to do a further modification to reach a satisfactory answer within the domain of classical field theory.

We have to worry about what happens in triple overlaps. The situation is depicted in Figure 5. One has to see whether the introduction of the triple overlap introduces some $E_{\alpha\beta}$ dependence and a modification of the above is required. The modification is obtained by applying the ideas of Wu and Yang one more time. By using the conditions on the overlaps one can rewrite the previous equation as

$$\int_{R'_\alpha} T_\alpha - \int_{E_{\alpha\gamma}} J_{\alpha\gamma} + \int_{R'_\gamma} T_\gamma - \int_{E_{\gamma\beta}} J_{\gamma\beta} + \int_{R'_\beta} T_\beta$$
$$- \int_{E_{\alpha\beta}} (J_{\alpha\beta} + J_{\beta\gamma} + J_{\gamma\alpha})$$

The form of the above is reminiscent of equation (1). There is an ambiguity in the classical action when one looks at triple overlaps. The above appears to depend on $E_{\alpha\beta}$. Previously we found that the ambiguity was a constant. In the present case we will have to work a little harder.

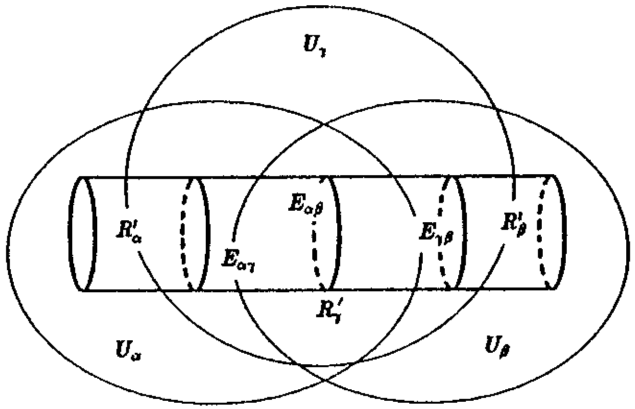

Figure 5: The generalization of Figure 2 to one higher dimension. Note that R_α and R_β have been subdivided into regions R'_α, R'_β and R'_γ. Also note the new edges.

According to the tic-tac-toe box, on the triple overlaps there exist distinct functions $K_{\alpha\beta\gamma}$ such that

$$dK_{\alpha\beta\gamma} = J_{\alpha\beta} + J_{\beta\gamma} + J_{\gamma\alpha}$$

The term involving $E_{\alpha\beta}$ may be rewritten as

$$\int_{E_{\alpha\beta}} dK_{\alpha\beta\gamma} = \int_{\partial E_{\alpha\beta}} K_{\alpha\beta\gamma} = 0$$

This vanishes since $E_{\alpha\beta}$ is boundaryless.

The triple overlaps introduce no ambiguities into the two dimensional field theory. This is unlike the electromagnetic example of Wu and Yang. We will see that a more careful analysis of triple overlaps requires some modifications of the Wu-Yang procedure. We must analyze a new feature of two dimensional field theories which is not present in the one dimensional example. A suitable modification of the Wu-Yang procedure will lead to conditions on quadruple overlaps.

The new feature of the two dimensional field theory is the existence of Y junctions when one subdivides the image of S, see Figure 6. The Wu-Yang prescription can be generalized in a simple manner to incorporate the physics of the Y junctions. The correct way to define

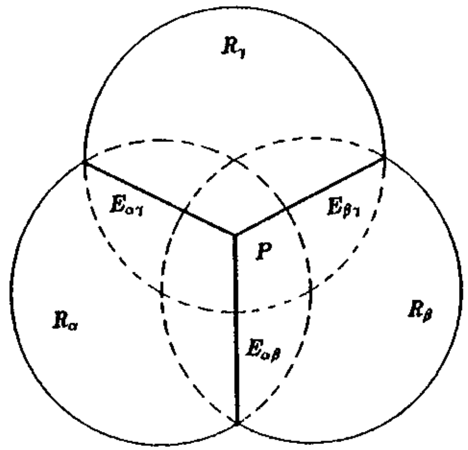

Figure 6: The appearance of Y-junctions when one subdivides a two dimensional integration region into three distinct non-overlapping sets.

the action in the case of Y junctions is

$$\begin{aligned}
&\int_{R_\alpha} T_\alpha - \int_{E_{\alpha\beta}} J_{\alpha\beta} \\
&+ \int_{R_\beta} T_\beta - \int_{E_{\beta\gamma}} J_{\beta\gamma} \\
&+ \int_{R_\gamma} T_\gamma - \int_{E_{\gamma\alpha}} J_{\gamma\alpha} \\
&- K_{\alpha\beta\gamma}(P)
\end{aligned}$$

We have used the notation of Figure 6. One can verify that a small movement of the Y junction leaves the value of the action invariant. This is the modification of the Wu-Yang prescription that is required.

Let us now see what happens when one introduces a fourth patch, U_δ, as in Figure 7 . The above may be rewritten as

$$\begin{aligned}
&\int_{R'_\alpha} T_\alpha + \int_{R'_\beta} T_\beta + \int_{R'_\gamma} T_\gamma + \int_{R'_\delta} T_\delta \\
&- \int_{E_{\alpha\delta}} J_{\alpha\delta} - \int_{E_{\beta\delta}} J_{\beta\delta} - \int_{E_{\gamma\delta}} J_{\gamma\delta} \\
&- K_{\alpha\delta\gamma}(Q_1) - K_{\alpha\beta\delta}(Q_2) - K_{\beta\gamma\delta}(Q_3) \\
&- \Big(K_{\alpha\beta\gamma}(P) + K_{\beta\gamma\delta}(P) + K_{\gamma\delta\alpha}(P) + K_{\delta\alpha\beta}(P) \Big)
\end{aligned}$$

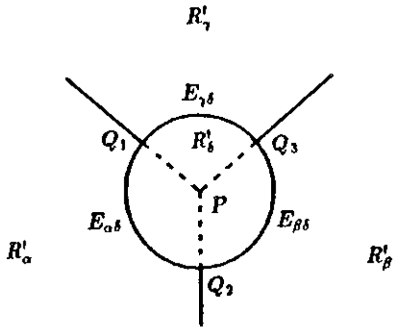

Figure 7: Introduction of a fourth coordinate patch into Figure 6. The R regions have been subdivided into R' regions.

The last line is a constant (independent of P) since $d\delta\{K\} = 0$. We learn that just as in the electromagnetic case the classical action is defined up to an additive constant. A path integral quantization immediately tell us that a consistent quantum theory is only possible if

$$K_{\alpha\beta\gamma}(P) + K_{\beta\gamma\delta}(P) + K_{\gamma\delta\alpha}(P) + K_{\delta\alpha\beta}(P) = 2\pi n_{\alpha\beta\gamma\delta}$$

where the n's are integers. Conditions have to be imposed on quadruple overlaps. Namely, one has to be able to choose a collection of integers $n_{\alpha\beta\gamma\delta}$ on any good cover of the manifold. The three cocycle $\{n_{\alpha\beta\gamma}\}$ is a representative of the third Cech cohomology class of the configurations space: $H^3_C(M, \mathbf{Z})$.

Finally, we mention that there is an analogue of the magnetic field in this problem given by the closed global three form $\mathcal{G}$. This three form is a representative of the third DeRham cohomology class of the manifold. If one calculates the 'flux' by integrating $\mathcal{G}$ over a boundaryless three dimensional region then one discovers that the total flux is given by the sum of $2\pi n_{\alpha\beta\gamma\delta}$ over the patches that intersect the region of interest.

These ideas generalize to higher dimensional field theories. If the dimension of spacetime is d then the *possibility* of a topological Lagrangian with a quantized coupling is determined by whether or not the $(d+1)$ DeRham cohomology class is non-trivial. Remember that there is an isomorphism between the DeRham classes and the Cech classes. It is true that if the $(d+1)$ DeRham cohomology class vanishes then there is no topological quantization. The precise requirements for topological quantization of the Lagrangian require a case by case study. For example, if spacetime is the two sphere S^2 and if the configuration space $M = S^1 \times S^1 \times S^1$ then there is no quantization condition. The reason is subtle.[5] The cohomology of M is the product of the cohomology of the respective circles. Therefore the third cohomology class of M is the product of the one class for each circle. The pullback of each of these one classes to spacetime S^2 is a trivial form. For example, one would reach a different conclusion if spacetime is $S^1 \times S^1$.

[5] I would like to thank P. Ginsparg and E. Witten for discussions on this point.

4 Classification of Wavefunctions

The classification of wavefunctions in a quantum field theory is not a well understood subject. The problem is completely understood in quantum mechanics. I believe that one of the underlying difficulties involves the issue of locality in a quantum field theory. This is intimately related to the notion of *local cohomology*. We use the standard physics convention of writing the dimension of spacetime as $(n+1)$ where n is the spatial dimensionality.

The space of field configurations $\mathcal{C}$ is given by:

$$\mathcal{C} = \{\varphi : \Sigma \to M\} ,$$

where Σ is the spatial manifold and M is the space where the fields reside, i.e., the target space for the map. In the case of quantum mechanics, a $(0+1)$ dimensional field theory, the space of field configurations is the same as M, the space where the fields reside. Naively, a Schrödinger wavefunction Ψ is a map that assigns to each element $\varphi(\mathbf{x})$ in $\mathcal{C}$ a complex number $\Psi[\varphi(\mathbf{x})]$. More precisely, a wavefunction is a local section of a line bundle over the field configurations $\mathcal{C}$. Since quantum mechanics requires a Hilbert space type of structure, the structure group of the line bundle reduces to $U(1)$. There is a theorem [10] that states that over a sufficiently nice space the line bundles are classified by $H^2_C(\mathcal{C}, \mathbf{Z})$. This may be seen by the following simple argument. Let $\{U_\alpha\}$ be a cover for $\mathcal{C}$. Let $\{\exp(i\xi_{\alpha\beta})\}$ be the transitions functions[6] that define the bundle:

$$\Psi_\alpha = \exp(i\xi_{\alpha\beta})\,\Psi_\beta ,$$

where Ψ_α and Ψ_β are local coordinates on the bundle over the corresponding coordinate patches. Consider what happens as one changes coordinates from patches U_α to U_β to U_γ and back to U_α:

$$\begin{aligned}\Psi_\alpha &= \exp(i\xi_{\alpha\beta})\,\Psi_\beta \\ &= \exp(i\xi_{\alpha\beta})\,\exp(i\xi_{\beta\gamma})\,\Psi_\gamma \\ &= \exp(i\xi_{\alpha\beta})\,\exp(i\xi_{\beta\gamma})\,\exp(i\xi_{\gamma\alpha})\,\Psi_\alpha .\end{aligned}$$

One immediately concludes that

$$\xi_{\alpha\beta} + \xi_{\beta\gamma} + \xi_{\gamma\alpha} = 2\pi n_{\alpha\beta\gamma} ,$$

where the n's are integers. In fact, the $\{n_{\alpha\beta\gamma}\}$ define a two cocycle. In conclusion[7] the transition functions for the line bundle lead to a representative in the second Cech cohomology class $H^2_C(\mathcal{C}, \mathbf{Z})$. If one puts a connection $\mathcal{A}$ on a line bundle over $\mathcal{C}$ then the associated magnetic field $\mathcal{B} = d\mathcal{A}$ when integrated over a two cycle will have a quantized flux which is related to collection of integers in the Cech two cocycle.[8]

[6] I will refer to both ξ and its exponential $\exp(i\xi)$ as the transition function.

[7] This is valid only if the space $\mathcal{C}$ is sufficiently nice.

[8] Mathematically, the above may be sound but it might not necessarily have anything to do with physics.

For example, in the $(0+1)$ dimensional case, the space C is the same as the target space M. Line bundles in quantum mechanics are characterized by $H^2_C(M,\mathbf{Z})$. The associated integral Cech cocyles represent the quantization of ordinary magnetic flux.

The situation becomes much more interesting in the true field theory case. Note that the transition functions $\exp(i\xi_{\alpha\beta})$ are completely arbitrary. They can be non-local functionals of the fields. From a physical standpoint, this seems to be too strong a requirement. For example, reasonable local changes of variables do not affect the S–matrix but nothing is guaranteed by a non-local change of variables. It is not clear to me whether such non-local transition functions are allowed by nature. This means that $H^2_C(C,\mathbf{Z})$ might not be the relevant object in the classification of the physical wavefunctions [3].

The notion of locality in cohomology theory has arisen in several different ways. There appear to be ways to define *local cohomology* rigorously [13]. We will not worry about these technical details but just discuss the main ideas. Consider an abelian gauge theory in $(3+1)$ dimensions with a left handed Weyl fermion. This theory has the standard chiral anomaly. Since $\pi_5(U(1))=0$, there is no topological obstruction to defining the fermion determinant. There is a physical obstruction that is imposed by locality. Namely, the anomaly cannot be eliminated by the addition of local counter terms to the Lagrangian. A non-local counter term can be used to eliminate the abelian anomaly.

Similar ideas enter in the discussion of the classification of wavefunctions. The relevant object for the classification problem is probably not $H^2_C(C,\mathbf{Z})$ but a "local" version of the second cohomology group. I conjecture that in the type of theories discussed in this paper, the relevant local cohomology is closely related to $H^{n+2}_C(M,\mathbf{Z})$ where n is the spatial dimensionality. My argument is based on an analysis of the gauge transformation properties of the path integral. The path integral is a representation of the time evolution operator $\exp(-itH)$ where H is the Hamiltonian. For simplicity, I will discuss a $(1+1)$ dimensional theory of the type discussed in Section 3. One begins with the initial wave function and one evolves it forward in time by using the path integral to compute the evolution kernel. The discussion of Section 3 explains how to write the the action in such a way that things are well defined. Consider a history where one begins in patch U_β with a wave function Ψ_β. At time t_1 one goes to a patch U_α with wave function Ψ_α, analogous to the situation depicted in Figure 8. The discussion of Section 3 may be interpreted as saying that at time t_1 we are required to make a gauge transformation given by

$$\Psi_\alpha = \exp\left(-i\int_{E_{\alpha\beta}} J_{\alpha\beta}\right)\Psi_\beta\ ,$$

where $E_{\alpha\beta}$ is a constant time surface. This situation is is very similar to the quantum mechanical example of Section 2. The only difference has to do with the transition function. In the quantum mechanics case the transition function $\psi_{\alpha\beta}$ was a function of t and $\mathbf{x}$ only. In this case the transition function $\xi_{\alpha\beta}$ is a local functional:

$$\int_{E_{\alpha\beta}} J_{\alpha\beta}\ .$$

This suggests that one has a line bundle over C with local transition functions. The situation becomes more interesting when one looks at the Y junction case depicted in Figure 9. In

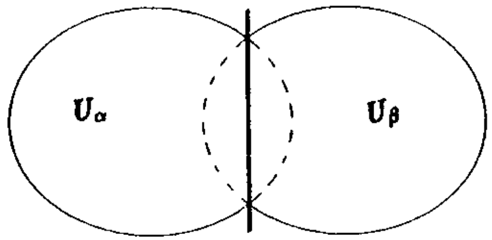

Figure 8: A gauge transformation from one patch to a second patch at time t_1. The solid line denotes an equal time surface.

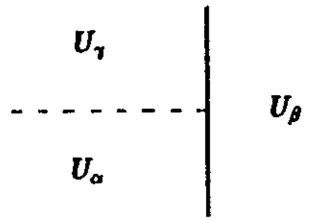

Figure 9: Going from a single patch to a double patch at time t_1. The equal time surface is the solid line. The two "future" patches are separated by the dashed line.

this case the correct transition function at time t_1 is given by:

$$\Psi_{\alpha+\beta} = \exp\left(-i\int_{E_{\alpha\beta}} J_{\alpha\beta}\right)\ \exp\left(-iK_{\alpha\beta\gamma}(P)\right)\ \exp\left(-i\int_{E_{\beta\gamma}} J_{\beta\gamma}\right)\ \Psi_\beta\ ,$$

where $\Psi_{\alpha+\beta}$ reflects the fact that the wavefunction for $t>t_1$ has to be specified in two coordinate patches. Note that the above is local. If one now looks at sequential changes of patches one discovers the constraint on the third cohomology class of M.

The following interesting question arises. One is constructing a line bundle over C using local transition functions. What is a connection on this line bundle? My guess is that the answer is related to the recent work of Wu and Zee [14]. These authors point out that there is some type of abelian structure in certain intrinsically non-abelian problems.

References

[1] P.A.M. Dirac, "Quantized Singularities in the Electromagnetic Field", *Proc. Royal Soc. London*, A133, 60(1931).

[2] O. Alvarez, "Topological Quantization and Cohomology", Berkeley preprint, to appear in *Commun. Math. Phys.*.

[3] O. Alvarez, unpublished.

[4] S. Deser, R. Jackiw and S. Templeton, "Topologically Massive Gauge Theories", *Ann. Phys.* 140, 372(1982).

[5] S. Deser, R. Jackiw and S. Templeton, "Three Dimensional Massive Gauge Theories", *Phys. Rev. Lett.* 48, 975(1982); J. Schoenfeld, "A Mass Term for Three Dimensional Gauge Fields", *Nucl. Phys.* B185, 157(1981); W. Siegel, "Unextended Superfields in Extended Supersymmetry", *Nucl. Phys.* B156, 135(1979).

[6] E. Witten, "Global Aspects of Current Algebra", *Nucl. Phys.* B223, 422(1983).

[7] J. Wess and B. Zumino, "Consequences of Anomalous Ward Identities", *Phys. Lett.* 37B, 95(1971).

[8] T.T. Wu and C.N. Yang, "Dirac's Monopole without Strings: Classical Lagrangian Theory", *Phys. Rev.* D14, 437(1976).

[9] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965.

[10] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, New York, 1982.

[11] I.M. Singer and J. Thorpe, *Lecture Notes on Elementary Topology and Geometry*, Springer-Verlag, New York, 1967.

[12] C. Misner, K. Thorne and J.A. Wheeler, *Gravitation*, Freeman, San Francisco, 1970.

[13] L. Bonora and P. Cotta-Ramusino, "Some Remarks on BRS Transformation Anomalies and the Cohomology of the Lie Algebra of the Group of Gauge Transfomations", *Commun. Math. Phys* 87, 589(1983).

[14] Y-S Wu and A. Zee, "Abelian Gauge Structure Inside Non-abelian Gauge Theories", Univesity of Washington preprint 40048-38 P4.