

Algebraic Topology and Operators in Hilbert Space

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Introduction.

In recent years considerable progress has been made in the global theory of elliptic equations. This has been essentially of a topological character and it has brought to light some very interesting connections between the topology and the analysis. I want to talk here about a simple but significant aspect of this connection, namely the relation between the index of Fredholm operators and the theory of vector bundles.

§1. Fredholm operators.

Let H be a separable complex Hilbert space. A bounded linear operator

$$F : H \rightarrow H$$

is called a Fredholm operator if its kernel and cokernel are both finite-dimensional. In other words the homogeneous equation $Fu = 0$ has a finite number of linearly independent solutions, and the inhomogeneous equation $Fu = v$ can be solved provided v satisfies a finite number of linear conditions. Such operators occur frequently in various branches of analysis particularly in connection with elliptic problems. A useful criterion characterizing Fredholm operators is the following

PROPOSITION 1.1. F is a Fredholm operator if and only if it is invertible modulo compact operators, i. e. $\exists G$ with $FG-1$ and $GF-1$ both compact.

This proposition is essentially a reformulation of the classical result of Fredholm that, if K is compact, $1+K$ is a Fredholm operator. In the theory of elliptic differential equations the approximate inverse G

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is called a "parametrix."

To reformulate (1.1) in geometrical terms let \mathcal{A} denote the Banach algebra of all bounded operators on H . The compact operators form a closed 2-sided ideal \mathcal{K} of \mathcal{A} so that $\mathcal{B} = \mathcal{A}/\mathcal{K}$ is again a Banach algebra. Let \mathcal{B}^* denote the group of invertible elements in \mathcal{B} . Then (1.1) asserts that the space \mathcal{F} of Fredholm operators is just the inverse image $\pi^{-1}(\mathcal{B}^*)$ under the natural map $\pi : \mathcal{A} \rightarrow \mathcal{B}$. In particular this shows that \mathcal{F} is an open set in \mathcal{A} , closed under composition, taking adjoints and adding compact operators.

The index of a Fredholm operator F is defined by

$$\text{index } F = \dim \text{Ker } F - \dim \text{Coker } F .$$

Remark. A simple application of the closed-graph theorem (see Lemma (2.1)) shows that a Fredholm operator has a closed range. Thus

$$\text{Coker } F = H/F(H) \cong \text{Ker } F^*$$

where F^* is the adjoint of F . Hence

$$\text{index } F = \dim \text{Ker } F - \dim \text{Ker } F^* .$$

As we have observed the composition $F' \circ F$ of two Fredholm operators is again Fredholm. A simple algebraic argument shows at once that

$$\text{index } F' \circ F = \text{index } F' + \text{index } F .$$

Suppose now that $\{e_0, e_1, \dots\}$ is an orthonormal base for H and let H_n denote the closed subspace spanned by the e_i with $i \geq n$. The orthogonal projection P_n onto H_n is then Fredholm and, since it is self-adjoint, it has index zero. Thus $F_n = P_n \circ F$ is Fredholm and has the same index as F . Now choose n so that e_0, e_1, \dots, e_{n-1} and $F(H)$

span H -- which is possible because $\dim H/F(H)$ is finite. Then $F_n(H) = H_n$ and so $\text{Ker } F_n^* = H_n^\perp$. Thus

$$(1.2) \quad \text{index } F = \text{index } F_n = \dim \text{Ker } F_n - n .$$

This is a convenient way of calculating the index because we have fixed the dimension of the cokernel to be n , and so we have only one unknown dimension to compute, namely $\dim \text{Ker } F_n$.

§2. Fredholm families.

We want now to investigate families of Fredholm operators depending continuously on some parameter. Formally this means that we have a topological space X (the space of parameters) and a continuous map

$$F : X \longrightarrow \mathcal{F}$$

where \mathcal{F} is topologized as a subspace of the bounded operators \mathcal{A} (with the norm topology). Thus for each $x \in X$ we have a Fredholm operator $F(x)$ and

$$\|F(x) - F(x_0)\| < \varepsilon$$

for x sufficiently close to x_0 .

We now ask how the vector spaces $\text{Ker } F(x)$ and $\text{Ker } F(x)^*$ vary with x . It is easy to see that their dimensions are not continuous (i. e. locally constant) functions of x : they are only semi-continuous, that is

$$\dim \text{Ker } F(x_0) \geq \dim \text{Ker } F(x)$$

for all x sufficiently close to x_0 . On the other hand, as we shall see below in (2.1), $\text{index } F(x)$ is locally constant. Thus, although $\dim \text{Ker } F(x)$ and $\dim \text{Ker } F(x)^*$ can jump as $x \longrightarrow x_0$, they always jump by the same amount so that their difference remains constant. This invariance of the index under perturbation is its most significant property: it brings it into

the realm of algebraic topology.

To go beyond questions of dimension it will be convenient to introduce the operators

$$F_n(x) = P_n \circ F(x)$$

defined as in §1. For large n this has the effect of fixing the dimension of the spaces $\text{Ker } F_n(x)$. This is a necessary preliminary if we want these spaces to vary reasonably with x . More precisely we have the following result on the local continuity properties of these kernels:

LEMMA (2.1). Let $F : X \rightarrow \mathcal{F}$ be a continuous family of Fredholm operators and let $x_0 \in X$. Choose n so that $F_n(x_0)(H) = H_n$. Then there exists a neighborhood U of x_0 so that, for all $x \in U$,

$$F_n(x)(H) = H_n .$$

Moreover $\dim \text{Ker } F_n(x)$ is then a constant d (for $x \in U$) and we can find d continuous functions

$$s_i : X \rightarrow H$$

such that, for $x \in U$, $s_1(x), \dots, s_d(x)$ is a basis of $\text{Ker } F_n(x)$.

The proof of the lemma is shorter than its enunciation. Let $S = \text{Ker } F_n(x_0)$ and define an operator

$$G(x) : H \rightarrow H_n \oplus S$$

by $G(x)u = (F_n(x)u, P_S u)$ where P_S is the projection on S . Clearly $G(x_0)$ is an isomorphism.* Since $G(x)$ is continuous in x it follows that† $G(x)$ is an isomorphism for all x in some neighborhood U of x_0 . This proves the lemma. In fact we clearly have $F_n(x)(H) = H_n$ for $x \in U$ and, if e_1, \dots, e_d is a basis of S ,

* Algebraically and hence, by the closed-graph theorem, topologically.

† Modulo an identification of $H_n \oplus S$ with H this is just the assertion that the invertible elements in the Banach algebra \mathcal{A} form an open set.

$$s_i(x) = G(x)^{-1} e_i \quad i \leq i \leq d$$

give a basis for $\text{Ker } F_n(x)$.

In view of (1.2) Lemma (2.1) shows at once that $\text{index } F(x) = d-n$ is a locally constant function of x . However it gives more information which we proceed to exploit. Suppose now that our parameter space X is compact. The first part of (2.1), combined with the compactness of X , implies that we can find an integer n so that

$$F_n(x)H = H_n \quad \text{for all } x \in X .$$

Choose such an integer and consider the family of vector spaces $S(x) = \text{Ker } F_n(x)$ for $x \in X$. If we topologize $S = \bigcup_{x \in X} S(x)$ in the natural way as a subspace of $X \times H$ the last part of (2.1) implies that S is a locally trivial family of d -dimensional vector spaces, i. e. in a neighborhood of any point x_0 , we can find d continuous maps $s_i : X \rightarrow S$ such that $s_1(x), \dots, s_d(x)$ lie in $S(x)$ and form a basis.

A locally trivial family of vector spaces parametrized by X is called a vector bundle over X . Thus a vector bundle consists of a topological space S mapped continuously by a map π onto X so that each 'fibre' $S(x) = \pi^{-1}(x)$ has a vector space structure and locally we can find continuous bases $s_1(x), \dots, s_d(x)$ as above.

There is[†] a natural notion of isomorphism for vector bundles over X . Let us denote by $\text{Vect}(X)$ the set of isomorphism classes of all vector bundles over X . If X is a point a vector bundle over X is just a single vector space and $\text{Vect}(X)$ is the set of non-negative integers \mathbb{Z}^+ . For general spaces X however one can give simple examples of non-trivial vector bundles so that the isomorphism class of a vector bundle is not determined by its dimension.

Given two vector bundles S, T over the same space X one can form their direct sum $S \oplus T$. This is again a vector bundle over X and

[†] For definitions and elementary properties of vector bundles see [1, Chapter I].

the fibre of $S \oplus T$ at x is just $S(x) \oplus T(x)$. This induces an abelian semi-group structure on $\text{Vect}(X)$, generalizing the semi-group structure on \mathbb{Z}^+ .

Let us return now to the consequences of (2.1). We have seen that, given the Fredholm family

$$F : X \longrightarrow \mathcal{F}$$

(with X compact), we can choose an integer n so that $\bigcup_{x \in X} \text{Ker } F_n(x)$ is a vector bundle over X . We denote this vector bundle by $\text{Ker } F_n$. Moreover $\text{Ker } F_n^*(x) = H_n^+$ for all x so that $\text{Ker } F_n^*$ is the trivial bundle $X \times H_n^+$. In view of formula (1.2) for the index of a single Fredholm operator it is now rather natural to try to define a more general notion of index for a Fredholm family F by putting

$$(2.2) \quad \begin{aligned} \text{index } F &= [\text{Ker } F_n] - [\text{Ker } F_n^*] \\ &= [\text{Ker } F_n] - [X \times H_n^+] \end{aligned}$$

where $[\]$ denotes the isomorphism class in $\text{Vect}(X)$. Unfortunately this does not quite make sense because $\text{Vect}(X)$, like \mathbb{Z}^+ , is only a semi-group and not a group so that subtraction is not admissible. However the way out is fairly clear: we must generalize the construction of passing from the semi-group \mathbb{Z}^+ to the group \mathbb{Z} and associate to the semi-group $\text{Vect}(X)$ an abelian group $K(X)$. There is in fact a routine construction which starts from an abelian semi-group A and produces an abelian group B . One way is to define B to consist of all pairs (a_1, a_2) with $a_i \in A$ modulo the equivalence relation generated by

$$(a_1, a_2) \sim (a_1 + a, a_2 + a) \quad a \in A$$

(we think of (a_1, a_2) as the difference $a_1 - a_2$). The only point to watch is that the natural map $A \longrightarrow B$ need not be injective.

Having introduced our group $K(X)$ formula (2.2) now makes sense and defines the index of a Fredholm family as an element of $K(X)$. In this definition we had to choose a sufficiently large integer n . However if we replace n by $n+1$ it is easy to see that

$$\begin{aligned} \text{Ker } F_{n+1} &\cong \text{Ker } F_n \oplus E_n \\ \text{Ker } F_{n+1}^* &\cong \text{Ker } F_n^* \oplus E_n \end{aligned}$$

where E_n is the trivial 1-dimensional vector bundle generated by the extra basis vector e_n . Thus

$$\text{index } F \in K(X)$$

is well-defined independent of the choice of n .

Remarks. 1. Our definition of index F was dependent on a fixed orthonormal basis. It is a simple matter, which we leave to the reader, to show that the choice of basis is irrelevant.

2. Our introduction of $K(X)$ here was motivated by the requirement of finding a natural "value group" for the index of Fredholm families. Historically the motivation for $K(X)$ was somewhat different and arose from the work of Grothendieck in algebraic geometry. For this reason $K(X)$ is referred to as the Grothendieck group of vector bundles over X .

Although our definition of the index as an element of $K(X)$ may seem plausible, it is not clear at first how trivial or non-trivial it is. To show that it really is significant I will mention the following result.

THEOREM (2.3). Let X be a compact space and let $[X, \mathcal{F}]$ denote the set of homotopy classes of continuous maps $F : X \rightarrow \mathcal{F}$. Then $F \mapsto \text{index } F$ induces an isomorphism

$$\text{index} : [X, \mathcal{F}] \rightarrow K(X) .$$

This theorem shows that our index is the only deformation invariant of a

Fredholm family. For example if X is a point the theorem asserts that the connected components of \mathcal{F} correspond bijectively to the integers, the correspondence being given by the ordinary (integer) index. The proof of (2.3) is not deep. It can be reduced quite easily (see [1; Appendix (A6)]) to Kuiper's theorem that the group \mathcal{Q}^* of invertible operators on H is contractible. The proof of Kuiper's theorem, though ingenious, involves only elementary properties of Hilbert space.

§3. K-Theory.

Having introduced the group $K(X)$ I shall now review briefly its elementary properties. In the first place we observe that one can form the tensor product $S \otimes T$ of two vector bundles and this induces a ring structure on $K(X)$. Thus $K(X)$ becomes a commutative ring with an identity element 1 -- corresponding to the trivial bundle $X \times \mathbb{C}^1$. When X is a point this is of course the usual ring structure of the integers \mathbb{Z} .

Next we investigate the behavior of K under change of parameter space. If $f : Y \rightarrow X$ is a continuous map and if S is a vector bundle over X we obtain an induced vector bundle f^*S over Y , the fibre $f^*S(y)$ being $S(f(y))$. Passing to K we get a homomorphism of rings

$$f^* : K(X) \rightarrow K(Y) .$$

Thus $K(X)$ is a contravariant functor of X . In this and other important respects $K(X)$ closely resembles the cohomology ring $H(X)$.

Finally we have the homotopy invariance of $K(X)$ which asserts that $f^* : K(X) \rightarrow K(Y)$ depends only on the homotopy class of the map $f : Y \rightarrow X$. This follows from the fact that, in a continuous family of vector bundles over X , the isomorphism class is locally constant. For a simple proof of this see [1, (1.4.3)].

It is convenient to extend the definition of $K(X)$ to locally compact spaces X by putting

$$K(X) = \text{Ker}\{K(X^+) \xrightarrow{i^*} K(+)\}$$

where X^+ is the one-point compactification of X and $i : + \rightarrow X^+$ is the inclusion of this "one-point." If X is not compact then $K(X)$ is a ring without identity: it is functorial for proper maps.

In addition to the ring structure in $K(X)$ it is convenient also to consider "external products." First of all, when X, Y are compact, we have a homomorphism

$$K(X) \otimes K(Y) \rightarrow K(X \times Y)$$

obtained by assigning to a vector bundle E over X and a vector bundle F over Y the vector bundle $E \boxtimes F$ over $X \times Y$ whose fibre at a point (x, y) is $E_x \otimes F_y$. This extends at once to locally compact X, Y in view of the fact that we have an exact sequence

$$(3.1) \quad 0 \rightarrow K(X \times Y) \rightarrow K(X^+ \times Y^+) \rightarrow K(X^+) \otimes K(Y^+)$$

which identifies $K(X \times Y)$ with the subgroup of $K(X^+ \times Y^+)$ vanishing on the two "axes" X^+, Y^+ . For a proof of (3.1) see [1, (2.48)].

If $x \in K(X), y \in K(Y)$ the image of $x \otimes y$ in $K(X \times Y)$ is called their external product and written simply as xy .

As an important example of a locally compact space X we can take the Euclidean space \mathbb{R}^n , so that X^+ is the sphere S^n . Then we have

$$K(S^n) \cong K(\mathbb{R}^n) \oplus \mathbb{Z}$$

the \mathbb{Z} summand being given by the dimension of a vector bundle. Thus $K(\mathbb{R}^n)$ is really the interesting part of $K(S^n)$. More generally for any X , (3.1) implies a decomposition

$$(3.2) \quad K(S^n \times X) \cong K(\mathbb{R}^n \times X) \oplus K(X) .$$

The fundamental result of K-theory is the Bott periodicity theorem which asserts that the groups $K(\mathbb{R}^n)$ are periodic in n with period 2, so that

$$K(\mathbb{R}^{2m}) \cong K(\text{point}) = \mathbb{Z}$$

$$K(\mathbb{R}^{2m+1}) \cong K(\mathbb{R}^1) = 0$$

(the last equality depends on the fact that all vector bundles over the circle S^1 are trivial). This periodicity follows by induction from the more general result:

THEOREM 3.3. For any locally compact space X we have a natural isomorphism

$$K(\mathbb{R}^2 \times X) \cong K(X) .$$

In §2 we explained the significant connection between the functor K and the space \mathcal{F} of Fredholm operators. In view of this it is rather natural to raise the following two questions:

- a) Can we use the index to prove the periodicity theorem (3.3)?
- b) Can we use (3.3) to help study and compute indices?

The answer to both questions is affirmative. The first is remarkably simple and I will explain it in detail. The second is considerably deeper and I shall not comment on it further. What I have in mind will be found in [4]. A general discussion of the relationship between questions (a) and (b) is given in [3].

To carry out the program answering question (a) I will show, in the next section, how to define a homomorphism

$$\alpha : K(\mathbb{R}^2 \times X) \longrightarrow K(X)$$

using the index of certain simple Fredholm families. Once α has been constructed it is not difficult to show that it is an isomorphism, thus proving Theorem (3.3). This will be done in §5.

§4. The Wiener-Hopf operator.

I will recall here the discrete analogue of the famous Wiener-Hopf equation.†

Let $f(z)$ be a continuous complex-valued function on the unit circle $|z| = 1$. Let H be the Hilbert space of square-integrable functions on the circle $|z| = 1$ and let H_n be the closed subspace spanned by the functions z^k with $k \geq n$. In particular H_0 is the space of functions $u(z)$ with Fourier series of the form

$$u(z) = \sum_{k \geq 0} u_k z^k .$$

Let P denote the projection $H \rightarrow H_0$ and consider the operator

$$F = Pf : H_0 \rightarrow H_0$$

where f here stands for multiplication by f . Thus the Fourier coefficients $(Fu)_n$ ($n \geq 0$) for $u \in H_0$ are given by

$$(4.1) \quad (Fu)_n = \sum_{k \geq 0} f_{n-k} u_k \quad (n \geq 0)$$

where f_m are the Fourier coefficients of f . The classical Wiener-Hopf equation is the integral counterpart of the discrete convolution equation (4.1), the summation being replaced by $\int_0^\infty \hat{f}(x-y)\hat{u}(y)dy$.

It is clear from the definition of F that

$$\|F\| \leq \sup |f(z)|$$

so that F depends continuously on f (for the norm topology of F and the sup norm topology of f). The basic result about these operators F is

PROPOSITION (4.2). If $f(z)$ is nowhere zero (on $|z| = 1$) then $F = Pf : H_0 \rightarrow H_0$ is a Fredholm operator.

† For an exhaustive treatment of this topic see [5].

Proof. Let \mathcal{C} denote the Banach algebra of all complex-valued functions f on the circle. Our construction $f \mapsto F$ defines a continuous linear map

$$\mathcal{C} \rightarrow \mathcal{A}$$

where \mathcal{A} is the Banach algebra of bounded operators on H_0 . Passing to the quotient $\mathcal{B} = \mathcal{A}/\mathcal{K}$ by the ideal \mathcal{K} of compact operators we obtain a continuous linear map $T : \mathcal{C} \rightarrow \mathcal{B}$. Suppose now that $f, g \in \mathcal{C}$ and have finite Fourier series

$$f(z) = \sum_{-n}^n f_k z^k \quad g(z) = \sum_{-m}^m g_k z^k$$

Let $l = fg$ and let F, G, L denote the corresponding elements of \mathcal{A} . It is then clear that

$$FG(z^k) = L(z^k) \text{ for } k \geq m+n$$

i. e. the operators FG and L coincide on the subspace H_{m+n} of H_0 . Thus $FG - L$ has finite rank and so is compact. Passing to the quotient algebra \mathcal{B} this means that

$$T(fg) = T(f)T(g) .$$

Thus $T : \mathcal{C} \rightarrow \mathcal{B}$ is a homomorphism on the subalgebra consisting of finite Fourier series. Since this subalgebra is dense in \mathcal{C} and since T is continuous it follows that T itself is a homomorphism. Since $T(1) = 1$ this implies that T takes invertible elements into invertible elements. Thus, if $f(z)$ is nowhere zero, $T(f)$ is invertible in \mathcal{B} and so, by (1.1), F is a Fredholm operator.

Remark. For the Fredholm operator $F = Pf$ of this proposition an integer n for which $F_n = P_n f : H_0 \rightarrow H_n$ is surjective can be found explicitly. It is enough to take a finite Fourier series

$$g(z) = \sum_{-n}^n g_k z^k$$

which approximates $f(z)^{-1}$ sufficiently so that

$$\sup |f(z)g(z) - 1| < 1 .$$

I shall omit the simple verification.

As a simple example consider the function $f(z) = z^m$. It is clear that for $F = Pf$ we then have

$$\text{index } F = -m .$$

Since the index is a locally constant function it follows that, for any continuous map

$$f : S^1 \longrightarrow \mathbb{C}^*$$

(where \mathbb{C}^* denotes the non-zero complex numbers), we have

$$(4.3) \quad \text{index } F = -w(f)$$

where $w(f)$ is the "winding number" of f -- intuitively the number of times the path f goes round the origin in the complex plane. This simple observation is of fundamental importance for us. In fact the Bott periodicity theorem is, in a sense, a natural generalization of (4.3). We proceed now to introduce, by steps, the generalizations of the Wiener-Hopf operator which are required.

First we make an obvious extension, replacing the scalar-valued functions by vector-valued functions. The function f , which plays a multiplicative role, must be replaced by a matrix and the non-zero condition in (4.2) is replaced by the non-singularity of the matrix. Thus we start from a continuous map

$$f : S^1 \longrightarrow GL(N, \mathbb{C})$$

of the circle into the general linear group of \mathbb{C}^N . We take the Hilbert space of L^2 -functions on S^1 with values in \mathbb{C}^N : if H is our original Hilbert space of scalar functions our new Hilbert space is $H \otimes \mathbb{C}^N$. As before we denote by P the projection

$$H \otimes \mathbb{C}^N \longrightarrow H_0 \otimes \mathbb{C}^N ,$$

and we define the operator

$$F = Pf : H_0 \otimes \mathbb{C}^N \longrightarrow H_0 \otimes \mathbb{C}^N$$

to be the composition of P and multiplication by the matrix function $f(z)$. The proof of (4.2) extends at once and shows that F is a Fredholm operator. The index of F is then a homotopy invariant of f . It determines in fact the element of the fundamental group of $GL(N, \mathbb{C})$ represented by f .

Next we generalize the situation by introducing a compact parameter space X . Thus we now consider a continuous map

$$(4.4) \quad f : S^1 \times X \longrightarrow GL(N, \mathbb{C})$$

so that $f(z, x)$ is a non-singular matrix depending continuously on two variables z, x . For each $x \in X$ we therefore get a Fredholm operator $F(x)$, acting on the Hilbert space $H_0 \otimes \mathbb{C}^N$. Moreover $F(x)$ is a continuous function of x so that we have a Fredholm family

$$F : X \longrightarrow \mathcal{F} .$$

By the construction of §2 this family has an index in $K(X)$. Thus

$$f \longmapsto F \longmapsto \text{index } F$$

assigns to each continuous map f , as in (4.4), an element of $K(X)$. Moreover this element depends only on the homotopy class of f .

Our final generalization is to allow the vector space \mathbb{C}^N to vary

continuously with the parameter x . More precisely we fix an N -dimensional complex vector bundle V over X and we suppose given a function

$$(4.5) \quad f(z, x) \in \text{Aut } V(x)$$

(where $V(x)$ denotes the fibre of V over x) which is continuous in z, x . Since V is locally trivial our function f is locally of the type we had previously (with $V(x) = \mathbb{C}^N$ for all x), so that there is no problem in defining continuity. Our Hilbert space will now be $H_0 \otimes V(x)$ and so varies from point to point. The operator

$$F(x) : H_0 \otimes V(x) \longrightarrow H_0 \otimes V(x)$$

is defined as before and is a Fredholm operator depending continuously on x . Since our Hilbert space varies this is a somewhat more general kind of family than the Fredholm families of §2, but we can still define the index in $K(X)$ by essentially the same method. Since V is locally trivial we can still apply the purely local Lemma (2.1), replacing H_n by $H_n \otimes V(x)$, and obtaining locally an operator $F_n(x)$ with the properties described in (2.1). The compactness of X then leads to a fixed n for which $\text{Ker } F_n$ is a vector bundle and $\text{Ker } F_n^*$ is a trivial vector bundle. We now define

$$\text{index } F = [\text{Ker } F_n] - [\text{Ker } F_n^*] \in K(X)$$

and prove as before that this is independent of n .

Finally therefore we have given a construction

$$(V, f) \longmapsto F \longmapsto \text{index } F$$

which assigns to each pair (V, f) — consisting of a vector bundle V over X and an f as in (4.5) — an element of $K(X)$. Again this depends only

on the homotopy class of f .

Now a pair (V, f) as above can be used to construct a vector bundle $E(V, f)$ over $S^2 \times X$. This is done as follows. We regard S^2 as the union of two closed hemi-spheres B^+ , B^- meeting on the equator S^1 . The vector bundle E is then constructed from the two vector bundles

$$\begin{array}{ccc} B^+ \times V & & B^- \times V \\ \downarrow & & \downarrow \\ B^+ \times X & & B^- \times X \end{array}$$

by identifying, over points $(z, x) \in S^1 \times X$,

$$(z, v) \in B^+ \times V(x) \text{ with } (z, f(z, x)v) \in B^- \times V(x) .$$

When X is a point this is a well known construction for defining vector bundles over the sphere S^2 . The parameter space X plays a quite harmless role.

It is not hard to show (see [1; p. 47]) that every vector bundle E over $S^2 \times X$ arises in this way from some pair (V, f) . Given E we first take V to be the vector bundle over X induced from E by the inclusion map $x \mapsto (1, x)$ of X into $S^2 \times X$ (where 1 denotes the point $z = 1$ on $S^1 \subset S^2$). We then observe that, since $B^+ \times X$ retracts onto $\{1\} \times X$, the part E^+ of E over $B^+ \times X$ is isomorphic to $B^+ \times V$. Similarly $E^- \cong B^- \times V$. Then, over $S^1 \times X$, the identification of E^+ and E^- defines f . The map f obtained this way is normalized so that $f(1, x)$ is always the identity of $V(x)$, and its homotopy class is then uniquely determined by E . Hence our construction

$$E \mapsto (V, f) \longrightarrow F \mapsto \text{index } F$$

defines a map

$$\text{Vect}(S^2 \times X) \longrightarrow K(X) .$$

This is clearly additive and so it extends uniquely to a homomorphism of groups

$$\alpha' : K(S^2 \times X) \longrightarrow K(X) .$$

We recall now formula (3.2) which identifies $K(\mathbb{R}^2 \times X)$ with a subgroup of $K(S^2 \times X)$. Restricting α' to this subgroup we therefore obtain a homomorphism

$$\alpha : K(\mathbb{R}^2 \times X) \longrightarrow K(X) .$$

Essentially therefore α is given by taking the index of a Wiener-Hopf family of Fredholm operators. From its definition it is clear that it has the following multiplicative property. Let Y be another compact space, then we have a commutative diagram

$$(4.6) \quad \begin{array}{ccc} K(\mathbb{R}^2 \times X) \otimes K(Y) & \longrightarrow & K(\mathbb{R}^2 \times X \times Y) \\ \downarrow \alpha_X \otimes 1 & & \downarrow \alpha_{X \times Y} \\ K(X) \otimes K(Y) & \longrightarrow & K(X \times Y) \end{array}$$

where the horizontal arrows are given by the external product discussed in §3.

We can now easily extend α to locally compact spaces X by passing to the one-point compactification X^+ and using (3.1). The commutative diagram (4.6) continues to hold for X, Y locally compact — again by appealing to (3.1).

In the next section we shall show how to prove that α is an isomorphism, thus establishing the periodicity theorem (3.3).

§5. Proof of periodicity.

We begin by defining a basic element b in $K(\mathbb{R}^2)$. We take the 1-dimensional vector bundle E_m over S^2 defined, as in §4, by the function $f(z) = z^m$. We put

$$b = [E_{-1}] - [E_0] \in K(S^2) .$$

Since E_{-1} and E_0 both have dimension 1 it follows that b lies in the summand $K(\mathbb{R}^2)$ of $K(S^2)$. As we have already observed, if F_m is the Wiener-Hopf operator defined by the function z^m , we have

$$\text{index } F_m = -m .$$

Thus

$$\begin{aligned} a(b) &= \text{index } F_{-1} - \text{index } F_0 \\ &= 1 . \end{aligned}$$

We now define, for any X , the homomorphism

$$\beta : K(X) \longrightarrow K(\mathbb{R}^2 \times X)$$

to be external multiplication by $b \in K(\mathbb{R}^2)$. Thus, for $x \in K(X)$, $\beta(x)$ is the image of $b \otimes x$ under the homomorphism

$$K(\mathbb{R}^2) \otimes K(X) \longrightarrow K(\mathbb{R}^2 \times X) .$$

With these preliminaries we can state a more precise version of the periodicity theorem.

THEOREM (5.1). For any locally compact space X , the homomorphisms

$$\beta : K(X) \longrightarrow K(\mathbb{R}^2 \times X)$$

$$\alpha : K(\mathbb{R}^2 \times X) \longrightarrow K(X)$$

are inverses of each other.

As we shall see the proof of (5.1) is now a simple consequence of the formal properties of α , β . First we apply the diagram (4.6) with $X = \text{point}$, $Y = X$. This gives, for any $x \in K(X)$,

$$\begin{aligned} a\beta(x) &= a(b)x \\ &= x \quad \text{since } a(b) = 1 . \end{aligned}$$

Thus a is a left inverse of β . To prove that it is also a right inverse we apply (4.6) with $Y = \mathbb{R}^2$. Thus we have the commutative diagram

$$\begin{array}{ccc} K(\mathbb{R}^2 \times X) \otimes K(\mathbb{R}^2) & \longrightarrow & K(\mathbb{R}^2 \times X \times \mathbb{R}^2) \\ \downarrow a_X & & \downarrow a_{X \times \mathbb{R}^2} \\ K(X) \otimes K(\mathbb{R}^2) & \longrightarrow & K(X \times \mathbb{R}^2) \end{array} .$$

Hence for any element $u \in K(\mathbb{R}^2 \times X)$ we have

$$(5.2) \quad a(ub) = a(u)b \in K(X \times \mathbb{R}^2) .$$

Consider now the map τ of $\mathbb{R}^2 \times X \times \mathbb{R}^2$ into itself which interchanges the two copies of \mathbb{R}^2 . On $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ this is given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} .$$

Since this has determinant +1 it lies in the identity component of $GL(4, \mathbb{R})$. Thus the map τ is homotopic to the identity (through homeomorphisms) and so τ^* is the identity of $K(\mathbb{R}^2 \times X \times \mathbb{R}^2)$. On the other hand we have

$$\tau^*(ub) = b\tilde{u}$$

where $\tilde{u} \in K(X \times \mathbb{R}^2)$ corresponds to $u \in K(\mathbb{R}^2 \times X)$ under the obvious identification. Thus

$$a(ub) = a(\tau^*(ub)) = a(b\tilde{u}) = a\beta(\tilde{u}) = \tilde{u}$$

since α is a left inverse of β . Combined with (5.2) this gives

$$\tilde{u} = \alpha(u)b \in K(X \times \mathbb{R}^2) .$$

Switching back to $K(\mathbb{R}^2 \times X)$ this is equivalent to

$$u = \beta\alpha(u) = \beta\alpha(u) \in K(\mathbb{R}^2 \times X) ,$$

proving that α is also a right inverse of β . This completes the proof of the theorem.

Concluding remarks.

I have given this proof of the periodicity theorem in such detail because I wanted to show how simple it really was. I hope I have demonstrated that the index of Fredholm families, which was used to construct the map α , plays a natural and fundamental role in K-theory.

This proof of periodicity has the advantage that it extends, with little effort, to various generalizations of the theorem. For full details on this I refer to [2].

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