Let \( \Omega \) be an open set of \( \mathbb{R}^n \), star-shaped (at 0), then \( \Omega \) is \( C^\infty \) diffeomorphic to \( \mathbb{R}^n \).

**Proof**

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a \( C^\infty \) function such that \( F = \{ x \mid f(x) = 0 \} \)

We set \( \varphi : \Omega \to \mathbb{R}^+ \) by

\[
\varphi(x) = \left[ 1 + \left( \int_0^1 \frac{dt}{\varphi(t x)} \right)^2 \right]^{1/2} \| x \|_2,
\]

(whence \( \| x \|_2 = (\sum x_i^2)^{1/2} \))

\( \varphi \) is smooth on \( \Omega \). We set \( A(x) = \sup \{ t > 0, \frac{x}{\| t x \|_2} \in \Omega \} \). \( \varphi \) sends injectively \( [0, A(x)] \cdot \frac{x}{\| x \|_2} \) into \( \mathbb{R}^+ \cdot \frac{x}{\| x \|_2} \).

Moreover, if we set \( u = \frac{x}{\| x \|_2} \), then

\[
\| \varphi(0) \|_2 = 0 \quad \text{and} \quad \lim_{t \to A(x)} \| \varphi(t u) \|_2 = \left[ 1 + \left( \int_0^1 \frac{dt}{\varphi(t u)} \right)^2 \right] A(x) = +\infty.
\]

Indeed, if \( A(x) = +\infty \) it is obvious

\[
\text{if } A(x) < +\infty \text{ then } \int_0^A \varphi(A(x) u) du = 0 \Rightarrow \varphi(t u) = 0 (t < A(x))
\]

\( \varphi \) smooth and so \( \int_0^A \varphi(A(u)) du \) diverges.

We infer that \( \varphi([0, A(x)] \cdot \frac{x}{\| x \|_2}) = \mathbb{R}^+ \cdot \frac{x}{\| x \|_2} \) and so \( \varphi(\Omega) = \mathbb{R}^n \).

To conclude, we have

\[
\frac{d}{dx} \left( \frac{d}{dx} \right) = A(x) \frac{d}{dx} + A(x) \frac{d}{dx},
\]

so if \( \xi \) with \( \xi \cdot x = 0 \), then there exists \( \mu \in \mathbb{R} \) such that \( \eta = \mu x \)

and we set \( \left[ \lambda(\xi) + A(x) \frac{d}{dx} \right] \xi = 0 \) (note that \( \lambda(0) = 1 \) so \( x \to 0 \)).

But we have \( \lambda(x) > 1 \) and \( g(x) = \lambda(\xi x) \) increasing so \( g'(1) = \frac{d}{dx} \lambda(\xi x) \)

which gives a contradiction.

**Nota bene** - The Whitney Theorem is a classical result. In the case \( n = 2 \), the Hopf-Rinow Theorem implies that \( \mathbb{R}^2 \) is holomorphically diffeomorphic to \( \mathbb{C} \).