Equilogical Spaces

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Abstract

It is well known that one can build models of full higher-order dependent type theory (also called the calculus of constructions) using partial equivalence relations (PERs) and assemblies over a partial combinatory algebra (PCA). But the idea of categories of PERs and ERs (total equivalence relations) can be applied to other structures as well. In particular, we can easily define the category of ERs and equivalence-preserving continuous mappings over the standard category $\mathbf{Top}_0$ of topological T_0-spaces; we call these spaces (a topological space together with an ER) *equilogical spaces* and the resulting category **Equ**. We show that this category—in contradistinction to $\mathbf{Top}_0$—is a cartesian closed category. The direct proof outlined here uses the equivalence of the category **Equ** to the category **PEqu** of PERs over algebraic lattices (a full subcategory of $\mathbf{Top}_0$ that is well known to be cartesian closed from domain theory). In another paper with Carboni and Rosolini (cited herein) a more abstract categorical generalization shows why many such categories are cartesian closed. The category **Equ** obviously contains $\mathbf{Top}_0$ as a full subcategory, and it naturally contains many other well known subcategories. In particular, we show why, as a consequence of work of Ershov, Berger, and others, the Kleene-Kreisel hierarchy of countable functionals of finite types can be naturally constructed in **Equ** from the natural numbers object $\mathbb{N}$ by repeated use in **Equ** of exponentiation and binary products. We also develop for **Equ** notions of modest sets (a category equivalent to **Equ**) and assemblies to explain why a model of dependent type theory is obtained. We make some comparisons of this model to other, known models.

1 Introduction

The genesis of this paper is the manuscript [38] "A New Category?" privately circulated by Dana Scott in December of 1996. During the last part of his

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graduate course on Domain Theory he had realized that by using some basic and well-known properties of domains (specifically, algebraic lattices) the category of equivalence relations on T_0-spaces not only was an extension of the topological category but was cartesian closed.

The present paper incorporates original motivation, definitions, and proofs of the earlier manuscript, and we then give an equivalent definition suggesting relationships to the extensive work on partial equivalence relations over partial combinatory algebras (hereafter, PCAs). In our conference paper [9], the reader will find an abstract framework due to Carboni and Rosolini in which the categories of equilogical spaces and partial equivalence relations over PCAs fit. Indeed, it is shown that there is a larger category than that of equilogical spaces that is cartesian closed. However, we shall not discuss the abstract categorical framework here (namely, that of exact completions of categories).

As in the earlier manuscript, our desire here is to give a fairly concrete description of the structures involved and the constructions from them. By extending the first treatment, we use an alternate equivalent definition of the category of equilogical spaces to give a definition of a model of dependent type theory and logic, analogous to the work over PCAs. We also discuss how far that analogy extends.

The final section of the paper shows how the work of Y. Ershov and E. Berger concerning the Kleene-Kreisel hierarchy of countable functionals and extensions can be incorporated into the category of equilogical spaces. In terms of the type theory, it turns out that the higher types over the integers $\mathcal{N} \to \mathcal{N}$, $(\mathcal{N} \to \mathcal{N}) \to \mathcal{N}$, $((\mathcal{N} \to \mathcal{N}) \to \mathcal{N}) \to \mathcal{N}$, etc., are indeed the countable functionals, as expected. In order to see this, we have to add appropriate categorical definitions to Berger's work.

Note added in February, 2001. Since the writing of this paper in 1998, much progress has been made in understanding equilogical spaces and their relationship to other categories. The relationship to tripos theory hinted at in the discussion in Section 4 has been worked out [7,8]; in particular, the open problem mentioned at the end of the discussion in Section 4 has been solved, see [7,8]. Also, the relation between equilogical spaces and domains with totality described in Section 5 has been extended to hierarchies of *dependent* types [4,3], and a relation to type-two effectivity has been discovered [3]. Also other researchers have contributed greatly to the study of equilogical spaces; see the papers cited here for references and discussions of their related work.

2 Motivation

The familiar categories **Set** and **Top**, consisting of sets and arbitrary mappings and of topological spaces and continuous mappings, have many well known closure properties. For example, they are both *complete* and *cocomplete*, meaning that they have all (small) limits and colimits. They are *well-powered* and *co-well-powered*, meaning that collections of subobjects and quotients of objects can be represented by sets. They are also nicely related, since **Set** can be regarded as a *full subcategory* of **Top**, and the *forgetful functor* that takes a topological space to its underlying set *preserves limits and colimits* (but reflects neither).

The category **Set** is also a *cartesian closed category*, meaning that the *function-space construct* or the *internal hom-functor* is very well behaved, in the sense that the functor $\cdot \times B$ is adjoint to $B \to \cdot$ for all objects B. However, it has been known for a long time that in **Top** no such assertion is available, because in general it is not possible to assign a topology to the set of continuous functions making this adjointness valid—except under some special conditions on the space B. Many remedies have been proposed, notably, (a) cutting down to *compactly generated spaces*, or (b) expanding the category to the category of *filter spaces* (or a related kind of *limit space*). These are interesting suggestions, but both have some drawbacks. Suggestion (a) applies only to Hausdorff spaces, and suggestion (2)—which the authors consider the more interesting from a logical point of view—introduces very unfamiliar spaces at the higher types (*i.e.*, after iterating the function-space construct several times). It remains to be seen whether the suggestion of this paper can be regarded as more concrete or more helpful than either (a) or (b).

Our solution to the problem of cartesian closedness is motivated by domain theory. The new category is formed from the category $\mathbf{Top}_0$ of topological T_0-spaces by using spaces together with *arbitrary* equivalence relations, to form the category, to be called called **Equ**, where the mappings are (suitable equivalence classes of) continuous mappings which preserve the equivalence relations. (A more precise definition will be given below.) Let us call these spaces *equilogical spaces* and the mappings *equivariant*. It seems surprising that this category has not been noticed before—if in fact it has not. It is easy to see that **Equ** is complete and cocomplete and that it embeds $\mathbf{Top}_0$ as a full and faithful subcategory (by taking the equivalence relation to be the identity relation).

What is perhaps not so obvious is that **Equ** is indeed *cartesian closed*. The proof of cartesian closedness outlined here uses old theorems in domain theory originally discovered by Scott: in particular, an injective property of algebraic lattices treated as topological spaces and the fact that they form a cartesian closed category (along with continuous functions). A more abstract, categorical proof can be found in [9] or in [37]. Also, in Section 4 we give an alternative concrete proof. Of course, algebraic lattices are just one of many cartesian closed categories proposed for domain theory—and not the most popular one. They allow, however, for some helpful embeddings of T_0-spaces.

For a long time Scott has been distressed that there are *too many* proposed categories of domains and that their study has become too arcane. It was hoped that the idea of *synthetic domain theory* would be the natural solution—but that theory has been slowed by many technical problems. The related idea of *axiomatic domain theory* is likewise hampered by the need to overcome technical difficulties. Despite very good work in both these directions, he does not feel that a final theory has emerged. Perhaps some of the ideas that have been used in these other approaches can be transplanted to the study of **Equ**, which seems to be a rich and fairly natural category with many subcategories. The basic idea of the synthetic approach is to establish a typed λ-calculus once and for all, and then to single out useful types (or domains) by means of special properties—just as is done in several other branches of mathematics. As far as **Equ** is concerned, the possibilities seem good, but this is still work in progress. We are encouraged, however, by the results so far obtained, some of which are presented here.

3 Equilogical Spaces

We begin by defining some notation and calling to mind some basic definitions and theorems concerning T_0-spaces and algebraic lattices. We then turn to the definition of equilogical spaces.

T_0-Spaces and Algebraic Lattices. Topological spaces will be considered as structures $\mathcal{T} = \langle T, \Omega_{\mathcal{T}} \rangle$, where T is the set of points of the space, and where $\Omega_{\mathcal{T}}$ is the set of open sets of $\mathcal{T}$. We shall often write $|\mathcal{T}| = T$, so as not to have to use a special letter for the points of a space. Complete lattices (and, more generally, posets) will be considered as structures $\mathcal{L} = \langle |\mathcal{L}|, \leq_{\mathcal{L}} \rangle$, where $\leq_{\mathcal{L}}$ is the partial ordering of the set $|\mathcal{L}|$. Completeness of course demands that every subset $S \subseteq |\mathcal{L}|$ has a least upper bound $\bigvee S \in |\mathcal{L}|$.

Definition 3.1 The **neighborhood filter** of a point $x \in |\mathcal{T}|$ of a topological space $\mathcal{T}$ is defined by the equation:

$$\mathcal{T}(x) = \{\, U \in \Omega_{\mathcal{T}} \mid x \in U \,\}.$$

The spaces we shall be concerned with are the T_0-spaces, where the topology distinguishes the points.

Definition 3.2 A topological space is a **T_0-space** provided that for every pair of distinct points there is an open set that contains one but not the other. Another way to say this condition is to say that for all $x, y \in |\mathcal{T}|$, if $\mathcal{T}(x) = \mathcal{T}(y)$, then $x = y$. The category of all such spaces and continuous mappings between them is denoted by $\mathbf{Top}_0$.

Definition 3.3 The **specialization ordering** of a topological space $\mathcal{T}$ is defined by:

$$x \leq_{\mathcal{T}} y \iff \mathcal{T}(x) \subseteq \mathcal{T}(y),$$

for all $x, y \in |\mathcal{T}|$.

Definition 3.4 Let $\mathcal{L}$ be a complete lattice. The **Σ-topology** on the lattice is defined as the collection of all upward closed subsets $U \subseteq |\mathcal{L}|$ such that whenever $S \subseteq |\mathcal{L}|$ and $\bigvee S \in U$, then $\bigvee S_0 \in U$ for some finite subset $S_0 \subseteq S$. The collection of all such subsets is denoted by $\Sigma_{\mathcal{L}}$.

The following theorems are now well-known. Proofs can, *e.g.*, be found in [16].

Theorem 3.5 *Given a complete lattice $\mathcal{L}$, the structure $\langle |\mathcal{L}|, \Sigma_{\mathcal{L}} \rangle$ is a T_0-space whose specialization ordering is exactly $\leq_{\mathcal{L}}$.*

For the powerset spaces $\mathcal{P}\,A$ the Σ-topology is very easy to describe: the open sets $U \subseteq \mathcal{P}\,A$ are the families of "finite character"; that is, a subset $X \subseteq A$ belongs to U if, and only if, some finite subset of X belongs to U. This is the same as giving $\mathcal{P}\,A$ the topology that corresponds to the product topology on 2^A where the two-element set has the topology with one open point and one closed point. The powerset spaces have an important role as being able to embed every T_0-space. The following elementary result is key to the subsequent development.

Theorem 3.6 (The Embedding Theorem) *Given a T_0-space $\mathcal{T}$, the mapping $x \mapsto \mathcal{T}(x)$ is a topological embedding of $\mathcal{T}$ into $\mathcal{P}\,\Omega_{\mathcal{T}}$ considered as a space with the Σ-topology.*

Powerset spaces also have another important property concerning continuous functions which allows for the transfer of functions over to the powerset space.

Theorem 3.7 (The Extension Theorem) *If $\mathcal{Y}$ is a subspace of a topological space $\mathcal{X}$, and if $f\colon |\mathcal{Y}| \to \mathcal{P}\,A$ is continuous, then the function f has a continuous extension to all the points of $\mathcal{X}$.*

Scott noticed the above theorems in 1970/71 and also pointed out that it in fact holds for all continuous retracts of the powerset spaces—these are the continuous lattices—but for our purposes here, the above suffices.

Powerset lattices can be generalized to **algebraic lattices**, namely those complete lattices that can be represented isomorphically as complete sublattices of a powerset closed under arbitrary intersections and directed unions. (These lattices can be characterized in other ways as well; see, *e.g.*, [13,16].) The Σ-topology on an algebraic lattice is just the restriction of the topology of the powerset space. An algebraic lattice is a continuous retract of the powerset containing it, but not all such retracts are algebraic.

The reason for considering algebraic lattices is that the lattice of continuous functions between powerset spaces is not usually a powerset space, but it is an algebraic lattice. And this extends to all algebraic lattices. Hence, we have the well known theorem (see [13,16]):

Theorem 3.8 *The category* **ALat** *is cartesian closed.*

The Category of Equilogical Spaces. We have now reviewed sufficient material to be able to give two definitions of the category of equilogical spaces and to show that the two definitions are equivalent. We will then prove that the category is cartesian closed.

Definition 3.9 The category **Equ** of **equilogical spaces** is defined as follows.

(1) Objects are structures $\mathcal{E} = \langle |\mathcal{E}|, \Omega_{\mathcal{E}}, \equiv_{\mathcal{E}} \rangle$, where $\langle |\mathcal{E}|, \Omega_{\mathcal{E}} \rangle$ is a T_0-space and $\equiv_{\mathcal{E}}$ is an (arbitrary) equivalence relation on the set $|\mathcal{E}|$.
(2) The mappings between equilogical spaces are the equivalence classes of continuous mappings between the topological spaces that preserve the equivalence relation (**equivariant** mappings), where the equivalence relation on mappings is defined by

$$f \equiv_{\mathcal{E} \to \mathcal{F}} g \iff \forall x, y \in |\mathcal{E}|. \Big(x \equiv_{\mathcal{E}} y \implies f(x) \equiv_{\mathcal{F}} g(y) \Big).$$

We remark that it has to be proved that $\equiv_{\mathcal{E} \to \mathcal{F}}$ actually is an equivalence relation, but this is an elementary exercise. It also has to be proved that the equilogical spaces and equivariant maps form a category, but this can also be safely left to the reader.

One odd feature of this definition is that the equivalence relation of an equilogical space may have very little to do with the topology. This means that in some cases the only equivariant mappings between two spaces might be the constant maps, or the only automorphisms of a given space might be the identity—despite a rich underlying topology. Thus, future investigations may suggest limiting the equivalence relations. But, for now, the general properties of the category seem to work out well for arbitrary equivalence relations, so we have not been motivated to make any further restrictions in this paper.

Recall that a category is complete if it has all (small) products and equalizers of all pairs of parallel arrows. Similarly, a category is cocomplete if it has all (small) coproducts and coequalizers of all pairs of parallel arrows. Also recall that a regular subobject is a subobject which arises as the equalizer of a pair of parallel arrows and that a category is regular well-powered if the regular subobjects of every object constitute a *set*. Dually, a regular quotient is a quotient which arises as the coequalizer of a pair of parallel arrows and a category is regular co-well-powered if no object has a proper class of non-isomorphic regular quotients.

Theorem 3.10 *The category* **Equ** *is complete, cocomplete, and it is regular well-powered, and regular co-well-powered.*[4]

[4] The authors are indebted to Peter Johnstone for pointing out that, contrary to the assertion made in Scott's original unpublished manuscript, **Equ** is *not* well powered, for there are fairly simple examples of objects in the category with an

Proof. The proof proceeds along standard lines making use of the corresponding properties of topological spaces.

Take *products* first. The product (of any number) of topological spaces is a space with a product topology. The product of equivalence relations is an equivalence relation. The projection mappings are clearly equivariant. And, if we have a family of (equivalence classes of) equivariant mappings into the various factor spaces, then (after applying the Axiom of Choice to pick representatives) we can obtain in the usual way *one* equivariant mapping into the product that combines all the separate mappings.

Next, take *equalizers*. Suppose $f, g\colon |\mathcal{E}| \to |\mathcal{F}|$ are two (representatives of) equivariant mappings. Form the set $\{\, x \in |\mathcal{E}| \mid f(x) \equiv_{\mathcal{F}} g(x) \,\}$. Endow this set with the subspace topology and with the restriction of the equivalence relation $\equiv_{\mathcal{E}}$. This structure, along with the obvious inclusion mapping into $\mathcal{E}$, is the desired equalizer. Thus, **Equ** is a complete category.

On to *coproducts*. The coproduct of topological spaces is just a disjoint union of the underlying sets with the topology on the union generated by the union of all the topologies. For equivalence relations, we have only to note that the union of equivalence relations on disjoint sets is indeed an equivalence relation. The injection mappings from the separate spaces into the union are obvious, as well as is the lifting property of a family of mappings from the separate spaces into a given target space.

Next, we discuss *coequalizers*. Suppose $f, g\colon \mathcal{E} \to \mathcal{F}$ are two (representatives of) equivariant mappings. On $|\mathcal{F}|$ we form the least equivalence relation containing both $\equiv_{\mathcal{F}}$ and the set of pairs $\{\, (f(x), g(x)) \mid x \in |\mathcal{E}| \,\}$. Using this equivalence relation on $|\mathcal{F}|$, we form the equilogical space $\mathcal{G}$. There is an obvious equivariant mapping $c\colon \mathcal{F} \to \mathcal{G}$ represented by the identity. This is the desired coequalizer. Thus, **Equ** is a cocomplete category.

Finally, we turn to well-poweredness. The properties of being regular well-powered and regular co-well-powered follow from the corresponding properties of $\mathbf{Top}_0$ and the category of equivalence relations; one just has to be careful to check that the regular subobjects are obtained by selecting some equivalence classes and taking the union of them to form a subspace; likewise, forming a regular quotient is just making the equivalence relation coarser (putting equivalence classes together). And, be warned that there are subobjects and quotients which are not formed in this simple way. ∎

The proof just given is sketchy in the handling of equivalence classes of maps, and, in the construction of the equalizer and coequalizer, it has to be checked that the structures suggested have the required universal properties. But, this argument—modulo equivalence classes—is exactly similar to what is done for the category $\mathbf{Top}_0$. We remark that the category of equivalence relations on sets is included here: a set is just a discrete topological space (and these form

unbounded number of non-isomorphic subobjects.

a full subcategory of $\mathbf{Top}_0$). Of course, with the aid of the Axiom of Choice, it is quickly shown that the category of equivalence relations is *equivalent* to the category of sets (*via* the obvious use of quotient sets). However, the category **Equ** introduced here is *not* equivalent to the category $\mathbf{Top}_0$. For one thing, no topology is being put on the quotient space $|\mathcal{E}|/{\equiv_{\mathcal{E}}}$. And this category has a property—cartesian closure—that $\mathbf{Top}_0$ does not share.

To investigate **Equ** further, we introduce a closely connected category.

Definition 3.11 The category **PEqu** of **partial equilogical spaces** is defined as follows.

(1) Objects are structures $\mathcal{A} = \langle |\mathcal{A}|, \Omega_{\mathcal{A}}, \equiv_{\mathcal{A}} \rangle$, where $\langle |\mathcal{A}|, \Omega_{\mathcal{A}} \rangle$ is the Σ-topology of an algebraic lattice, and where $\equiv_{\mathcal{A}}$ is a *partial* equivalence relation, i.e., reflexive only on a subset of $|\mathcal{A}|$.
(2) The mappings between partial equilogical spaces are the equivalence classes of continuous mappings between the algebraic lattices that preserve the partial equivalence relation, where the equivalence relation on mappings is defined as before by

$$f \equiv_{\mathcal{A} \to \mathcal{B}} g \iff \forall x, y \in |\mathcal{A}|. \big(x \equiv_{\mathcal{A}} y \implies f(x) \equiv_{\mathcal{B}} g(y) \big).$$

These mappings will also be called **equivariant**.

If we consider the relation $f \equiv_{\mathcal{A} \to \mathcal{B}} g$ as being defined between arbitrary continuous functions, then equivariant maps for the category **PEqu** are the (equivalence classes of) the functions f satisfying $f \equiv_{\mathcal{A} \to \mathcal{B}} f$, since that means that the function preserves the underlying equivalence relation. This remark gives a hint as to how we will define function spaces, but first we want to check the equivalence of categories.

Theorem 3.12 *The categories* **Equ** *and* **PEqu** *are equivalent.*

Proof. The naturally suggested functor from **PEqu** to **Equ** is the one that takes $\langle |\mathcal{A}|, \Omega_{\mathcal{A}}, \equiv_{\mathcal{A}} \rangle$ and restricts the topology to the subspace on the subset $\{ x \in |\mathcal{A}| \mid x \equiv_{\mathcal{A}} x \}$. On this subset the equivalence relation is "total". The mappings are likewise restricted. Call the functor R (for "restriction"). Now, if $f \colon \mathcal{A} \to \mathcal{B}$ is a map of **PEqu**, then $R(f) = f \restriction |R(\mathcal{A})| \colon R(\mathcal{A}) \to R(\mathcal{B})$ is valid as a map of **Equ**, and identities and compositions are preserved.

We note first that the functor R is faithful by definition. Then, the functor R is full in view of The Extension Theorem (because continuous functions between T_0-spaces can be extended to any algebraic lattices embedding them). Finally, the functor R is essentially surjective on objects by virtue of The Embedding Theorem (and note that the equivalence relation on the T_0-space does not have to be extended but remains partial). This is enough to show that the categories are equivalent. ∎

The idea of partial equivalence relations has been very widely employed. Scott believes he first called general attention to it in the late '60s after extracting it from the studies by G. Kreisel and A. Troelstra on extensional theories of higher-type functionals in recursion theory. However, it has been mostly used recently in the context of giving types to (quotients of) subsets of a universal model of some sort. We think allowing partial equivalence relations over a large category (such as algebraic lattices) is possibly a new idea; but, certainly, many familiar proofs get reused in the new context. The following theorem is an example of this reuse.

Theorem 3.13 *The category* **Equ** *is cartesian closed.*

Proof. In view of the previous theorem, we will show that **PEqu** is cartesian closed. Given structures $\mathcal{A}$ and $\mathcal{B}$ in **PEqu** we define the structure $\mathcal{A} \to \mathcal{B}$ so that

(i) $|\mathcal{A} \to \mathcal{B}|$ is the set of continuous functions between the lattices $|\mathcal{A}|$ and $|\mathcal{B}|$;
(ii) $\Omega_{\mathcal{A}\to\mathcal{B}}$ is the Σ-topology on this algebraic lattice;
(iii) $\equiv_{\mathcal{A}\to\mathcal{B}}$ is the partial equivalence defined previously.

We have to show, that for any three structures in **PEqu**, say, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, there is a one-one correspondence between functions in the two spaces:

$$(\mathcal{A} \times \mathcal{B} \to \mathcal{C}) \text{ and } (\mathcal{A} \to (\mathcal{B} \to \mathcal{C})).$$

As we know, there is a particular one-one correspondence that is an isomorphism of the underlying algebraic lattices (and a homeomorphism of topological spaces). It only remains to show that the isomorphism preserves the partial equivalence relation on the compound space. This is a "self-proving" theorem, in the sense that once the question is stated it is just a matter of unpacking the definitions to finish it off. ■

4 Equilogical Spaces, Type Theory and Logic

We have now already seen that the category of equilogical spaces provides a model of the simply-typed λ-calculus, inasmuch as **Equ** is cartesian closed. In this section we show that **Equ** in fact supports a much more expressive type theory and logic, which can be introduced by using the method of assemblies. Here, as elsewhere in the paper, we have favored a concrete exposition over a more abstract and economical presentation.

For simplicity, we sometimes write an object $\mathcal{A} = \langle |\mathcal{A}|, \Omega_{\mathcal{A}}, \equiv_{\mathcal{A}} \rangle$ of **PEqu** as $(A, \equiv_A)$ with A the algebraic lattice $\langle |\mathcal{A}|, \Omega_{\mathcal{A}} \rangle$ and $\equiv_A$ the partial equivalence relation $\equiv_{\mathcal{A}}$. We then write $|A|$ for the underlying set of the algebraic lattice A.

Modest Sets and Assemblies. We first introduce yet another equivalent definition of the category **Equ**, which will allow us to proceed by analogy to the category of partial equivalence relations over a PCA (see, *e.g.*, [11]).

Definition 4.1 The category **Assm**(**ALat**) of **assemblies over the category of algebraic lattices** is defined as follows.

(1) Objects are triples (X, A, E) with $X \in \mathbf{Set}$, $A \in \mathbf{ALat}$, and the mapping $E \colon X \to \mathcal{P}\,|A|$ in **Set** is such that $E(x)$ is non-empty for all $x \in X$. We call the elements in $E(x)$ *realizers* for x.
(2) The morphisms from an object (X, A, E) to an object (X', A', E') are functions $f \colon X \to X'$ in **Set** for which there exists a continuous function $g \colon A \to A'$ in **ALat** such that

$$\forall x \in X. \forall a \in E(x). g(a) \in E'(f(x)).$$

We call such a function g a *realizer* for f, and say that g *tracks* f.

Definition 4.2 An object (X, A, E) of **Assm**(**ALat**) is called *modest* if, and only if,

$$\forall x, x' \in X. \Big(x \neq x' \implies E(x) \cap E(x') = \emptyset\Big).$$

The full subcategory of **Assm**(**ALat**) formed by the modest objects is referred to as the category of **modest sets over algebraic lattices** is denoted **Mod**(**ALat**).

Roughly speaking a modest set is an assembly where a realizer $a \in E(x)$ carries enough information to determine the element $x \in X$ uniquely. An example of an assembly which is *not* isomorphic to any modest set is $(\{0, 1\}, \mathcal{P}\{0\}, E)$, where $E(0) = E(1) = \mathcal{P}\{0\}$. Here, the realizers tell us nothing at all about the differences between 0 and 1. (A term such as "separated" might have been more descriptive than "modest" — but see the further comments on terminology below.)

Readers familiar with categories of realizability models based on PCAs will immediately note the similarity of the above definitions to the well-known definitions of the categories of modest sets and assemblies over a PCA (see, *e.g.*, [19,11,28,26]). Those categories both embed into the so-called *realizability topos* over the PCA [19]. We do not get a corresponding embedding into a topos, however; we shall discuss why below.

One useful intuition is to think of the category of algebraic lattices as providing a *typed universe of realizers* (*cf.* the untyped universe of realizers provided by a PCA). Indeed for many conclusions we do not use any properties of algebraic lattices beyond the fact that it is a cartesian closed category. For example, we might use the cartesian closed category $\aleph_0$**ALat** of *countably based* algebraic lattices, equivalent to the category of algebraic sublattices of $\mathcal{P}\,\mathbb{N}$. In this case, modest sets are really modest in the sense of having their cardinality bounded by $2^{\aleph_0}$. It turns out also that one can obtain more general results based on only a weakly cartesian closed category of realizers [9]; we shall not go into that here, preferring for concreteness to stay with the example of all algebraic lattices.

Theorem 4.3 *The categories* **Equ**, **PEqu**, *and* **Mod**(**ALat**) *are all equivalent.*

Proof. Define a functor $F\colon \mathbf{Mod}(\mathbf{ALat}) \to \mathbf{PEqu}$ by $F(X, A, E) = (A, \equiv_A)$, where $a \equiv_A a' \iff \exists x \in X. a, a' \in E(x)$. When applied to a morphism $f\colon (X, A, E) \to (X', A', E')$ in **Mod**(**ALat**), the functor F gives the equivalence class of a realizer $g\colon A \to A'$ (g in **ALat**) for f which exists by virtue of f being a morphism in **Mod**(**ALat**). The definition of F is clearly independent of the choice of g. It is straightforward to verify that the functor F is full and faithful and essentially surjective on objects. For the latter, given an object $(A, \equiv_A) \in \mathbf{PEqu}$, consider the object $(\{\, a \in |A| \mid a \equiv_A a \,\}/{\equiv_A}, A, E) \in \mathbf{Mod}(\mathbf{ALat})$ with E the identity function on equivalence classes. ∎

We now use the alternative description of **Equ** provided by the above theorem to present some of its categorical properties in a different way. Some of the properties we have already seen, but the alternative descriptions below are useful. Along the way, we consider **Assm**(**ALat**), since the constructions are basically the same and we shall make use of **Assm**(**ALat**) below.

First, let us denote that inclusion functor from **Mod**(**ALat**) to **Assm**(**ALat**) by $\mathcal{I}$. We now check some categorical properties directly.

Theorem 4.4 *Both* **Assm**(**ALat**) *and* **Mod**(**ALat**) *are cartesian closed and the inclusion preserves the cartesian closed structure:*

Proof. The terminal object of **Assm**(**ALat**) is $(1_{\mathbf{Set}}, 1_{\mathbf{ALat}}, E_1)$ with $1_{\mathbf{Set}} = \{*\}$, $1_{\mathbf{ALat}} = \{*'\}$, and $E_1(*) = \{*'\}$. Clearly it is modest and terminal in **Mod**(**ALat**).

The binary product of (X, A, E_X) and (Y, B, E_Y) is $(X \times Y, A \times B, E)$ with $E(x, y) = E_X(x) \times E_Y(y)$. Here we make use of the binary products in the category of algebraic lattices, in analogy with the way in which the product operation of a PCA is used to prove that the category of assemblies and modest sets over such has binary products. If (X, A, E_X) and (Y, B, E_Y) are both modest, then also their product so defined is modest.

The exponential of (X, A, E_X) and (Y, B, E_Y) is (Z, B^A, E) with $Z = \{\, f \in Y^X \mid \exists g\colon A \to B. g \text{ tracks } f \,\}$; $E(f)$ the set of elements of B^A which track f, *i.e.*, $E(f) = \{\, g \in B^A \mid \forall x \in X. \forall a \in E_X(x). g(a) \in E_Y(f(x)) \,\}$. If (X, A, E_X) and (Y, B, E_Y) are both modest, then also (Z, B^A, E) is modest. ∎

Theorem 4.5 *Both* **Assm**(**ALat**) *and* **Mod**(**ALat**) *have finite limits and the inclusion preserves the finite limits.*

Proof. By the previous theorem it suffices to consider equalizers. The equalizer of $f, g\colon (X, A, E_X) \to (Y, B, E_Y)$ is $(\{\, x \in X \mid f(x) = g(x') \,\}, A, E'_X)$, where E'_X is E_X restricted to the subset, together with the obvious inclusion map. Let us also write out the pullback of f and g in

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, B, E_Y) \arrow[d, "g"] \\ (X, A, E_X) \arrow[r, "f"'] & (Z, C, E_Z) \end{tikzcd}$$

The object P is $(\{ (x, y) \in X \times Y \mid f(x) = g(y) \}, A \times B, E)$ with $E(x, y) = E_X(x) \times E_Y(y)$. ■

A morphism $f \colon (X, A, E_X) \to (Y, B, E_Y)$ is a monomorphism in $\mathbf{Assm}(\mathbf{ALat})$ (or in $\mathbf{Mod}(\mathbf{ALat})$) exactly if f is an injective function of sets; it is an epimorphism exactly if f is a surjective function. Let us now consider regular subobjects.

Recall that a regular category is a category with finite limits and (stable under pullback) image factorizations (see, *e.g.*, [10]).

Theorem 4.6 *Both* $\mathbf{Assm}(\mathbf{ALat})$ *and* $\mathbf{Mod}(\mathbf{ALat})$ *are regular categories.*

Proof. By the previous theorems, it suffices to show that we have stable image factorizations. The image factorization of $f \colon (X, A, E_X) \to (Y, B, E_Y)$ is

$$\begin{tikzcd} (X, A, E_X) \arrow[rr, "f"] \arrow[dr, "e"'] & & (Y, B, E_Y) \\ & (X/{\sim}, A, E'_X) \arrow[ur, "m"'] & \end{tikzcd}$$

where

$$\forall x, x' \in X. \left(x \sim x' \iff f(x) = f(x') \right) \qquad \text{and} \qquad E'_X([x]) = \bigcup_{x' \in [x]} E_X(x').$$

For the mappings, we set $e(x) = [x]$ (which is tracked by the identity), and $m([x]) = f(x)$ (which is tracked by a realizer for f). ■

Theorem 4.7 *The regular subobjects of an object* (X, A, E_X)*, both in the category* $\mathbf{Assm}(\mathbf{ALat})$ *and in* $\mathbf{Mod}(\mathbf{ALat})$ *are in bijective correspondence with the powerset of* X*.*

Proof. This follows easily from the description of equalizers. ■

In terms of **PEqu**, a regular subobject of an object $(A, \equiv_A)$ consists of the algebraic lattice A together with a partial equivalence relation corresponding to a collection of the equivalence classes of $\equiv_A$.

The well-known relationship between the category of assemblies over a PCA and the category of sets (see, *e.g.*, [20,19]) can easily be generalized to our situation as well: The category **Set** of sets embeds into the category of assemblies by the functor $\nabla \colon \mathbf{Set} \to \mathbf{Assm}(\mathbf{ALat})$ where $\nabla(X) = (X, 1_{\mathbf{ALat}}, E)$ with $E(x) = *$, for all $x \in X$, and $\nabla(f \colon X \to Y) = f$, trivially realized. Then one can show that ∇ is full and faithful, preserves finite limits, and coequalizers of kernel pairs (hence is exact in the sense of Barr [2]) and exponentials. Define the "global sections" functor $\Gamma\colon \mathbf{Assm}(\mathbf{ALat}) \to \mathbf{Set}$ by $\Gamma(X, A, E) = X$ and $\Gamma(f) = f$. Then Γ is faithful and exact. Moreover, one can easily prove the following theorem.

Theorem 4.8 *The functor Γ is left adjoint to ∇ with $\Gamma\nabla = id$.*

The categorical relationship between modest sets and assemblies is given by this theorem:

Theorem 4.9 *The category* $\mathbf{Mod}(\mathbf{ALat})$ *is a reflective subcategory of the category* $\mathbf{Assm}(\mathbf{ALat})$.

Proof. The reflection functor $R\colon \mathbf{Assm}(\mathbf{ALat}) \to \mathbf{Mod}(\mathbf{ALat})$ is defined as follows. On objects (X, A, E), let $R(X, A, E) = (X/{\sim}, A, E')$ where $x \sim x'$ if, and only if, $E(x) \cap E(x') \neq \emptyset$ and $E'([x]) = \bigcup_{x \in [x]} E(x')$. On morphisms f, let $R(f)$ be the mapping $[x] \mapsto [f(x)]$. ■

Modeling Dependent Type Theory. In this subsection we show that the category $\mathbf{Mod}(\mathbf{ALat})$, and thus $\mathbf{PEqu}$, models dependent type theory. Types are indexed objects of $\mathbf{Mod}(\mathbf{ALat})$; the indexing is by objects of $\mathbf{Mod}(\mathbf{ALat})$. The regular subobjects can be used to give us logic to reason about the types and with respect to which we have full subset types and full quotient types. See [18,24,26] for more on subset types and quotient types. The same holds for $\mathbf{Assm}(\mathbf{ALat})$, but here, in addition, the logic is higher order — in short, the point is that the regular subobject classifier is not an object of $\mathbf{Mod}(\mathbf{ALat})$ but it is an object of $\mathbf{Assm}(\mathbf{ALat})$; we explain this in more detail below.

All this works by analogy to the situation for modest sets and assemblies over a PCA. But the analogy seems to stop here; for example, the modest sets over a PCA form essentially an internal category in the corresponding category of assemblies and can be used to give a model of the calculus of constructions with an impredicative universe of types. We do not have a corresponding result with modest sets and assemblies over the category of algebraic lattices as we will explain.

Before embarking on the technical development, let us consider an example. Let Y be a closed type (an object of $\mathbf{Mod}(\mathbf{ALat})$) and let $\mathcal{N}$ denote the type of natural numbers. Further assume $u\colon Y \to \mathcal{N}$ in $\mathbf{Mod}(\mathbf{ALat})$. In the dependent type theory we can then form the type

$$\prod y\colon Y.\, \{\, n \in \mathcal{N} \mid n \geq u(y) \,\}$$

consisting of all functions, which, given a y produces an n greater or equal to $u(y)$. Here $\{\, n \in \mathcal{N} \mid n \geq u(y) \,\}$ is a well-formed (subset) type in the context $y\colon Y$.

For the technical development, we make use of B. Jacobs' fibrational description of models of dependent type theory [23,25,26], which is related to the D-categories [14], categories with attributes [12,30], display-map categories [40,21], and comprehensive fibrations [32]. See [23] for a comprehensive introduction. We make a point of describing the models in a so-called "split" way, so as to avoid problems with interpreting dependent type theory. See, for example, [29,34,31,35,17] for a discussion of this issue. As this section progresses, we assume more and more familiarity with the categories of modest sets, assemblies and realizability toposes over PCAs. See, for example, [19,22,33] for background on these categories.

We first define a category of uniform families of objects of the category $\mathbf{Mod}(\mathbf{ALat})$. Uniformity refers to the fact that each object of the family will have the same underlying algebraic lattice. The idea is that a dependent type, in a context interpreted as the object I, will be a family of objects indexed by the object I in $\mathbf{Mod}(\mathbf{ALat})$.

Definition 4.10 The category $\mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))$ is defined as follows.

(1) Objects are triples of the form $(I, A, (X_i, E_i)_{i \in X_I})$, where

$$I = (X_I, A_I, E_I) \in \mathbf{Mod}(\mathbf{ALat}) \quad \text{and} \\ (X_i, A, E_i) \in \mathbf{Mod}(\mathbf{ALat}), \text{ for all } i \in X_I.$$

(2) Morphisms from $(I, A, (X_i, E_i)_{i \in X_I})$ to $(J, B, (Y_j, E'_j)_{j \in X_J})$, with

$$I = (X_I, A_I, E_I) \qquad \text{and} \qquad J = (X_J, A_J, E_J),$$

are pairs of the form $(f, (f_i)_{i \in X_I})$, with

$$f \colon I \to J \text{ in } \mathbf{Mod}(\mathbf{ALat}) \qquad \text{and} \qquad f_i \colon X_i \to Y_{f(i)} \text{ in } \mathbf{Set},$$

for which there exists a $g \colon A_I \to A \to B$ in $\mathbf{ALat}$ such that g tracks f uniformly, that is,

$$\forall i \in X_I. \forall a_i \in E_I(i). \forall x \in X_i. \forall a \in E_i(x). g(a_i)(a) \in E'_{f(i)}(f_i(x));$$

(3) The identity morphism on an object $I = (X_I, A_I, E_I)$ is $(id, (id)_{i \in X_I})$.
(4) The composition of $(f, (f_i)_{i \in X_I})$ and $(g, (g_j)_{j \in X_J})$ is $(g \circ f, (g_{f(i)} \circ f_i)_{i \in X_I})$.

We think of a family $(I, A, (X_i, E_i)_{i \in X_I})$ as a type in context I, whose fiber at i in X_I is $(X_i, A, E_i)_{i \in X_I}$. There is an obvious forgetful functor

$$U \colon \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat})) \to \mathbf{Mod}(\mathbf{ALat})$$

given by $(I, A, (X_i, E_i)_{i \in X_I}) \mapsto I$ and $(f, (f_i)_{i \in X_I}) \mapsto f$.

Theorem 4.11 *The functor* $U \colon \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat})) \to \mathbf{Mod}(\mathbf{ALat})$ *is a split fibration which is equivalent, as a fibration, to the codomain fibration over* $\mathbf{Mod}(\mathbf{ALat})$.

Proof. First define split cartesian liftings: Suppose $u\colon I \to J$ in $\mathbf{Mod}(\mathbf{ALat})$ and let $(J, B, (Y_j, E'_j)_{j \in X_J})$ be an object over J. Then

$$(u, (id)_{i \in X_I})\colon (I, B, (Y_{u(i)}, E'_{u(i)})_{i \in X_I}) \to (J, B, (Y_j, E'_j)_{j \in X_J})$$

is the cartesian lifting over u.

Now consider the standard codomain fibration

$$\mathrm{cod}\colon \mathbf{Mod}(\mathbf{ALat})^{\to} \longrightarrow \mathbf{Mod}(\mathbf{ALat})$$

where, as usual, $\mathbf{Mod}(\mathbf{ALat})^{\to}$ is the category of commutative squares, with objects morphisms $\varphi\colon X \to I$ of $\mathbf{Mod}(\mathbf{ALat})$ and with morphisms from $\varphi\colon X \to I$ to $\psi\colon Y \to J$ pairs (u, f) of morphisms in $\mathbf{Mod}(\mathbf{ALat})$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

commutes.

Define the functor $\mathcal{P}$ as in

$$\begin{tikzcd} \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat})) \arrow[rr, "\mathcal{P}"] \arrow[dr] & & \mathbf{Mod}(\mathbf{ALat})^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathbf{Mod}(\mathbf{ALat}) & \end{tikzcd}$$

by mapping an object $(I, A, (X_i, E_i)_{i \in X_I})$, with $I = (X_I, A_I, E_I)$, to

$$(\textstyle\coprod_{i \in X_I} X_i, A_I \times A, E) \xrightarrow{\pi} I,$$

with $E(i, x) = E_I(i) \times E_i(x)$. The functor $\mathcal{P}$ maps a morphism

$$(u, (f_i)_{i \in X_I})\colon (I, A, (X_i, E_i)_{i \in X_I}) \to (J, B, (Y_j, E'_j)_{j \in X_J}),$$

with $I = (X_I, A_I, E_I)$ and $J = (X_J, A_J, E_J)$, to the square

$$\begin{tikzcd} (\coprod_{i \in X_I} X_i, A_I \times A, E) \arrow[r, "\{u,f\}"] \arrow[d, "\pi"'] & (\coprod_{j \in X_I} Y_j, A_J \times B, E') \arrow[d, "\pi"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

where $\{u, f\}$ is the function $(i, x) \mapsto (u(i), f_i(x))$ tracked by

$$\lambda(a_i, a).\, (r_u(a_i), g(a_i)(a))\colon A_I \times A \to A_J \times B,$$

with $r_u\colon A_I \to A_J$ a realizer for $u\colon I \to J$ and g a realizer for the family $(f_i)_{i \in X_I}$. This is, of course, a morphism in $\mathbf{ALat}$ since it is defined in the internal typed lambda calculus language of $\mathbf{ALat}$.

One can now verify that $\mathcal{P}$ is a full and faithful fibered functor. Moreover we can define a fibered functor $Q\colon \mathbf{Mod}(\mathbf{ALat})^{\to} \to \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))$ mapping $\varphi\colon X \to I$, with $I = (X_I, A_I, E_I)$ and $X = (X_X, A_X, E_X)$ to the family $(I, A_X, (X_i, E_i)_{i\in X_I})$ with $X_i = \varphi^{-1}(i)$ and $E_i(x) = E_X(x)$; a morphism (u, f) as in

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

is mapped by Q to $(u, (f)_{i\in X_I})$. It can then be verified that Q is also a fibered functor and that $PQ \cong id$ vertically and that $QP \cong id$ vertically. ■

Consider a type-in-context $(I, A, (X_i, E_i)_{i\in X_I})$. The functor $\mathcal{P}$, from the proof above, applied to this type-in-context yields the projection

$$(\textstyle\coprod_{i\in X_I} X_i, A_I \times A, E) \xrightarrow{\pi} I$$

morphism in $\mathbf{Mod}(\mathbf{ALat})$. This projection morphism gives rise to a substitution functor

$$\pi^*\colon \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))_I \to \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))_{(\coprod_{i\in X_I} X_i, A_I\times A, E)}.$$

We think of this functor as follows. It takes a type in context I and views it as a type in the extended context $(\coprod_{i\in X_I} X_i, A_I \times A, E)$, corresponding to the weakening rule

$$\frac{I \vdash X\colon \mathsf{Type} \qquad I \vdash Y\colon \mathsf{Type}}{I, x\colon X \vdash Y\colon \mathsf{Type}}$$

The interpretation of $I, x\colon X \vdash Y\colon \mathsf{Type}$ is the functor π^* applied to the interpretation of $I \vdash Y\colon \mathsf{Type}$. To model dependent sums and dependent products, we need to have left adjoints $\coprod$ and right adjoints $\prod$ to the functor π^*.

It is easy to see that $(I_{\mathbf{Set}}, 1_{\mathbf{ALat}}, (1_{\mathbf{Set}}, E_1)_{i\in X_I})$ is a terminal object in the fiber over $I = (X_I, A_I, E_I)$, where $E_1(*) = \{*\}$. The terminal object functor $1\colon \mathbf{Mod}(\mathbf{ALat}) \to \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))$ maps an object $I = (X_I, A_I, E_I)$ to the terminal object over I and a morphism $u\colon I \to J$ to the morphism $(u, (\lambda x.\, *)_{i\in X_I})$. This terminal object functor has a right adjoint

$$\{\}\colon \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat})) \to \mathbf{Mod}(\mathbf{ALat})$$

defined by, for $I = (X_I, A_I, E_I)$, $\{(I, A, (X_i, E_i)_{i\in X_I})\} = (\coprod_{i\in X_I} X_i, A_I \times A, E)$ with $E(i, x) = E_I(i) \times E_i(x)$. That is, $\{\} = \mathrm{dom} \circ \mathcal{P}$ where $\mathcal{P}$ was defined in the proof of the previous theorem. Briefly, if $(u, (f_i)_{i\in X_I})$ is a morphism from $1(I)$ to $(J, B, (Y_j, E_j)_{j\in X_J})$, with $I = (X_I, A_I, E_I)$ and $J = (X_J, A_J, E_J)$ then its adjoint transpose from I to $\{(J, B, (Y_j, E_j)_{j\in X_J})\}$ is $\lambda i.\, (u(i), f_i(*))$, realized by

$$\lambda a_i.\, \lambda a.\, (r_u(a), r_f(a_i)(*'))\colon A_I \to A \to B,$$

where r_u is a realizer for r and r_f is a realizer for the family $(f_i)_{i\in X_I}$. Thus the constructions are exactly analogous to the case for modest sets over a PCA. In summa, since the terminal object functor has a right adjoint and the projection functor $\mathcal{P}$ is full we have a *split full comprehension category with unit.*

Next, we argue that the compression category has split products. What this means is that, for any family $\mathcal{X} = (I, A, (X_i, E_i)_{i\in X_I})$ over $I = (X_I, A_I, E_I)$ with projection $\pi_{\mathcal{X}}\colon \{\mathcal{X}\} = (\coprod_{i\in X_I} X_i, A_I \times A, E) \to I$, the reindexing functor $\pi_{\mathcal{X}}^*$ has a right adjoint $\prod_{\mathcal{X}}$, which satisfies a Beck-Chevalley condition. Define

$$\prod\nolimits_{\mathcal{X}} \left((\textstyle\coprod_{i\in X_I} X_i, A_I \times A, E), C, (Z_k, E_k)_{k\in\coprod_{i\in X_I} X_i} \right)$$

to be

$$\Big(I, A \to C, (\{\, f\colon X_i \to \bigcup_{x\in X_i} Z_{(i,x)} \mid \forall x \in X_i. f(x) \in Z_{(i,x)} \,\}, E_i')_{i\in X_I}\Big),$$

where

$$\begin{aligned} E_i'(f) &= \{\, g\colon A \to C \mid \text{“g tracks f”} \,\} \\ &= \{\, g\colon A \to C \mid \forall x \in X_i. \forall a \in E_i(x). g(a) \in E_{(i,x)}(f(x)) \,\}. \end{aligned}$$

It is easy to verify that E_i' is modest. The adjoint transposes are defined essentially as for the case of the family of sets fibration; one just has to verify that one has the required realizers, but that is simple using the internal typed lambda calculus of **ALat**. Now for the Beck-Chevalley condition, we are to show that for a pullback

$$\begin{array}{ccc} (\coprod_{i\in X_I} X_{u(i)}, A_I \times B, E) & \xrightarrow{\{u,id\}} & (\coprod_{j\in X_J} X_j, A_J \times B, E') \\ \Big\downarrow{\scriptstyle \pi_{\mathcal{X}}} & & \Big\downarrow{\scriptstyle \pi_{\mathcal{Y}}} \\ I & \xrightarrow[u]{} & J \end{array}$$

in $\mathbf{Mod}(\mathbf{ALat})$, we have that the canonical natural transformation

$$u^* \prod\nolimits_{\mathcal{Y}} \to \prod\nolimits_{\mathcal{X}} \{u, id\}^*$$

is an *identity* (not only iso, because we claim to have *split* products). This is straightforward to verify.

For the comprehension category to have strong split coproducts (modeling dependent sums) we need, with notation as in the previous paragraph, first to have left adjoints $\coprod_{\mathcal{X}}$ to $\pi_{\mathcal{X}}^*$, for projections $\pi_{\mathcal{X}}$, satisfying a Beck-Chevalley condition. Define

$$\coprod\nolimits_{\mathcal{X}} \left((\textstyle\coprod_{i\in X_I} X_i, A_I \times A, E), C, (Z_k, E_k)_{k\in\coprod_{i\in X_I} X_i} \right)$$

to be

$$\Big(I, A \times C, (\{\, (x,z) \mid x \in X_i, z \in Z_{(i,x)} \,\}_i, E'_i)_{i \in X_I}\Big),$$

with $E'_i(x,z) = E_i(x) \times E_{(i,x)}(z)$, easily seen to be modest. On a morphism $(id, (f_{(i,x)})_{(i,x)\in\coprod_{i\in X_I} X_i})$ we define $\coprod_{\mathcal{X}}$ to give $(id, ((x,z) \mapsto (x, f_{(i,x)}(z)))_{i\in X_I})$, which is clearly realizable. Again it is straightforward to verify that the Beck-Chevalley condition holds, *i.e.*, referring to the pullback in the previous paragraph, that $\coprod_{\mathcal{X}}\{u, id\}^* \to u^* \coprod_{\mathcal{Y}}$ is an identity. This shows then that we have split coproducts. To have *strong* split coproducts, we have to show that the canonical map κ in the following diagram is an iso:

$$\begin{tikzcd} P \arrow[r, "\kappa"] \arrow[d, "\pi"'] & Q \arrow[d, "\pi"] \\ R \arrow[r, "\pi_{\mathcal{X}}"'] & I \end{tikzcd}$$

where

$$\begin{aligned} P &= \Big(\textstyle\coprod_{(i,x)\in\coprod_{i\in X_I} X_i} X_i, (A_I \times A) \times C, E\Big)\,, \\ Q &= \Big(\coprod_{i\in X_I} \{\, (x,z) \mid x \in X_i, z \in Z_{(i,x)} \,\}, A_I \times (A \times C), E'\Big)\,, \\ R &= (\coprod_{i\in X_I} X_i, A_I \times A, E'')\,. \end{aligned}$$

But κ is just the map $((i,x),z) \mapsto (i,(x,z))$, which is clearly realizable by the corresponding map on algebraic lattices, and obviously has an inverse. Hence we have strong coproducts.

We have thus shown the following theorem, with notation as in Theorem 4.11 and its proof.

Theorem 4.12 $\mathcal{P}\colon \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat})) \to \mathbf{Mod}(\mathbf{ALat})^{\to}$ *is a split closed comprehension category. Hence, we have a model of dependent type theory.*

We can use the regular subobjects to provide a logic with which one can reason about the types of the type theory. By Theorem 4.7, the regular subobjects of an object $I = (X_I, A_I, E_I)$ is isomorphic to $\mathcal{P}\,X_I$. Hence the category of regular subobjects of $\mathbf{Mod}(\mathbf{ALat})$, denoted $\mathbf{RegSub}(\mathbf{Mod}(\mathbf{ALat}))$, can be identified with the category with objects (I,K), where $I = (X_I, A_I, E_I) \in \mathbf{Mod}(\mathbf{ALat})$ and $K \subseteq X_I$ and with morphisms from (I,K) to (J,L) maps $u\colon I \to J$ in $\mathbf{Mod}(\mathbf{ALat})$ satisfying that $u(K) \subseteq L$. In the regular subobject fibration

$$\begin{tikzcd} \mathbf{RegSub}(\mathbf{Mod}(\mathbf{ALat})) \arrow[d] \\ \mathbf{Mod}(\mathbf{ALat}) \end{tikzcd}$$

reindexing of (J,L) along a map $u\colon I \to J$, *i.e.*, $u^*(J,L)$ is given by taking the inverse image of L along u.

One can use this regular subobject fibration to get a (classical) logic, essentially as for sets and for regular subobjects of the modest sets over a PCA. Moreover, with regard to this logic, the comprehension category $\mathcal{P}$ admits full (dependent) subset types and full (dependent) quotient types. However, for reasons of space, we do not spell that out here. Instead, let us mention that the above models of type theory can be also be defined, in the exact same way, for the category $\mathbf{Assm}(\mathbf{ALat})$ of assemblies over algebraic lattices. For this case, the logic of regular subobjects will be higher-order: the regular subobject fibration has a generic object, a regular subobject classifier, namely the object $\nabla 2 \in \mathbf{Assm}(\mathbf{ALat})$. Note that this is an object in $\mathbf{Assm}(\mathbf{ALat})$ which is not in $\mathbf{Mod}(\mathbf{ALat})$ since it is not modest. Again, this is analogous to the situation of modest sets and assemblies over a partial combinatory algebra [19,33,26].

Discussion. We should mention that the analogy with categories defined over a PCA can be made mathematically precise in the sense that there is a notion of a "weak tripos" — a tripos as in [20] except for the requirement of a generic object. For such a fibered preorder, one can define a category of assemblies and modest sets and show that they model dependent type theory. The tripos for a PCA will then provide an example, as will the weak tripos constructed over the category of algebraic lattices. The details will appear elsewhere.

We can also discuss just how far one can consider the analogy with categories defined over a PCA in an informal way and aimed at the reader already familiar with the situation for the categories defined over a PCA. We mainly highlight a couple of interesting questions.

One of the nice features of the modest sets and assemblies over a PCA is that they can be used to give a model of the calculus construction (see, *e.g.*, [22,29,35]). In fact, instead of the category of modest sets one uses the equivalent category of partial equivalence relations to get a *small* category. The crucial point is that this small category can be seen as an internal category in the category of assemblies and that the externalization of this internal category is a fibration equivalent to the fibration of uniform modest sets over the assemblies, which thus has a generic object allowing us to get an impredicative small universe of types as in the calculus of constructions.

An obvious next question is whether we can get something similar in our case with modest sets and assemblies over algebraic lattices. It turns out that, in our case working over algebraic lattices (or indeed any cartesian closed category), the fibration of uniform modest sets over assemblies is complete, but we cannot show that it is essentially small. This is not surprising since the category of algebraic lattices is not small. However, even if we only consider a *small* cartesian closed category as our category of realizers, the corresponding fibration is not small (is not equivalent to the externalization of an internal category).

The obvious solution to try, by analogy with the situation over a PCA, is to consider the small category of partial equivalence relations as an internal category in the category of assemblies (simply by embedding it via ∇ as is done for the case of PCAs), *but* then the externalization does not consist of *uniform* families: each set in the family will have a different underlying object of realizers. In fact, we have not been able to show that the fibration of partial equivalence relations is small and, indeed, we believe that it is not, unless further assumptions are made about the underlying category of realizers (besides it being a small cartesian closed category).

Another obvious question to ask, following the analogy with categories over a PCA, is whether $\mathbf{PER}(\mathbf{ALat}) \simeq \mathbf{Mod}(\mathbf{ALat})$ and $\mathbf{Assm}(\mathbf{ALat})$ embed fully and faithfully into a big "realizability topos over algebraic lattices" (such as the exact completion of the regular category $\mathbf{Assm}(\mathbf{ALat})$). The answer is no because $\mathbf{PER}(\mathbf{ALat})$ is not well-powered. For note that it embeds fully, faithfully by a finite limit preserving functor into the exact completion of $\mathbf{Assm}(\mathbf{ALat})$, and so the latter is also non-well-powered and, hence, not a topos. Again, even if we take a small cartesian closed category as the universe of realizers, it does not appear to be enough. To overcome this problem we tried to mimic the proof of Robinson and Rosolini [36], but it cannot be easily generalized. In other words, it appears that something more needs to be assumed about the universe of realizers, and we have to leave that as an open question.

5 Equilogical Spaces and Domains with Totality

Kleene-Kreisel countable functionals of finite type [27] occur in various models of computation. Ershov [15] placed them in a domain-theoretic setting, and Berger [5] worked out a general notion of totality for domain theory which subsumes Ershov's hierarchy of finite types. He also extended this approach to dependent types in his Habilitationsschrift [6]. We show that Berger's codense and dense objects in domain theory embed fully and faithfully in **PEqu**, from which it follows directly by the previous work of Ershov and Berger that the Kleene-Kreisel functionals are constructed in **PEqu** by repeated use of exponentiation starting from the natural numbers object. We begin this section with a quick overview of totality as defined by Berger [5]. Please refer to the original paper for details.

Domains with Totality. For our purposes, a *domain* $\mathcal{D} = \langle |\mathcal{D}|, \leq_{\mathcal{D}} \rangle$ is an algebraic consistently-complete directed-complete partially ordered set with a least element. We may view domains as topological spaces with their Σ-topologies, just as we did with complete lattices. Let **Dom** be the category of domains and continuous functions. Domains can also be considered as topologically closed non-empty subsets of algebraic lattices. Thus **ALat** is a full subcategory of **Dom**. Additionally **Dom** is a cartesian closed category (see, *e.g.*, [39] or [1]), and **ALat** is a full cartesian closed subcategory of **Dom**. A domain becomes an algebraic lattice if a "top" element is added to the poset. This construction produces a functor which, however, is not a reflection and it does not preserve the ccc-structure.

The following definitions are taken from Berger [5]. We follow the terminology of Berger [6] in which the term *total* has been replaced by the term *codense*. A subset $M \subseteq |\mathcal{D}|$ of a domain $\mathcal{D}$ is *dense* if it is dense in the topological sense, *i.e.*, the closure of M is $|\mathcal{D}|$. We write $x \uparrow y$ when elements $x, y \in |\mathcal{D}|$ are bounded, and $x \not\uparrow y$ when they are unbounded.

A finite subset $\{x_0, \dots, x_k\} \subseteq |\mathcal{D}|$ is *separable* if there exist open subsets $U_0, \dots, U_k \subseteq |\mathcal{D}|$ such that $x_0 \in U_0, \dots, x_k \in U_k$ and $U_0 \cap \cdots \cap U_k = \emptyset$. We say that $U_0, \dots, U_k$ *separate* $x_0, \dots, x_k$. It is easily seen that a finite set is separable if, and only if, it is unbounded. A family of open sets $\mathcal{U}$ is *separating* if it separates every separable finite set, *i.e.*, for every separable $\{x_0, \dots, x_k\} \subseteq |\mathcal{D}|$ there exist members of $\mathcal{U}$ that separate it.

The *boolean domain* $\mathbb{B}_\perp$ is the flat domain for the boolean values *tt* and *ff*. A *partial continuous predicate* (pcp) on a domain $\mathcal{D}$ is a continuous function $p \colon |\mathcal{D}| \to \mathbb{B}_\perp$. The function-space domain $[\mathcal{D} \to \mathbb{B}_\perp]$ is denoted by $\mathrm{pcp}(\mathcal{D})$. With each pcp p we associate two disjoint open sets by inverse images:

$$p^+ = p^{-1}(\{\mathit{tt}\}) \qquad \text{and} \qquad p^- = p^{-1}(\{\mathit{ff}\}).$$

A subset $P \subseteq |\mathrm{pcp}(\mathcal{D})|$ is *separating* if the corresponding family $\left\{p^+ \;\middle|\; p \in P\right\}$ is separating.

Given a set $M \subseteq |\mathcal{D}|$ let

$$\mathcal{E}(M) = \left\{p \in |\mathrm{pcp}(\mathcal{D})| \;\middle|\; \forall x \in M.\, p(x) \neq \perp\right\}.$$

A set M is *codense* in $\mathcal{D}$ if the family $\mathcal{E}(M)$ is separating. An element $x \in |\mathcal{D}|$ is codense if the singleton $\{x\}$ is codense in $\mathcal{D}$. Every element of a codense set is codense, but not every set of codense elements is codense. If $M \subseteq |\mathcal{D}|$ is a codense set then the consistency relation $\uparrow$ is an equivalence relation on M. Thus, a codense set $M \subseteq |\mathcal{D}|$ can be viewed as a domain $\mathcal{D}$ together with a partial equivalence relation $\approx_M$, which is just the relation $\uparrow$ restricted to M.

A *totality* on a domain, in the sense of Berger [5], is a dense and codense subset of a domain. Note that in the original paper by Berger [5] codense sets are called *total*. Here we are using the newer terminology of Berger [6].

Given domains with totality $M \subseteq |\mathcal{D}|$ and $N \subseteq |\mathcal{E}|$, it is easily seen that the set $M \times N \subseteq |\mathcal{D}| \times |\mathcal{E}|$ is again a totality on the domain $\mathcal{D} \times \mathcal{E}$. Similarly, by the Density Theorem in Berger [5] the set

$$\langle M, N \rangle = \left\{f \in [\mathcal{D} \to \mathcal{E}] \;\middle|\; f(M) \subseteq N\right\}$$

is a totality on the function-space domain $[\mathcal{D} \to \mathcal{E}]$. This idea of totality generalizes the simple-minded connection between total and partial functions using flat domains. If A is any set, let $A_\perp$ be the flat domain obtained by adding a bottom element. Then A itself is a totality on $A_\perp$, and the total functions of $A \to B$ in **Set** correspond to (equivalence classes) of functions in $\langle A, B\rangle$ considered as elements of $[A_\perp \to B_\perp]$.

Partial Equivalence Relations. Let $\mathbf{PER}(\mathbf{Dom})$ be the category formed just like **PEqu** except that domains are used instead of algebraic lattices, *i.e.*, an object of $\mathbf{PER}(\mathbf{Dom})$ is a structure $\mathcal{D} = \langle |\mathcal{D}|, \leq_\mathcal{D}, \approx_\mathcal{D}\rangle$ where $\langle |\mathcal{D}|, \leq_\mathcal{D}\rangle$ is a domain and $\approx_\mathcal{D}$ is a partial equivalence relation on $|\mathcal{D}|$. Category $\mathbf{PER}(\mathbf{Dom})$ is cartesian closed, and for $\mathcal{D}, \mathcal{E} \in \mathbf{PER}(\mathbf{Dom})$ we choose the canonical product and exponential $\mathcal{D} \times \mathcal{E}$ and $\mathcal{D} \to \mathcal{E}$ whose underlying domains are the standard product and exponential in **Dom**, and the partial equivalence relations are defined by

$$
\begin{aligned}
(x_1, y_1) \approx_{\mathcal{D}\times\mathcal{E}} (x_2, y_2) &\iff x_1 \approx_\mathcal{D} x_2 \wedge y_1 \approx_\mathcal{E} y_2 \\
f \approx_{\mathcal{D}\to\mathcal{E}} g &\iff \forall x, y \in |\mathcal{D}|. \Big(x \approx_\mathcal{D} y \Longrightarrow f(x) \approx_\mathcal{E} g(y)\Big).
\end{aligned}
$$

We say that a partial equivalence relation $\approx_\mathcal{D}$ on a domain $\mathcal{D}$ is *dense* when its domain

$$\mathrm{dom}(\approx_\mathcal{D}) = \Big\{x \in |\mathcal{D}| \;\Big|\; x \approx_\mathcal{D} x\Big\}$$

is a dense subset of $\mathcal{D}$.

Because every algebraic lattice is a domain, **PEqu** is a full subcategory of $\mathbf{PER}(\mathbf{Dom})$. The top-adding functor $T\colon \mathbf{PER}(\mathbf{Dom}) \to \mathbf{PEqu}$ maps an object $\mathcal{D} \in \mathbf{PER}(\mathbf{Dom})$ to the object

$$T(\mathcal{D}) = \langle |\mathcal{D}| \cup \{\top\}, \Omega_{T(\mathcal{D})}, \approx_\mathcal{D}\rangle$$

where $\langle |\mathcal{D}| \cup \{\top\}, \Omega_{T(\mathcal{D})}\rangle$ is the algebraic lattice obtained from the underlying domain of $\mathcal{D}$ by attaching a compact top element. Functor T maps a morphism $[f]\colon \mathcal{D} \to \mathcal{E}$ to the morphism $T([f])$ represented by the map

$$T(f)(x) = \begin{cases} f(x) & x \neq \top \\ \top & x = \top. \end{cases}$$

The top-adding functor is a product-preserving reflection, hence **PEqu** is an exponential ideal and a sub-ccc of $\mathbf{PER}(\mathbf{Dom})$.

In category **Dom** it is *not* the case that every continuous map $f\colon D' \to |\mathcal{E}|$ defined on an arbitrary non-empty subset $D' \subseteq |\mathcal{D}|$ has a continuous extension to the whole domain $|\mathcal{D}|$. Because of this fact the category $\mathbf{PER}(\mathbf{Dom})$ has certain undesirable properties. However, it is true that every continuous map defined on a *dense* subset has a continuous extension; this is an easy consequence of the Extension Theorem and the fact that a domain becomes an algebraic lattice when a top element is added to it. These observations suggest that we should consider only the dense partial equivalence relations on domains.

Let **DPER**(**Dom**) be the full subcategory of **PER**(**Dom**) whose partial equivalence relations are either *dense* or *empty*. We are including the empty equivalence relation here because the only map from an empty subset always has a continuous extension. The objects whose partial equivalence relations are empty are exactly the initial objects of **DPER**(**Dom**). We have the following theorem.

Theorem 5.1 **DPER**(**Dom**) *and* **PEqu** *are equivalent.*

Proof. In one direction, the equivalence is established by the top-adding functor $T\colon \mathbf{DPER}(\mathbf{Dom}) \to \mathbf{PEqu}$. In the other direction, the equivalence functor $K\colon \mathbf{PEqu} \to \mathbf{DPER}(\mathbf{Dom})$ is defined as follows. When $\mathcal{A} = (|\mathcal{A}|, \Omega_{\mathcal{A}}, \emptyset)$ is an initial object, define $K(\mathcal{A}) = \mathcal{A}$. Otherwise K maps an object $\mathcal{A} \in \mathbf{PEqu}$ to an object $K(\mathcal{A})$ whose underlying domain is the set $|K(\mathcal{A})| = \overline{\mathrm{dom}(\approx_{\mathcal{A}})}$, which is the topological closure of $\mathrm{dom}(\approx_{\mathcal{A}})$ in $|\mathcal{A}|$, equipped with the subspace topology. The partial equivalence relation for $K(\mathcal{A})$ is just $\approx_{\mathcal{A}}$ restricted to $|K(\mathcal{A})|$. The functor K maps a morphism $[f]\colon \mathcal{A} \to \mathcal{B}$ to the morphism represented by the restriction $f \restriction_{|K(\mathcal{A})|}$. Here we assume that the morphism from an initial object $\mathcal{A} = (|\mathcal{A}|, \emptyset)$ is represented by the constant map $f\colon x \mapsto \bot$. If $\mathcal{A}$ is initial, $K([f])$ is obviously well defined. When $\mathcal{A}$ is not initial, $K([f])$ is well defined because continuity of f implies that

$$f(|K(\mathcal{A})|) = f(\overline{\mathrm{dom}(\approx_{\mathcal{A}})}) \subseteq \overline{f(\mathrm{dom}(\approx_{\mathcal{A}}))} \subseteq \overline{\mathrm{dom}(\approx_{\mathcal{B}})} = |K(\mathcal{B})|.$$

It is easily checked that K and T establish an equivalence between **PEqu** and **DPER**(**Dom**). ∎

We would like to represent domains with totality as equilogical spaces. If $M \subseteq |\mathcal{D}|$ is codense and dense in $\mathcal{D}$, let $\langle \mathcal{D}, \sim_M \rangle$ be the object of **PER**(**Dom**) whose underlying domain is $\mathcal{D}$ and the partial equivalence relation $\sim_M$ is the relation $\uparrow$ on M. This identifies domains with totality as objects of the category **DPER**(**Dom**). The following result shows that the morphisms of **DPER**(**Dom**) are the right ones, because the ccc structure of **DPER**(**Dom**) agrees with the formation of products and function-space objects with totality.

Theorem 5.2 *Let $M \subseteq |\mathcal{D}|$, $N \subseteq |\mathcal{E}|$ be codense and dense subsets in domains $\mathcal{D}$ and $\mathcal{E}$, respectively. Then in* **DPER**(**Dom**)

$$\langle \mathcal{D}, \sim_M \rangle \times \langle \mathcal{E}, \sim_N \rangle = \langle \mathcal{D} \times \mathcal{E}, \sim_{M \times N} \rangle, \qquad \textit{and}$$
$$\langle \mathcal{D}, \sim_M \rangle \to \langle \mathcal{E}, \sim_N \rangle = \langle [\mathcal{D} \to \mathcal{E}], \sim_{\langle M, N \rangle} \rangle.$$

Proof. Here it is understood that the product $\langle \mathcal{D}, \sim_M \rangle \times \langle \mathcal{E}, \sim_N \rangle$ and the exponential $\langle \mathcal{D}, \sim_M \rangle \to \langle \mathcal{E}, \sim_N \rangle$ are the canonical ones for **PER**(**Dom**). They are objects in **DPER**(**Dom**) by the Density Theorem in Berger [5]. The first equality follows from the observation that $(x_1, y_1) \uparrow (x_2, y_2)$ if, and only if, $x_1 \uparrow x_2$ and $y_1 \uparrow y_2$. Let $\mathcal{X} = \langle \mathcal{D}, \sim_M \rangle \to \langle \mathcal{E}, \sim_N \rangle$ and $\mathcal{Y} = \langle [\mathcal{D} \to \mathcal{E}], \sim_{\langle M,N \rangle} \rangle$. Objects $\mathcal{X}$ and $\mathcal{Y}$ have the same underlying domains, so we only have to show that the two partial equivalence relations coincide. The partial equivalence relation on $\mathcal{X}$ is

$$f \approx_{\mathcal{X}} g \iff f, g \in \langle M, N \rangle \text{ and } \forall x, y \in M. \big(x \uparrow y \Longrightarrow f(x) \uparrow g(y) \big).$$

Suppose $f \approx_{\mathcal{X}} g$. Then $f, g \in \langle M, N \rangle$ and it remains to be shown that $f \uparrow g$. For every $x \in M$, since $x \uparrow x$ and $f \approx_{\mathcal{X}} g$, $f(x) \uparrow g(x)$, thus by Lemma 7 in Berger [5] f and g are inseparable, which is equivalent to them being bounded. Conversely, suppose $f, g \in \langle M, N \rangle$ and $f \uparrow g$. For every $x, y \in M$ such that $x \uparrow y$, it follows that $f(x) \uparrow g(y)$ because $f(x) \leq (f \vee g)(x \vee y)$ and $g(y) \leq (f \vee g)(x \vee y)$. This means that $f \approx_{\mathcal{X}} g$. ∎

Higher Types. The category **PEqu** is a full sub-ccc of **PER**(**Dom**). Since **DPER**(**Dom**) is a full subcategory of **PER**(**Dom**) and is equivalent to **PEqu**, it is a full sub-ccc of **PER**(**Dom**) as well. Theorem 5.2 states that for codense and dense subsets $M \subseteq |\mathcal{D}|$ and $N \subseteq |\mathcal{E}|$, the exponential $\langle \mathcal{D}, \approx_M \rangle \to \langle \mathcal{E}, \approx_N \rangle$ coincides with the object $\langle [\mathcal{D} \to \mathcal{E}], \approx_{\langle M,N \rangle} \rangle$. We may use this to show that in **PEqu** the countable functionals of finite types arise as iterated function spaces of the natural numbers object. For simplicity we only concentrate on pure finite types ι, $\iota \to \iota$, $(\iota \to \iota) \to \iota$, ... and skip the details of how to extend this to the full hierarchy of finite types generated by ι, o, $\times$, and $\to$.

The natural numbers object in **DPER**(**Dom**) is the object

$$\mathcal{DN}_0 = \langle \mathbb{N}_\perp, \leq_{\mathbb{N}_\perp}, \approx_{\mathcal{DN}_0} \rangle$$

whose underlying domain is the flat domain of natural numbers $\mathbb{N}_\perp = \mathbb{N} \cup \{\perp\}$ and the partial equivalence relation $\approx_{\mathcal{DN}_0}$ is the restriction of identity to $\mathbb{N}$. Define the hierarchy $\mathcal{DN}_1, \mathcal{DN}_2, \ldots$ inductively by

$$\mathcal{DN}_{j+1} = \mathcal{DN}_j \to \mathcal{DN}_0$$

where the arrow is formed in **DPER**(**Dom**). By Theorem 5.2, this hierarchy is contained in **DPER**(**Dom**) and corresponds exactly to Ershov's and Berger's construction of countable functionals of pure finite types. It is well known that the equivalence classes of $\mathcal{DN}_j$ correspond naturally to the original Kleene-Kreisel countable functionals of pure type j, see Berger [5] or Ershov [15].

In **PEqu** the natural numbers object is

$$\mathcal{N}_0 = \langle \mathbb{N}_{\perp,\top}, \leq_{\mathbb{N}_{\perp,\top}}, \approx_{\mathcal{N}_0} \rangle,$$

where $\mathbb{N}_{\perp,\top} = \mathbb{N} \cup \{\perp, \top\}$ is the algebraic lattice of flat natural numbers with bottom and top, and $\approx_{\mathcal{N}_0}$ is the restriction of identity to $\mathbb{N}$. The iterated function spaces $\mathcal{N}_1, \mathcal{N}_2, \ldots$ are defined inductively by

$$\mathcal{N}_j = \mathcal{N}_{j-1} \to \mathcal{N}_0.$$

The hierarchies $\mathcal{DN}_0, \mathcal{DN}_1, \ldots$ and $\mathcal{N}_0, \mathcal{N}_1, \ldots$ correspond to each other in view of the equivalence between $\mathbf{DPER}(\mathbf{Dom})$ and $\mathbf{PEqu}$, because they are both built from the natural numbers object by iterated use of exponentiation, hence the equivalence classes of $\mathcal{N}_j$ correspond naturally to the Kleene-Kreisel countable functionals of pure type j.

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