HSP type theorems in the category of posets

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1 Introduction

The origin of this paper was a question by Vaughan Pratt about possible HSP theorems for theories based on the category of posets. As it happens, he had a different (more special) notion of theory based on posets than the one used here, but it is possible to specialize the results here to Pratt's notion.

Pratt gives the following explanation of his interest in the results:

> As a computer science application of our techniques we may take the action logic of [Pratt, 1990]. Action logic is a finitely based equational theory conservatively extending the equational theory of regular expressions. An action algebra (model of action logic) is implicitly partially ordered via $a \le b$ just when $a + b = b$. However $a + b$ plays no other essential role in action logic, and may be dropped provided we introduce the relation $\le$ explicitly into the language. The equations defining the residuals $a \longrightarrow b$ and $a \leftarrow b$ and for star a^* may then be rephrased as inequalities. This then raises the sorts of foundational questions addressed in [the present] paper; indeed [the present] paper was in direct response to my having raised them myself.

A theory is a category equipped with certain structures and a model of that theory is a functor (usually to sets, although various kinds of generalizations are possible) that preserves that structure. Basically, the theory is a kind of prototype of a certain kind of mathematical entity.

For example, the theory of monoids is a category $\mathscr{T}$ with finite limits that contains a monoid object M_0 with the property that for any other category $\mathscr{C}$ with finite limits and any monoid object M in $\mathscr{C}$ there is one and only one functor $T\colon \mathscr{T} \longrightarrow \mathscr{C}$ that preserves finite limits such that $T(M_0) = M$ and such that the monoid structure in M is consistent with that of M_0.

In the case at hand, we wish to deal with theories that take advantage of the special properties of ordered sets. Among other things, this means that the domains of

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operations can be specified in terms of the ordered structure and that the output of an operation can also be an ordered structure.

An HSP subcategory of the category of models is a subcategory that is closed under quotients ('homomorphic images'), subobjects and products. For example, the subcategory of commutative monoids is an HSP subcategory of the category of monoids. Then the HSP theorem says that HSP subcategories are determined by equations in the theory (and conversely, but the converse is trivial).

When the theory is based on posets, things get rather more complicated. For one thing, there are now choices to be made. What is a subobject? A quotient? And what replaces equation? Of course, perhaps it is equation, but that seems too closely tied to sets. After all, equality is the only first order predicate in set theory, while posets have inequality as well. And indeed, for the appropriate notion of HSP, you may need inequalities to cut out HSP subcategories. Unfortunately, that is not always sufficient. You may also have to iterate the construction, even transfinitely often.

We adopt the useful convention of denoting the set of morphisms in category $\mathscr{C}$ from an object C to C' by $\mathscr{C}(C, C')$.

2 Functorial semantics for posets

2.1 Theories on posets. Let **Pos** be the category of posets and monotone (increasing) functions. By a **theory** over **Pos** we mean a category $\mathscr{T}$ and a functor $F\colon \mathbf{Pos} \longrightarrow \mathscr{T}^{\mathrm{op}}$ that is an isomorphism on objects and that has a left adjoint. We will suppose one more property. To explain this property, let $\mathscr{K} = \mathscr{T}^{\mathrm{op}}$. In view of the fact that F is an isomorphism on objects, we can treat $\mathscr{T}$ as a category whose objects are all posets and whose morphisms include, but are not limited to the monotone functions. This will become much clearer after we see examples. (Actually, since F is not assumed faithful, it would also be possible that two distinct functions become equal as morphisms in $\mathscr{K}$; it is nonetheless useful to think of the situation as has just been described.) The additional condition we want is that the countable (including finite) objects are a regular generating family. This means that in order to have an arrow $A \longrightarrow B$ it is sufficient to have arrows $f_i\colon A_i \longrightarrow B$ for every subobject $A_i \rightarrowtail A$ with A_i countable subject only to the condition that f_i agree with f_j on $A_i \cap A_j$. Moreover, two morphisms $A \longrightarrow B$ that agree on every countable subobject of A are equal. This condition is far more stringent that what is really necessary (which is that the category $\mathscr{K}$ and the functor F be accessible in the sense of Makkai and Paré, [1989]), but is certainly satisfied by any theory of interest to computer science.

2.2 Models. Classically one worked with theories, but we will find it convenient to work with the dual category $\mathscr{K}$. Let $F\colon \mathbf{Pos} \longrightarrow \mathscr{K}$ be a theory as above. A **model** of the theory is a pair (P, M) in which $M\colon \mathscr{K}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a functor and P is a poset such that $M \circ F^{\mathrm{op}} = \mathbf{Pos}(-, P)$. A morphism $(f, \phi): (P, M) \longrightarrow (P', M')$ is a pair such that ϕ is a natural transformation and f is a monotone function such that for any poset A the square

$$\begin{array}{ccc} M(F(A)) & \xrightarrow{\phi F(A)} & M'(F(A)) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ \mathbf{Pos}(A, P) & \xrightarrow[\mathbf{Pos}(A, f)]{} & \mathbf{Pos}(A, P') \end{array}$$

commutes. The Yoneda lemma insures that given a natural transformation $\phi: M \longrightarrow M'$ there is a unique morphism between the representing objects. Hence

$$\mathrm{Mod}(\mathscr{K})((P, M), (P', M')) \cong NT(M, M')$$

the set of natural transformations.

We call the resultant category $\mathrm{Mod}(\mathscr{K})$, the category of models of $\mathscr{K}$. The functor $U: \mathrm{Mod}(\mathscr{K}) \longrightarrow \mathbf{Pos}$ that takes (M, P) to P and (f, ϕ) to f we call the **underlying poset** functor.

2.3 Seeds. By a **seed** of a theory, we mean a diagram

$$\begin{array}{ccccc} & & \mathscr{P} & & \\ & {\scriptstyle V}\swarrow & & \searrow{\scriptstyle F} & \\ \mathbf{Pos} & & & & \mathscr{K}_0 \end{array}$$

in which $\mathscr{P}$ is a graph and $V: \mathscr{P} \longrightarrow \mathbf{Pos}$ is the inclusion of a subgraph and $\mathscr{K}_0$ is a graph with diagrams and $F: \mathscr{P} \longrightarrow \mathscr{K}_0$ is a graph morphism that is an isomorphism on nodes (objects). By a graph, we mean a "category without composition". That is it has objects (often called nodes) and arrows, but no way of composing them. Of course, one can compose them freely to produce paths made up from these arrows and a diagram is a pair of paths that start at the same node and finish at the same node. A model in a category of a graph with diagrams consists of functions that takes objects to objects and arrows to arrows and preserves source and target. Moreover, it must take diagrams to commutative diagrams. The details are found in Barr & Wells [1985, 1990].

Given a seed of a theory as above, a **model** is a pair (P, M) where M is a model of $\mathscr{K}_0^{\mathrm{op}}$ and P a poset such that for P_0 in $\mathscr{P}$, we have $MFP_0 = \mathbf{Pos}(P_0, P)$.

A seed is called **small** if the nodes are a set. A small seed gives rise to a theory as we will see.

2.4 The theory associated with a small seed. Let

$$\mathbf{Pos} \xleftarrow{\;V\;} \mathscr{P} \xrightarrow{\;F\;} \mathscr{K}_0$$

be a small seed. Let $\mathrm{Mod}(\mathscr{K}_0)$ denote the category of models. There is a functor $U\colon \mathrm{Mod}(\mathscr{K}_0) \longrightarrow \mathbf{Set}$ defined by $U(P, M) = P$ and $U(f, \phi) = f$. This is called the **underlying** functor of the model category.

2.5 Proposition. *Let* $\mathbf{Pos} \longleftarrow \mathscr{P} \longrightarrow \mathscr{K}_0$ *be a small seed. The underlying functor* $\mathrm{Mod}(\mathscr{K}_0) \longrightarrow \mathbf{Pos}$ *has a left adjoint.*

This is proved in Barr [to appear]. The proof is based on results contained in the remarkable book of Makkai & Paré [1990] on accessible categories.

Given this fact, we can now construct the theory associated to any seed. Let $F\colon \mathbf{Pos} \longrightarrow \mathrm{Mod}(\mathscr{K}_0)$ be the free functor. Let $\mathscr{K}$ be the full subcategory of $\mathrm{Mod}(\mathscr{K})$ consisting of all objects of the form FP for P a poset. This category is known as the **Kleisli category** of the adjunction. By an abuse of notation we will let $F\colon \mathbf{Pos} \longrightarrow \mathscr{K}$ also denote the functor to that subcategory. Then it is easy to see that $F^{\mathrm{op}}\colon \mathbf{Pos}^{\mathrm{op}} \longrightarrow \mathscr{T} = \mathscr{K}^{\mathrm{op}}$ is a theory as defined. Less obvious, but also proved in the forthcoming paper, [Barr, to appear] is the fact that $\mathrm{Mod}(\mathscr{K})$ is isomorphic to $\mathrm{Mod}(\mathscr{K}_0)$. Thus a seed is equivalent to a theory.

2.6 Example. Here is an example. We define a seed $\mathscr{K}_0$. Here and elsewhere, we let $\mathbf{n}$ stand for the chain of length n. So that $\mathbf{2}$ stands for the 2 element chain, while $2 = 1 + 1$ is the discrete set with 2 elements. We let $\dot{+}$ stand for the ordinal sum of ordered sets, so that $\mathbf{2} = 1 \dot{+} 1$.

This seed consists of $\mathscr{P}$ which is the graph with two objects $1 = \mathbf{1}$ and $\mathbf{2}$ of $\mathbf{Pos}$ and no arrows. The graph $\mathscr{K}_0$ is $\mathscr{P}$ together with just one arrow $\sigma\colon 1 \longrightarrow \mathbf{2}$. A model (P, M) is determined by $M(\sigma)\colon M(\mathbf{2}) \longrightarrow M(1)$ since M is already determined by its value on maps in $\mathbf{Pos}$. Now $M(\mathbf{2}) = \mathbf{Pos}(\mathbf{2}, P)$ and $M(1) = \mathbf{Pos}(1, P)$. The former is the set of pairs of elements $x \le y$, while the latter is simply the set of elements of P. Thus to have an M is to have a function $M(\sigma)$ that assigns to each pair of elements $x \le y$ an element $M(\sigma(x, y))$. Since M takes values in the category of sets, it is not assumed that $M(\sigma)$ preserves order. As we will see, that can be added as a condition, if wanted.

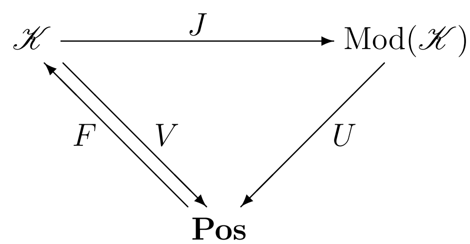
2.7 We have indicated that the underlying functor from the category of models of a small seed has an adjoint. The same is true for a full theory; the proof is entirely different.

2.8 Proposition. *Let* $F\colon \mathbf{Pos} \longrightarrow \mathscr{K}$ *be a theory. Then the underlying functor* $U\colon \mathrm{Mod}(\mathscr{K}) \longrightarrow \mathbf{Pos}$ *has a left adjoint.*

Proof. We begin by defining a functor $J\colon \mathscr{K} \longrightarrow \mathrm{Mod}(\mathscr{K})$ which is, up to equivalence, the Yoneda embedding. Let V be right adjoint to F. For an object FA of $\mathscr{K}$, we let $J(FA)$ be the pair (VFA, M) where $M(FB) = \mathbf{Pos}(B, VFA) \cong \mathscr{K}(FB, FA)$. If $f\colon FA' \longrightarrow FA$ is an arrow, then Jf is the composite

$$\mathbf{Pos}(B, VFA') \cong \mathscr{K}(FB, FA') \xrightarrow{\mathscr{K}(FB,f)} \mathscr{K}(FB, FA) \longrightarrow \mathbf{Pos}(B, VFA)$$

Then we have the situation



with F left adjoint to V. I claim that JF is left adjoint to U. In fact,

$$\begin{aligned}\mathrm{Mod}(\mathscr{K})(JFA, (M'P')) &\cong NT(\mathscr{K}(-, FA), M') \cong M'FA \\ &\cong \mathbf{Pos}(A, P') \cong \mathbf{Pos}(A, U(P', M'))\end{aligned}$$

This completes the proof of the adjunction. □

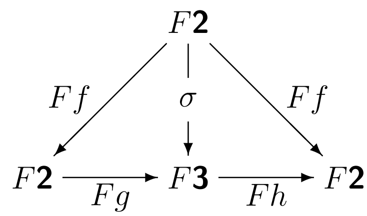
3 HSP subcategories

We now come to the main results of this paper. In traditional universal algebra an HSP subcategory is a full subcategory of a category of algebras that is closed under homomorphic images, subobjects and products. In fact, the idea predates category theory, when these were called HSP classes. The main theorem on the subject is that HSP classes are exactly the classes of algebras that satisfy a set of equations. An equation is asserts the equality of two terms constructed from the theory.

We will give examples to show that the proposition in this form is not true. Moreover if it were, it wouldn't be as interesting. What is true is almost as nice and is considerably more interesting.

The reason is that in a poset one wants to be able to say not merely that two terms are equal, but also be able to say that one term is less than or equal to another. In other words, to specify inequalities as well as equalities. On the other hand, the result is false because although an HSP subcategory of an HSP subcategory is an HSP subcategory, it can happen that one can impose equalities or inequalities on a theory and get a new theory that allows new equalities or inequalities involving terms that weren't even in the previous theory. This in turn results from the fact that, as soon as you leave the category of sets as a base, it is no longer necessarily the case that the image of a morphism between two models is a model.

This is best shown by example. Consider a seed that has four arrows as shown in the diagram:

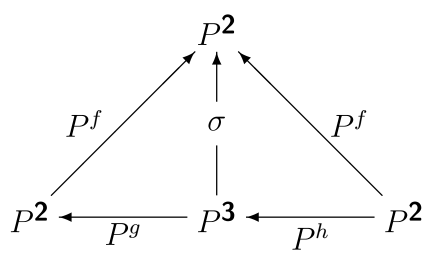


The arrow $f: \mathbf{2} \longrightarrow \mathbf{2}$ takes both elements of $\mathbf{2}$ to the larger element; the arrows $g, h: \mathbf{3} \longrightarrow \mathbf{2}$ are the two surjective arrows in **Pos**. The two paths in the diagram above are each paths in this seed.

An algebra for this seed is a poset P equipped with an operation

$$M(\sigma): \mathbf{Pos}(\mathbf{3}, P) \longrightarrow \mathbf{Pos}(\mathbf{2}, P)$$

which we will abbreviate $\sigma: P^{\mathbf{3}} \longrightarrow P^{\mathbf{2}}$ such that the diagram



commutes. When this is all sorted out, it means that there is a pair of $\mathbf{3}$-ary operations, say σ_0 and σ_1 such that for all $x \le y \le z$,

$$\sigma_0(x, y, z) \le \sigma_1(x, y, z)$$

and that for $x \le y$,

$$\sigma_0(x, x, y) = \sigma_1(x, x, y) = \sigma_0(x, y, y) = \sigma_1(x, y, y) = y$$

One easily discovers from these data that the free model generated by the three element poset $x \le y \le z$ contains just five elements, $x \le y \le z$ and $\sigma_0(x, y, z) \le \sigma_1(x, y, z)$. The equations that a model must satisfy force that all other terms are equal to one of these five. Also freeness forces there to be no other inequalities than the ones already shown. Now let us add the equation $z = \sigma(x, y, z)$. The original five terms have shrunk to four, but now there are new terms such as $\sigma_1(x, y, \sigma_1(x, y, z))$,

$\sigma_1(x, z, \sigma_1(x, y, z))$, $\sigma_1(y, \sigma_1(x, y, z), \sigma_1(y, \sigma_1(x, y, z)))$ and infinitely many more. Moreover, there is a possibility of new equations being added, equations that could not be stated in the original theory because the necessary terms didn't exist. For example, such an equation as

$$\sigma_1(x, y, \sigma_1(x, y, z)) = \sigma_1(x, z, \sigma_1(x, y, z))$$

Conceivably, this new equation could introduce new terms and new possibilities, allowing new equations and so *ad infinitum*.

When the base category is sets, equations can never introduce new terms; they can only introduce identifications among terms already present.

One more point is that we might want to be able to consider subcategories determined by inequalities rather than just equalities. In order to do this, we have to look more carefully at what we mean by an HSP subcategory. The reason is that whereas a subobject of an object that satisfies an equality still satisfies that equality, the same is not true for inequalities, unless we restrict our notion of subobject. For example, 2 is a subobject of **2**, but the inequality of the latter is not satisfied in the former. One might decide to stick to regular subobjects (which means, in the context of poset, that they have the inherited order relation). But in fact, the best approach is to leave the notion of subobject as a parameter.

3.1 Factorization systems. By a factorization system on a category $\mathscr{C}$ we mean a pair $\mathscr{E}/\mathscr{M}$ of classes of arrows of the category such that

1. $\mathscr{E} \cap \mathscr{M}$ is the class of isomorphisms;

2. $\mathscr{E}$ and $\mathscr{M}$ are closed under composition;

3. Every arrow $f \in \mathscr{C}$ can be factored as $f = m \circ e$ with $m \in \mathscr{M}$ and $e \in \mathscr{E}$;

4. In any commutative square

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow[m]{} & D \end{array}$$

   with $m \in \mathscr{M}$ and $e \in \mathscr{E}$ there is an arrow $B \longrightarrow C$ making both triangles commute. This arrow is called the "diagonal fill-in".

5. Every morphism in $\mathscr{E}$ is an epimorphism and every morphism in $\mathscr{M}$ is a monomorphism.

The last condition is not always assumed. Many of the ideas can work without it and there are certain cases in which that is useful. In that case you will usually want to strengthen the third condition to say that the diagonal fill-in is unique.

Let us say that an arrow e is **orthogonal** to an arrow m if that diagonal fill-in condition is satisfied with respect to those two arrows. We will say that a class $\mathscr{E}$ of arrows is orthogonal to a class $\mathscr{M}$ if the diagonal fill-in condition is satisfied between all the arrows in the first and all the arrows in the second. Note that this orthogonality relation is not symmetric.

It is straightforward to show that in a factorization system, the two classes determine each other. Given $\mathscr{E}$, $\mathscr{M}$ consists of all the arrows in the category that $\mathscr{E}$ is orthogonal to and given $\mathscr{M}$, $\mathscr{E}$ consists of all the arrows that are orthogonal to $\mathscr{M}$. Similarly, if we have two distinct factorization systems, $\mathscr{E}/\mathscr{M}$ and $\mathscr{E}'/\mathscr{M}'$, then $\mathscr{E} \subseteq \mathscr{E}'$ if and only if $\mathscr{M}' \subseteq \mathscr{M}$. Moreover the one inclusion is strict if and only if the other is.

In well-behaved categories (categories with pullbacks and pushouts in which every class of subobjects of an object have an intersection and dually), there are two extreme factorization systems. The first has for $\mathscr{M}_0$ all monomorphisms and for $\mathscr{E}_0$ all arrows orthogonal to $\mathscr{M}_0$. The second has for $\mathscr{E}_1$ the class of all epimorphisms and for $\mathscr{M}_1$ all arrows that $\mathscr{E}_1$ is orthogonal to.

An epimorphism that is orthogonal to every monomorphism is called an **extremal** epimorphism. Assuming the category has pullbacks, an epimorphism is extremal if and only if it factors through no proper subobject of the codomain. The dual notion is called an extremal monomorphism. These are the two extreme cases for if $\mathscr{E}/\mathscr{M}$ is any other factorization system, from $\mathscr{M} \subseteq \mathscr{M}_0$ and $\mathscr{E} \subseteq \mathscr{E}_1$, it follows that $\mathscr{E}_0 \subseteq \mathscr{E}$ and $\mathscr{M}_1 \subseteq \mathscr{M}$.

In **Pos** we have the two extreme factorization systems. There is at least one intermediate one and may be more. Note first that all monics are injective and all epics are surjective. This puts certain constraints on factorization systems. For example, it is easy to see that the arrow $2 \longrightarrow \mathbf{2}$ is both epic and monic and cannot be factored into an epic followed by a monic unless one of the two is an isomorphism. Thus if we have an intermediate factorization system $\mathscr{E}/\mathscr{M}$, it must be that $2 \longrightarrow \mathbf{2}$ is in $\mathscr{E}$ or $\mathscr{M}$. If it is in $\mathscr{E}$, then you can show that only extremal monomorphisms are in $\mathscr{M}$, so that $\mathscr{M} \subseteq \mathscr{M}_1$, whence $\mathscr{M} = \mathscr{M}_1$ and this is the epic/extremal monic factorization. On the other hand, if $2 \longrightarrow \mathbf{2}$ is in $\mathscr{M}$, it does not follow that $\mathscr{E}_0 \subseteq \mathscr{E}$.

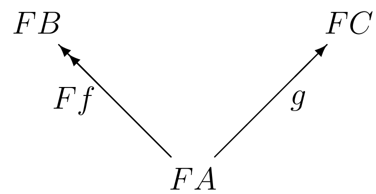
Every arrow in **Pos** factors as an extremal epi followed by a bijection followed by an extremal mono. Thus a factorization system is determined by its restriction to the bijections. But a bijection is simply a set with two partial orderings on it, the second finer than the first. So we have to say which refinements are in $\mathscr{E}$ and which are in $\mathscr{M}$. One possibility is to put a refinement in $\mathscr{E}$ if and only if all the new orderings are among elements in the same component and in $\mathscr{M}$ if and only if all the new orderings are between elements in different components. You have to show the diagonal fill-in condition is satisfied, but it is.

The examples below will refer only to the two extremal factorizations.

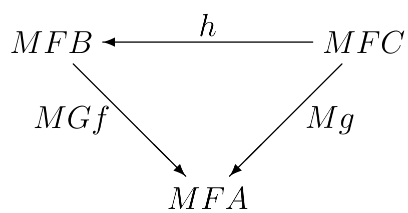
4 Subcategories defined by Horns

Throughout this section, we will suppose chosen a fixed factorization system $\mathscr{E}/\mathscr{M}$.

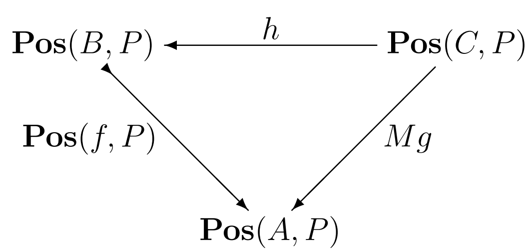
4.1 Horns. Let $F\colon \mathbf{Pos} \longrightarrow \mathscr{T}^{\mathrm{op}} = \mathscr{K}$ be a theory on posets. By a **Horn** we mean a diagram in $\mathscr{K}$



in which $f \in \mathscr{E}$. A model (P, M) of $\mathscr{K}$ **satisfies** the Horn if there is a function $h\colon MFC \longrightarrow MFB$ such that the diagram



commutes. This is equivalent to the commutation of



where $\mathbf{Pos}(f, P)$ is monic since f is epic. That implies that the factorization, if it exists, is unique. Thus there is a full subcategory defined by the existence of such a factorization.

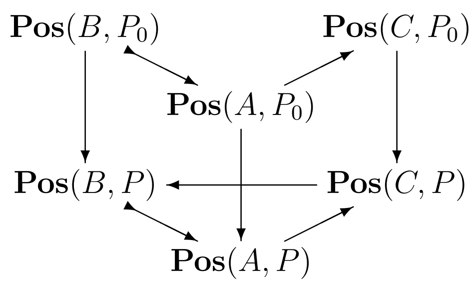
4.2 HSP subcategory. An arrow f of $\mathrm{Mod}(\mathscr{K})$ is called a U-split epimorphism if Uf is a split epimorphism. An arrow f is called a $U/\mathscr{M}$ monomorphism if $Uf \in \mathscr{M}$ We will say that a subcategory of $\mathrm{Mod}(\mathscr{K})$ is an **HSP subcategory** provided it is closed under U-split epimorphisms, $U/\mathscr{M}$-monomorphisms and products. This means, obviously, that the notion is dependent on the factorization system.

4.3 Theorem. *Let $F\colon \mathbf{Pos} \longrightarrow \mathscr{T}^{\mathrm{op}} = \mathscr{K}$ be a theory on posets and*

$$FB \xrightarrow{\;Ff\;} FA \xleftarrow{\;g\;} FC$$

be a Horn. The full subcategory of objects of $\mathrm{Mod}(\mathscr{K})$ *consisting of objects that satisfy the Horn is an HSP subcategory.*

Proof. Let $\mathscr{C}$ denote the category of models and let $\mathscr{C}_0$ denote the full subcategory. It is immediate that this subcategory is closed under limits. Suppose that (P, M) is an object of $\mathscr{C}_0$ and that $(P', M') \rightarrowtail (P, M)$ is a subobject with $P' \rightarrowtail P$ in $\mathscr{M}$. We must show we can fill in the top arrow in the diagram

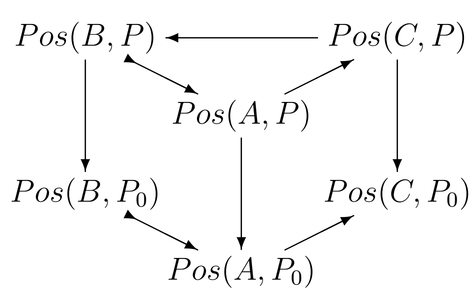


To do this it is sufficient to show that the right hand square is a pullback. But the diagonal fill-in condition can be restated as asserting that for any $A \twoheadrightarrow B$ and $P \rightarrowtail P_0$ the square

$$\begin{array}{ccc} \mathrm{Hom}(B, P_0) & \longrightarrow & \mathrm{Hom}(A, P_0) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(B, P) & \longrightarrow & \mathrm{Hom}(A, P) \end{array}$$

is a pullback.

Finally, let $(P, M) \longrightarrow (P_0, M_0)$ be such that $P \longrightarrow P_0$ is a split epi and suppose that (P, M) is in $\mathscr{C}_0$. We must be able to fill in the missing arrow in the diagram



This comes from the diagonal fill-in in the square

$$\begin{array}{ccc}
\mathbf{Pos}(C,P) & \twoheadrightarrow & \mathbf{Pos}(C,P_0) \\
\downarrow & & \\
\mathbf{Pos}(C,P_0) & & \Big\downarrow \\
\downarrow & & \\
\mathbf{Pos}(B,P) & \twoheadrightarrow & \mathbf{Pos}(A,P_0)
\end{array}$$

□

4.4 Corollary. *The full subcategory of objects that satisfy any set of Horns is an HSP subcategory of the category of models.*

The converse is not true. The example of 3 illustrates why. An equation asserts the equality of two terms. This means that there are two operations, say $\tau_0, \tau_1 \colon FA \longrightarrow FB$ that are to become equal. This is done with the Horn

$$FB \xleftarrow{\langle \mathrm{id}, \mathrm{id} \rangle} F(A+A) \xrightarrow{\tau_0 + \tau_1} FA$$

It is easy to see that an algebra satisfies this Horn if and only if it satisfies $\tau_0 = \tau_1$. Note that the arrow $\langle \mathrm{id}, \mathrm{id} \rangle$ is split epi and split epis are easily seen to be in the epi part of any factorization system.

We saw in that example that we could get an equation and then when that was satisfied get a new equation whose terms didn't even exist in the original theory. Thus the property of being the algebras satisfying a set of Horns is not transitive, while that of being an HSP category is. Thus the two cannot coincide. However, the next best thing is true.

4.5 Theorem. *For any accessible theory on* **Pos***, the HSP subcategories of the category of models of the theory are the transitive closure (possibly transfinite) of the subcategories defined by satisfying Horns.*

What this statement means is that if $\mathscr{D}$ is an HSP subcategory of $\mathscr{C}$, then there is an ordinal chain

$$\mathscr{D} = \mathscr{C}_\lambda \subseteq \ldots \mathscr{C}_{\kappa+1} \subseteq \mathscr{C}_\kappa \subseteq \ldots \subseteq \mathscr{C}_1 \subseteq \mathscr{C}_0 \subseteq \mathscr{C}$$

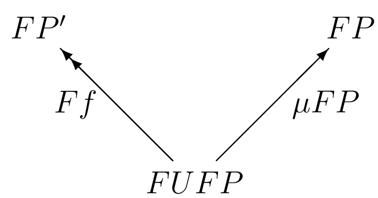
of subcategories such that for each cardinal κ, $\mathscr{C}_{\kappa+1}$ is derived from $\mathscr{C}_\kappa$ as the subcategory satisfying a set of Horns and if κ is a limit ordinal, then $\mathscr{C}_\kappa = \bigcap_{\mu<\kappa} \mathscr{C}_\mu$.

Proof. Suppose that (P,M) is an object of $\mathscr{C}$, (P',M') is an object of $\mathscr{D}$ and $(f,\phi)\colon (P,M) \longrightarrow (P',M')$ is an arrow. Let (P_0,M_0) be the intersection of all $(P'',M'') \longrightarrow (P',M')$

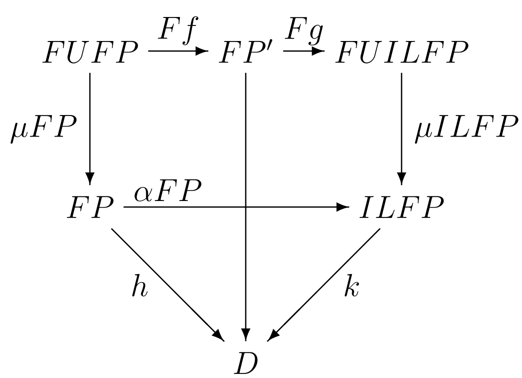
such that $P'' \rightarrowtail P'$ is in $\mathscr{M}$ and such that P'' contains the image of f. Then it is a standard property of factorization systems that $P_0 \longrightarrow P'$ is in $\mathscr{M}$. It is also clear that (f, ϕ) factors through (P_0, M_0). I claim that the factored map is an epimorphism. In fact, if it isn't, there would be two maps from (M_0, P_0) that were equal on that image and their equalizer would be a strictly smaller subobject of (P, M) that factored the map. But a regular monomorphism is automatically extremal and is thus in the monic part of every factorization system.

In Barr [to appear] it is shown how to use the results of Makkai & Paré [1990] on accessible categories to show that the category of models of an accessible theory is accessible. Makkai & Reyes have also shown that an object in an accessible category has only a set of subobjects and a set of quotient objects. From this it is easy to use the factorization just described to construct an adjoint for the inclusion of $\mathscr{D}$ into $\mathscr{C}$.

For similar reasons, the underlying functor $U: \mathscr{C} \longrightarrow$ **Pos** has a left adjoint F and from the adjunction identities, we get that for any E, the arrow $\epsilon E: FUE \longrightarrow E$ is U-split, since $U\epsilon E \circ \eta UE = \mathrm{id}_{UE}$. Thus every model is the target of a U-split epimorphism whose source is a free model. Hence if $\mathscr{D}$ is a proper subcategory, there is some free model FP that it doesn't contain. Suppose the front adjunction is $\alpha: FP \longrightarrow ILFP$. Since this is in $\mathscr{E}$ and not an isomorphism, it is not in $\mathscr{M}$ and thus $U\alpha$ is not in $\mathscr{M}$ either. Then we can factor $U\alpha = g \circ f$ with $f: UFP \longrightarrow P'$ in $\mathscr{E}$ and $g \in \mathscr{M}$ and f is not an isomorphism. Now we consider the Horn



I claim that every object of $\mathscr{D}$ satisfies this Horn. In fact, suppose $h: FP \longrightarrow D$ is a morphism. Then there is a map $k: ILFP \longrightarrow D$ such that $k \circ \alpha FP = h$. Now the required arrow $FP' \longrightarrow D$ is the composite $k \circ \mu ILFP \circ Fg$ in the diagram



Each instance of such a free model not in $\mathscr{D}$ gives such a Horn clause and the class of all of them determines an HSP subcategory $\mathscr{C}_0 \subseteq \mathscr{C}$ such that $\mathscr{D} \subseteq \mathscr{C}_0$. If this inclusion is proper, then $\mathscr{D}$ is an HSP subcategory of $\mathscr{C}_0$ and we may repeat the construction. In this way, we get a descending ordinal sequence of HSP subcategories. At limit ordinals, take the intersection of the subcategories. The only thing left is to show that this process terminates. The reason for that is that the triple is accessible and hence is determined by its values on a small subcategory. These objects are all well-co-powered and hence there can be only a set of these quotient triples. □

5 Connection with other works

There have been many generalizations of Birkhoff's theorem. Some citations are given in [Barr, to appear]. As far as I have been able to determine, they have all taken the point of view that equations are given using the equality predicate alone. If you are in a sup semilattice, for example, then a predicate of the form $a \le b$ can be replaced by the equality predicate $a \vee b = b$ and similarly in an inf semilattice. But if you just have a poset, then there is no alternative to using the inequality predicate directly. Thus this paper (and its generalization to arbitrary categories that is to appear), go in a different direction from the other generalizations. On the other hand, it is likely that in most cases, the results of this paper can be subjected to the same generalizations as Birkhoff's original theorem.

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