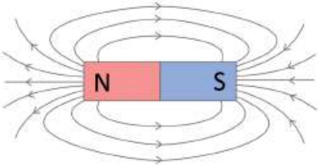
Mathematical Approach to Magnetic Monopoles with Yang–Mills theory

Samuel Adrian Antz

February 25, 2026

Magnetic monopole



A magnetic monopole is a magnet with only one pole (no closed magnetic flow lines). A simple description of its magnetic field $\mathbf{B}$ is:

$$\mathbf{B} = g\frac{\mathbf{r}}{r^3} \tag{1}$$

with the non-vanishing *magnetic charge* g. It solves Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

on $\mathbb{R}^3 \setminus \{0\}$.

Vector potential

Due to the non-vanishing magnetic charge g, a *vector potential* $\mathbf{A}$ with $\mathbf{B} = \nabla \times \mathbf{A}$ doesn't exist any more, since Stoke's theorem would imply:

$$\begin{aligned} 4\pi g &= \int_{S_R(0)} \frac{g}{r^2} \mathrm{d}S = \int_{S_R(0)} \mathbf{B} \cdot \mathrm{d}\mathbf{S} \\ &= \int_{S_R(0)} (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S_R(0) = \emptyset} \mathbf{A} \cdot \mathrm{d}\mathbf{s} = 0 \end{aligned} \tag{3}$$

independent of the radius R.

Vector potential

Expanding $\mathbf{B} = \nabla \times \mathbf{A}$ also yields no global solution, but nonetheless local solutions:

$$\frac{1}{r \sin\vartheta} \frac{\partial}{\partial\vartheta}(A_\varphi \sin\vartheta) = (\nabla \times \mathbf{A})_r \overset{!}{=} \mathbf{B}_r = \frac{g}{r^2}$$
$$\Rightarrow \mathbf{A}^\pm(\vartheta) = A^\pm_\varphi(\vartheta)\mathbf{e}_\varphi = \frac{g}{r \sin\vartheta}(-\cos\vartheta \pm 1)\mathbf{e}_\varphi. \tag{4}$$

All are undefined on the *Dirac strings* $\vartheta = 0$ and $\vartheta = \pi$, connecting the magnetic monopole with infinity, but choosing the integration constants as ± 1 makes it possible to continuously expand them over at least one Dirac string using L'Hôpital's rule:

$$\mathbf{A}^+(0) := \lim_{\vartheta \to 0} \mathbf{A}^+(\vartheta) = 0; \tag{5}$$

$$\mathbf{A}^-(\pi) := \lim_{\vartheta \to \pi} \mathbf{A}^-(\vartheta) = 0. \tag{6}$$

(If more magnetic monopoles are considered, then Dirac strings can also connect them with each other.)

Gauge

$\mathbf{B} = \nabla \times \mathbf{A}$ doesn't uniquely determine the vector potential $\mathbf{A}$. In fact, a *gauge transformation* $\mathbf{A} \mapsto \mathbf{A} + \nabla\Lambda$ leaves the magnetic field $\mathbf{B}$ fully invariant.

Gauge is the freedom to choose representatives.

It's therefore possible to impose restrictions on the vector potential $\mathbf{A}$ as needed, for example the *Coulomb gauge* $\nabla \cdot \mathbf{A} = 0$ by solving the Poisson equation:

$$\Delta\Lambda = -\nabla \cdot \mathbf{A}. \tag{7}$$

Such a gauge transformation also exists between the two local vector potentials $\mathbf{A}^{\pm}$ of a magnetic monopole:

$$\nabla_{\varphi} = \frac{1}{r \sin(\vartheta)} \frac{\partial}{\partial \varphi}; \tag{8}$$

$$\mathbf{A}^{+}(\vartheta) = \mathbf{A}^{-}(\vartheta) + \nabla(2g\varphi). \tag{9}$$

Quantization

An electron in the field of a magnetic monopole is described by the *Schrödinger equation*. Since not the magnetic field $\mathbf{B}$, but its vector potentials $\mathbf{A}^{\pm}$ enter, there are in fact two different Schrödinger equations for the exact same problem:

$$i\hbar\frac{\partial\Psi^{\pm}}{\partial t} = \frac{1}{2m_{\mathrm{e}}}\left(-i\hbar\nabla + e\mathbf{A}^{\pm}\right)^{2}\Psi^{\pm}. \tag{10}$$

Since the Schrödinger equation is invariant under the common gauge transformations $\mathbf{A} \mapsto \mathbf{A} + \nabla\Lambda$ and $\Psi \mapsto e^{i\frac{e}{\hbar}\Lambda}\Psi$,[1] the gauge transformation $\mathbf{A}^{+}(\vartheta) = \mathbf{A}^{-}(\vartheta) + \nabla(2g\varphi)$ enforces the gauge transformation:

$$\Psi^{+} = e^{i\frac{2e}{\hbar}g\varphi}\Psi^{-}. \tag{11}$$

Here, the *electromagnetic gauge group* U(1) appears.

---

[1] An additional $\Phi \mapsto \Phi - \partial_t\Lambda$ for the electric potential Φ is obmitted here since there is neither an electric field nor time dependence.

Quantization

Now the azimuthal angle φ being cyclic with $\varphi = 0$ and $\varphi = 2\pi$ describing the same coordinates leads to the magnetic charge being an integer multiple, called *quantum number*, of the *reduced magnetic flux quantum*:

$$g = n\Phi_0 = n\frac{\hbar}{2e}, n \in \mathbb{N}. \tag{12}$$

Topological influence

It turns out that the reason behind the Dirac strings is **topological**. It can be seen by using the homotopy equivalence $\mathbb{R}^3 \setminus \{0\} \simeq S^2$ and switching to a coordinante free description: One can write the vector potential $\mathbf{A}$ as a 1-form and the magnetic field $\mathbf{B}$ as a 2-form (both independent from the distance r in spherical coordinates):[2]

$$A^{\pm} = \mathbf{A}_x^{\pm}\mathrm{d}x + \mathbf{A}_y^{\pm}\mathrm{d}y + \mathbf{A}_z^{\pm}\mathrm{d}z = g\left(-\cos(\vartheta) \pm 1\right)\mathrm{d}\varphi; \tag{13}$$

$$B = \mathbf{B}_x\mathrm{d}y \wedge \mathrm{d}z + \mathbf{B}_y\mathrm{d}z \wedge \mathrm{d}x + \mathbf{B}_z\mathrm{d}x \wedge \mathrm{d}y = g\sin(\vartheta)\mathrm{d}\vartheta \wedge \mathrm{d}\varphi \tag{14}$$

Maxwell's equations simply become:

$$\mathrm{d}B = 0. \tag{15}$$

[2] Although the vector potential no longer contains a division by $\sin(\vartheta)$, the divergence still remains in $\mathrm{d}\varphi$, which is undefined on the Dirac strings.

de Rham cohomology

Hence there is a connection with *de Rham cohomology*:

$$[B] \in H^2_{\mathrm{dR}}(\mathbb{R}^3 \setminus \{0\}) \cong H^2_{\mathrm{dR}}(S^2) \cong \mathbb{R}. \tag{16}$$

An explicit isomorphism is then given by integration with:

$$4\pi g = \int_{S^2} B \mathrm{dvol}, \tag{17}$$

which shows that a *global* vector potential $A \in \Omega^1(S^2)$ with $B = \mathrm{d}A$ exists if and only if $g = 0$. Otherwise only *local* vector potentials exist.

Topological quantum numbers

Continuous view (in $\mathbb{R}$):

$$\text{closed differential form/de Rham cohomology class} \\ \leftrightsquigarrow \text{magnetic charge}$$

Discrete view (in $\mathbb{Z}$):

$$\text{lifted de Rham/integral cohomology class} \\ \leftrightsquigarrow \text{quantized magnetic charge}$$

Due to this connection to physics, characteristic classes, which can be expressed by integrals over certain differential forms according to Chern–Weil theory, are also called *topological quantum numbers*.

Topology in physics

A similar relation between topology, electrodynamics and quantum mechanics is demonstrated by the *Aharanov–Bohm effect*: A magnetic field, which exists **only inside** an infinite long cylinder, still influences electrons **outside of it**.

In topology, it comes down to $\mathbb{R}^3 \setminus (\{(0,0)\} \times \mathbb{R}) \simeq S^1$. In electrodynamics and quantum mechanics, it comes down to $B = 0$ but $A \neq 0$ (which is what enters the Schrödinger equation) outside the cylinder. (In other words: A is not just theoretical help, but can be observed.)

Similar topological effects are of fundamental importance in physics, for example for topological insulators or topological phase transitions (Nobel Prize in Physics 2016).

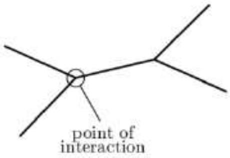
Principal bundles

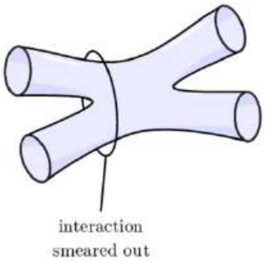
A question to motivate *principal bundles* is:

Are there surjective maps $P_n \twoheadrightarrow S^2$ for all quantum numbers $n \in \mathbb{Z}$, so that the local vector potentials $\mathbf{A}^{\pm}$ for the magnetic monopole with charge $g = n\Phi_0$ on S^2 can be pulled back from a global vector potential on P_n along local sections?

Topological quantum field theory

A very general idea here is that moving to larger spaces and dimensions often leads to common ground. For example, the interaction of particles is not described by a smooth manifold due to vertices, while the interaction of strings is:





Clutching construction

One can obtain the gauge transformation:

$$\mathbf{A}^+(\vartheta) = \mathbf{A}^-(\vartheta) + 2n\Phi_0, \tag{18}$$

which is formulated along the equator $S^1 \subset S^2$, from the clutching construction for the vector bundle

$$(\gamma_{\mathbb{C}}^{1,1})^{\otimes n} \twoheadrightarrow S^2 \tag{19}$$

with the tautological vector bundle $\gamma_{\mathbb{C}}^{1,1} \twoheadrightarrow \mathbb{C}P^1 \cong S^2$.
($n \cdot - \colon \mathfrak{u}(1) \to \mathfrak{u}(1)$ lifts to $-^n\colon\ \mathrm{U}(1) \to \mathrm{U}(1)$.) Now their frame bundles $P_n = \mathrm{Fr}_{\mathrm{U}}((\gamma_{\mathbb{C}}^{1,1})^{\otimes n}) \twoheadrightarrow S^2$ (by only keeping the electromagnetic gauge group $\mathrm{U}(1) \subset \mathbb{C}$ in every vector fiber) are the required spaces.

$n = 0$ yields $S^2 \times \mathrm{U}(1)$ and $n = 1$ yields S^3 with the complex Hopf fibration.

Principal bundles

For a topological group G, a fiber bundle $\pi\colon P \to M$ with a right group action $P \times G \to P$, which acts free and transitive on each fiber $P_x := \pi^{-1}(x)$ for $x \in M$, is a *principal G-bundle*. P is called *total space* and M is called *base space*.

If G is a Lie group and if P and M are smooth manifolds, then the principal bundle is also called *smooth*.

Structure of fibers

Important note:

The fibers of a principal G-bundle are not G, but G-torsors ("group without neutral element"). There is no canonical isomorphism $P_x \cong G$, unless a point p is chosen.

Principal connections

Principal connections are the desired differential forms on the total space from which the local vector potentials can be pulled back. A purely mathematical question to motivate principal connections is:

At a point $p \in P$, can the tangent space T_pP be split into vector subspaces for the fiber direction of G and parametrization direction of M?

Fiber direction of G

Due to the projection and the group action, understanding the fiber direction of G is very simple: There is a *vertical vector (sub)bundle*:

$$VP := \ker(\mathrm{d}\pi \colon TP \to TM) \subset TP. \tag{20}$$

Even better, there is a canonical vector space isomorphism:

$$\mathfrak{g} \xrightarrow{\cong} V_p P, \xi \mapsto \xi_p^\# := \frac{\mathrm{d}}{\mathrm{d}t}\big(p \cdot \exp(\xi t)\big)|_{t=0} \tag{21}$$

with the *fundamental vector field* $-^\# \colon \mathfrak{g} \to \mathfrak{X}(P)$. It is a Lie algebra homomorphism, meaning $[\xi, \zeta]^\# = [\xi^\#, \zeta^\#]$ for all $\xi, \zeta \in \mathfrak{g}$.

One has:

$$V_{pg} P = \mathrm{d}_p R_g (V_p P) \tag{22}$$

for all $p \in P$ and $g \in G$.

Parametrization direction of M

Due to no other structure than the topology, understanding the parametrization direction of M requires a choice: For example very straight forward a *horizontal vector (sub)bundle* $HP \subset TP$ with:

$$H_{pg}P = \mathrm{d}_p R_g(H_p P) \tag{23}$$

for all $p \in P$ and $g \in G$ (to be compatible with the principal bundle) and:

$$VP \oplus HP \cong TP. \tag{24}$$

Due to the second property, the decomposition in a Whitney sum for fiber and parametrization direction, this *Ehresmann connection* can be seen as connecting nearby fibers.

Horizontal lift

Since an Ehresmann connection $HP \subset TP$ gives P the parametrization direction of M, it becomes possible to relate vector fields on them.

For a vector field $X \in \mathfrak{X}(M)$, its *horizontal lift* $\widetilde{X} \in \Gamma^\infty(P, H) \subset \mathfrak{X}(P)$ is the vector field with:

$$\widetilde{X}_{pg} = \mathrm{d}_p R_g(\widetilde{X}_p) \tag{25}$$

for all $p \in P$ and $g \in G$ (to be compatible with the principal bundle) and:

$$\mathrm{d}\pi \circ \widetilde{X} = X \circ \phi. \tag{26}$$

Parametrization of M

An equivalent way to view an Ehresmann connection is as a differential form $A \in \Omega^1(P, \mathfrak{g})$ with:

$$\mathrm{Ad}_{g^{-1}} A_{pg} \mathrm{d}_p R_g(v) = A_p(v) \tag{27}$$

for all $p \in P$, $v \in T_p P$ and $g \in G$ (to be compatible with the principal bundle) and:

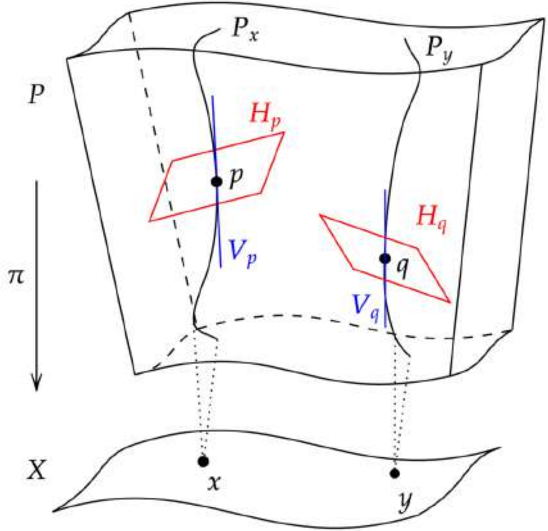
$$A_p(\xi_p^{\#}) = \xi \tag{28}$$

for all $p \in P$ and $\xi \in \mathfrak{g}$. Due to the second property, the entire image already being hit by fiber directions, this *principal connection* establishes the correspondence:

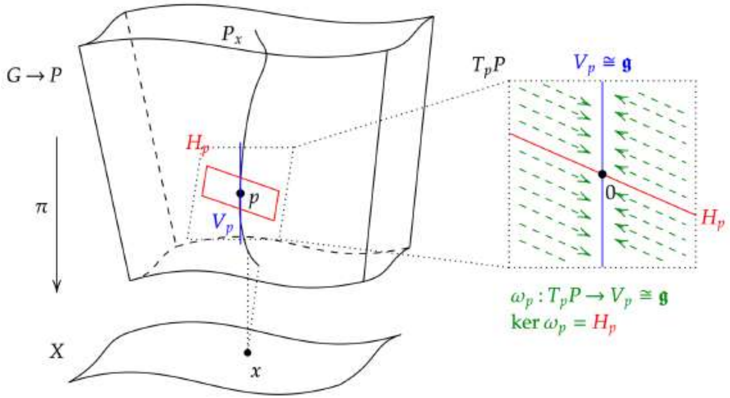
$$H_p P = \ker(A_p). \tag{29}$$

The space of connections is denoted $\Omega^1_{\mathrm{conn}}(P, \mathfrak{g})$.

Principal connections as horizontal subspace



Principal connections as vertical projections



Vector bundle valued differential forms

Describing principal connections requires differential forms with values not in the vector space $\mathbb{R}$ but the Lie algebra $\mathfrak{g}$. More generally, differential forms can also be considered with values in vector bundles and Lie algebra bundles.

Let $E \twoheadrightarrow M$ be a vector bundle and let:

$$\Omega^k(M, E) := \Gamma^\infty(\Lambda^k T^* M \otimes E). \tag{30}$$

A differential form $\omega \in \Omega^k(M, E)$ can be represented as:

$$\omega = \sum_{i \in I} \omega_i \otimes s_i \tag{31}$$

for ordinary differential forms $\omega_i \in \Omega^k(M)$ and smooth sections $s_i \in \Gamma^\infty(E)$. (Such a representation might not be unique since it might be impossible to find a family of smooth sections, which form a basis of every fiber. Stiefel–Whitney classes measure the failure of this property.)

Vector bundle connections

A question to motivate *vector bundle connections* is:

Can one define a differential $\mathrm{d}_\nabla \colon \Omega^k(M, E) \to \Omega^{k+1}(M, E)$, called *covariant derivative*, analogous (not necessarily similar) to the Cartan differential?

Vector bundle connections

A first attempt is just taking the Cartan differential itself:

$$\mathrm{d}\omega = \sum_{i \in I} \mathrm{d}\omega_i \otimes s_i. \tag{32}$$

But this expression is not well-defined due to the missing uniqueness of the representation. For a scalar function $f \in C^\infty(M)$, one has $(f\omega_i) \otimes s_i = \omega_i \otimes (fs_i)$, which conflicts the above definition. (f is derived in the left expression, but not the right expression.)

Vector bundle connections

It is therefore necessary to derive sections, which reduces the problem to $k = 0$ and a map $\nabla \colon \Gamma^\infty(E) \cong \Omega^0(M, E) \to \Omega^1(M, E)$, so that:

$$\mathrm{d}_\nabla \omega = \sum_{i \in I} \mathrm{d}\omega_i \otimes s_i + \omega_i \wedge \nabla s_i. \tag{33}$$

Usually, one writes $\nabla \colon \mathfrak{X}(M) \times \Gamma^\infty(E) \to \Gamma^\infty(E)$ with $\nabla_X s = (\nabla s)(X)$ for all smooth sections $s \in \Gamma^\infty(E)$ and vector fields $X \in \mathfrak{X}(M)$. For a scalar function $f \in C^\infty(M)$, one can now compensate the previous issue by requiring a similar product rule:

$$\nabla_X(fs) = (\mathcal{L}_X f)s + f \nabla_X s. \tag{34}$$

Furthermore, $\nabla_X s$ is $C^\infty(M)$-linear in X and $\mathbb{R}$-linear in s.

The space of connections is denoted $\mathcal{A}$.

Space of vector bundle connections

Important note:

The space $\mathcal{A}$ of connections on a vector bundle is an **affine** vector space, modelled over $\Omega^1(M, \mathrm{End}(E))$. There is no canonical isomorphism, unless a connection ∇ is chosen.

Wedge products

Gauge theory also requires wedge products, which need further structures on the vector bundle E to compute coefficients.

If E has a fiberwise scalar product (like the Killing form of a Lie algebra), then there is a wedge product:

$$\mathrm{tr}(- \wedge -) \colon \Omega^k(M, E) \times \Omega^l(M, E) \to \Omega^{k+l}(M). \tag{35}$$

If E has a fiberwise Lie bracket, then there is a wedge product:

$$[- \wedge -] \colon \Omega^k(M, E) \times \Omega^l(M, E) \to \Omega^{k+l}(M, E). \tag{36}$$

Adjoint bundle

In fact, a Lie algebra bundle arises naturally from a principal bundle: Similar to how the difference of two vector bundle connections in $\mathcal{A}$ lies in $\Omega^1(M, \mathrm{End}(E))$, where does the difference D of two principal connections (like $\mathbf{A}^+ - \mathbf{A}^- = 2n\Phi_0$ for the magnetic monopole) in $\Omega^1_{\mathrm{conn}}(P, \mathfrak{g})$ lie in?

Pulling back along local sections $s_i \colon M \supset U_i \hookrightarrow P$ for a covering $(U_i)_{i \in I} \subset M$ leads to local differential forms $s_i^* D \in \Omega^1(U_i, \mathfrak{g})$ glueing together non-trivial (not necessarily resulting in a differential form in $\Omega^1(M, \mathfrak{g})$). But the twisting of the principal bundle P can be transfered over to the Lie algebra $\mathfrak{g}$ using the action of the Lie group G on both.

Adjoint bundle

For a Lie group G with Lie algebra $\mathfrak{g}$ and a smooth principal G-bundle $P \twoheadrightarrow M$, its *adjoint (vector) bundle* is:

$$\mathrm{Ad}(P) := (P \times \mathfrak{g})/\sim \tag{37}$$

with $[(pg, \xi)] \sim [(p, \mathrm{Ad}_g(\xi))]$ for all $p \in P$, $g \in G$ and $\xi \in \mathfrak{g}$. It uses the *adjoint representation*:

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g}), \mathrm{Ad}_g(\xi) := \frac{\mathrm{d}}{\mathrm{d}t}\left(g \exp(\xi t) g^{-1}\right)|_{t=0} \tag{38}$$

Sections of the adjoint bundle can alternatively be described by smooth functions:

$$\begin{aligned} &\Gamma^\infty(\mathrm{Ad}(P)) \\ \cong &\{\tilde{s} \in C^\infty(P, \mathfrak{g}) | \forall p \in P \forall g \in G : \tilde{s}(eg) = \mathrm{Ad}_{g^{-1}}(\tilde{s}(e))\}. \end{aligned} \tag{39}$$

with $s(p) = [(p, \tilde{s}(p))]$ for all $s \in \Gamma^\infty(\mathrm{Ad}(P))$ and $p \in P$.

Relations between vector bundles

$\mathsf{Ad}(P)$ and TM are both vector bundles over M while VP and HP are both vector bundles over P. One has the following relations:

$$\mathrm{d}\pi^* \,\mathsf{Ad}(P) \cong VP; \tag{40}$$

$$\mathrm{d}\pi^* \,TM \cong HP. \tag{41}$$

Space of principal connections

Important note:

The space $\Omega^1_{\mathrm{conn}}(P, \mathfrak{g})$ of principal connections is an **affine** vector space ("vector space without zero"), modelled over $\Omega^1(M, \mathrm{Ad}(P))$. There is no canonical isomorphism, unless a connection A is chosen.

Connections between connections

With the adjoint bundle, vector bundle connections can also be considered for principal bundles with the important result:

Principal connections on P correspond bijectively to Lie bracket compatible vector bundle connections on $\mathrm{Ad}(P)$.

A connection $\nabla \colon \mathfrak{X}(M) \times \Gamma^\infty(\mathrm{Ad}(P)) \to \Gamma^\infty(\mathrm{Ad}(P))$ preserves the Lie bracket if and only if:

$$\nabla_X[s, t] = [\nabla_X s, t] + [s, \nabla_X t] \tag{42}$$

for all sections $s, t \in \Gamma^\infty(\mathrm{Ad}(P))$.

Connections between connections

For a principal connection $A \in \Omega^1_{\mathrm{conn}}(P, \mathfrak{g})$, its corresponding Lie bracket compatible vector bundle connection is denoted $\nabla^A \in \mathcal{A}$ and its covariant derivative is denoted $\mathrm{d}_A := \mathrm{d}_{\nabla^A}$. One has:

$$\left(\nabla^A_X s\right)(x) := [(p, \mathrm{d}_p \widetilde{s}(\widetilde{X}_p))] \in \mathrm{Ad}(P) \tag{43}$$

$$\widetilde{\nabla^A_X} s(p) := \mathrm{d}_p \widetilde{s}(\widetilde{X}_p) \in \mathfrak{g} \tag{44}$$

for all $X \in \mathfrak{X}(M)$, $s \in \Gamma^\infty(\mathrm{Ad}(P))$, $p \in P$ and $x \in M$ with $\pi(p) = x$.

Curvature form

For a principal connection $A \in \Omega^1_{\mathrm{conn}}(P, \mathfrak{g})$, its covariant derivative d_A can **not** be used to define a de Rham cohomology since $\mathrm{d}_A^2 \neq 0$ in general. Instead there exists a *curvature form* $F_A \in \Omega^2(M, \mathrm{Ad}(P))$ with:

$$\mathrm{d}_A^2 s = [F_A, s] \tag{45}$$

for all $s \in \Gamma^\infty(\mathrm{Ad}(P))$. It fulfills the *Bianchi identity*:

$$\mathrm{d}_A F_A = 0. \tag{46}$$

Pulling back along local sections $s_i \colon M \supset U_i \hookrightarrow P$ for a covering $(U_i)_{i \in I} \subset M$ leads to:

$$F_A|_{U_i} = \mathrm{d}(s_i^* A) + [s_i^* A, s_i^* A]. \tag{47}$$

(Compare with the curvature $R = \partial\Gamma + \Gamma\Gamma$ in Riemannian geometry.)

Interactions and gauge groups

In mathematics, this makes abelian gauge groups cause linear curvature equations and non-abelian gauge groups cause non-linear curvature equations.

In physics, the first term – differential and linear – describes a particle's own field while the second term – algebraic and quadratic – describes the particle's self-interaction.

Photons, which carry the electromagnetic interaction but no electric charge themselves, have abelian gauge group U(1). Gluons, which carry the strong interaction and have color charge themselves, have non-abelian gauge group SU(3).

Yang–Mills equations

A connection A is called *flat* if and only if $F_A = 0$. Since global topological information can often be computed from local geometric information, in the case of characteristic classes and curvature forms described by Chern–Weil theory, flat connections don't have to exist.

A fitting alternative are connections, which at least locally minimize the curvature form, called *Yang–Mills connections*. These arise as critical points of the *Yang–Mills action*:

$$\begin{aligned} S_{\mathrm{YM}} \colon \Omega^1_{\mathrm{conn}}(P, \mathfrak{g}) \to \mathbb{R}, S_{\mathrm{YM}}(A) &= \int_M \|F_A\|^2 \mathrm{dvol}_g \\ &= \int_M \mathrm{tr}(F_A \wedge \star F_A) \mathrm{dvol}_g \qquad (48) \end{aligned}$$

and are described by the *Yang–Mills equations*:

$$\mathrm{d}_A \star F_A = 0. \qquad (49)$$

Yang–Mills equations

Electromagnetism without magnetic monopoles uses $P \cong M \times \mathrm{U}(1)$. For the electromagnetic field strength $F \in \Omega^2(M)$ (with $F = B$ without electric field), Maxwell's equations are special cases of the Bianchi identity and the Yang–Mills equations:

$$\mathrm{d}F = 0; \tag{50}$$

$$\mathrm{d} \star F = 0. \tag{51}$$

Own research

For principal bundles $P \twoheadrightarrow M$ with a vector bundle isomorphism $a\colon TM \xrightarrow{\cong} \mathrm{Ad}(P)$, hence in particular $a \in \Omega^1(M, \mathrm{Ad}(P))$, what are the consequences for gauge theory?

In mathematics, one can ask: When taking the Killing form on $\mathfrak{g}$, when is $g_a(X, Y) = \langle a(X), a(Y) \rangle$ a Riemannian metric? How do metric-compatible or torsion-free connections ∇ on TM as well as Lie bracket compatible, Killing form compatible or Yang–Mills connections on $\mathrm{Ad}(P)$ relate to each other?

In physics, one can ask: What are concrete implications for the weak interaction with gauge group $\mathrm{SU}(2)$ over a spin 3 spacetime and the strong interaction with gauge group $\mathrm{SU}(3)$ over an eightdimensional spacetime?

Important works

- Paul Dirac, *Quantized Singularities in the Electromagnetic Field* (1931)
- Chen Ning Yang, Robert Mills: *Conservation of Isotopic Spin and Isotopic Gauge Invariance*
- Tai Tsun Wu and Chen Ning Yang: *Properties of Matter Under Unusual Conditions* (1968)
- Tai Tsun Wu, Chen Ning Yang: *Concept of nonintegrable phase factors and global formulation of gauge fields* (1975)