A Convenient Category of Domains

Dedicated to Gordon Plotkin on the occasion of his 60th birthday

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Abstract

We motivate and define a category of *topological domains*, whose objects are certain topological spaces, generalising the usual ω-continuous dcppos of domain theory. Our category supports all the standard constructions of domain theory, including the solution of recursive domain equations. It also supports the construction of free algebras for (in)equational theories, can be used as the basis for a theory of computability, and provides a model of parametric polymorphism.

1 Introduction

A strong theme in Gordon Plotkin's work on domain theory is an emphasis on presenting domain theory as a toolkit for the semanticist. In particular, in his "Pisa" notes [38] (an early version of which bears a title that explicitly reflects this perspective [37]), he highlights the variety of different constructions that domain theory supports, motivating each by its computational relevance, and discussing in detail how they may be combined for semantic purposes. Hand-in-hand with this is a mathematical emphasis on grouping domains collectively into categories, so that the constructions on them get explained in terms of their universal properties. This emphasis presumably reflects an early awareness by Plotkin that, should traditional domains turn out not to fulfil all semantic needs, one might nevertheless expect other candidate notions of domain to provide much the same in the way of category-theoretic structure. Later, such considerations lay at the core of the development of axiomatic domain theory in the 1990's — a theory to which Plotkin himself made important contributions, see, e.g., [11].

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The motivation for the present article lies in observations by Plotkin concerning deficiencies in the semantic toolkit provided by traditional domain theory. In domain theory, it is known how to model: (i) higher-order types (using cartesian closed categories of domains); (ii) computability (using ω-continuous dcpos); and (iii) general computational effects such as nondeterminism (as free algebras for inequational theories). Furthermore, it is possible to combine any *two* of these features. (For (i)+(ii), use any of the cartesian closed full subcategories of ω-continuous dcpos; for (ii)+(iii), use the category of ω-continuous dcpos itself; and, for (i)+(iii), use the category of all dcpos.) However, Plotkin observed that it is not possible to combine all three. (None of the cartesian closed subcategories of ω-continuous dcpos are closed under the formation of free algebras.) This observation led Plotkin to ask for someone to find a category of domains that does support all three features in combination. Indeed, at the 2002 meeting in Copenhagen honouring Dana Scott's 70th birthday, Plotkin publicly expressed the wish to receive such a category of domains as a future birthday present for himself. This article is the requested present.

Actually, it was clear to anyone with detailed knowledge of the work on synthetic domain theory from the 1990's [35,22,30,42,34] that such categories of domains were achievable, as long as one was willing to allow them to arise as not easily describable subcategories of realizability toposes. However, we took the main challenge of Plotkin's wish to be to obtain such a category as close in spirit to the familiar categories of domain theory as possible. The approach presented here began with Simpson's observation that one particular category of domains arising in synthetic domain theory has a straightforward alternative description as a category of topological spaces [49,2]. The purpose of the present paper is to show that this category can be derived from first principles without any reference to synthetic domain theory. Indeed, it is obtained as the result of a certain natural combination of topological and domain-theoretic concerns.

Since the early days of domain theory, it has enjoyed a symbiotic relationship with general topology, see [15] for an overview. This is no accident. As Smyth observed, cf. [50,53], there is a strong analogy between open sets in topology and observable properties of data, according to which one should expect mathematical models of datatypes to be topological spaces. We review this connection between topology and computation in Section 2, and we use it as the starting point for our investigations.

A limitation of the analogy between topology and computation is that the mathematical world of topology contains many weird and wonderful spaces for which no connection with computation can possibly be envisaged. It is natural then to seek to explicitly identify those topological spaces that can be argued to have some plausible connection with computation. This is the task we address in Section 3. The idea is to require elements of a topological space to be representable as infinite streams of discrete data, cf. [54]. This allows a notion of *physical feasibility* to be developed, following Plotkin's related terminology in [38]. Roughly speaking, physical feasibility captures the idea that, in computation, a finite amount of output must depend only on a finite amount of input. For those topological spaces which have *admissible quotient representations*, in the sense of [46,47], physical feasibility coincides with continuity, and so the topology of the space accurately reflects its computational behaviour. Such spaces thus provide a candidate for the restricted class of topological spaces we are looking for.

In Section 4, we study the topological spaces that arise in the above way. Such spaces have various characterisations, all due to Schröder [46,47]. Most concisely, they are exactly the T_0 topological *q*uotients of *c*ountably *b*ased spaces (henceforth qcb spaces). It turns out that the category of qcb spaces has excellent closure properties: it is countably complete, countably cocomplete, and cartesian closed.

Having identified qcb spaces as a reasonable topological notion of datatype, we turn to the concerns of domain theory in Section 5. There, we impose a further condition on qcb spaces, in order to identify a notion of *topological domain* enjoying the expected fixed point property: every continuous endofunction has a least (in the topological specialization order) fixed point. As usual, what is needed for this is a least element and an appropriate form of chain completeness. The category of topological domains possesses the expected categorical structure. In particular, it is cartesian closed and so models products and function spaces. In Section 6, we outline how it also supports the other standard constructions from domain theory, including the solution of recursive domain equations.

Our stated motivation for the above development was to address the weakness Plotkin identified in traditional domain theory. In Section 7, we describe Battenfeld's work on the construction of free algebras for (in)equational theories over topological domains [3]. In Section 8, we outline how computability may be incorporated. Finally, in Section 9, we discuss how topological domains provide a model of parametric polymorphism. The latter facility might even be added as a further requirement (iv) to the original wish list above. Parametric polymorphism is a feature that traditional domain theory has hitherto proved entirely incapable of handling.

Throughout the above development, some attention is paid to the fact that topological domains include all ω-continuous pointed dcpos (with their Scott topologies). This allows comparisons to be made between constructions (function spaces, free algebras, etc.) in ordinary and topological domain theory. We discuss, in detail, the circumstances in which such constructions agree, and also when they disagree. In particular, the combination of free algebras and function spaces can lead to topological domains in which the topology is not the Scott topology, and thus one is taken outside of the world of ordinary domain theory. It is this fact that allows topological domains to retain a countable *pseudo*base and thus still be amenable to the development of a theory of computability. See Sections 5–8 for details.

In this paper, we establish a category of domains that is "convenient" in two senses. First, as discussed above, it provides the desired toolkit for semantic constructions, and one that goes beyond what is available in traditional domain theory. Second, the development retains the connection with topology enjoyed by ordinary domain theory. More fundamentally, the material presented in Sections 2–5 shows the development of topological domain theory to be mathematically compelling in itself. Indeed, we believe that topological domains arise as an inevitable consequence of combining the requirement of modelling fixed points with the concerns of physical feasibility.

Notation and prerequisites

The purpose of the present paper is to present a high-level (and hopefully readable) overview of the development topological domains. In doing so, we gather together results from a number of sources, mainly [46,47,4,3]. Although proofs are omitted; where possible, we try to give some indication of why the stated results hold.

We do assume some knowledge of basic domain theory and topology, as in, e.g., [38,1,15,50]. In domain theory, we write *dcpo* for a directed-complete partial order, and *dcppo* for a *pointed* dcpo (i.e., one with least element).

Notationally, when working with the set X^ω of infinite sequences over X, we write a general $\alpha \in X^\omega$ as $\alpha_0\alpha_1\alpha_2\ldots$, and we write $\alpha\lceil_n$ for the n-symbol prefix $\alpha_0\ldots\alpha_{n-1} \in X^n$.

Acknowledgements

We are happy to acknowledge the enormous influence Gordon Plotkin has had on the development of this research. Even the title is taken from an invited talk he gave in 1987 at the "Sussex Computing Meeting" on the Isle of Thorns. [2] It is a pleasure to be able to return the title as the wrapping for a birthday present. We also thank the anonymous referee for helpful suggestions.

2 Datatypes as Topological Spaces

Our aim in this and the next section, is to work our way towards a mathematical model of the notion of datatype starting from first principles. In this short

[2] Presumably, the original source for the title can be traced back to [52].

section, we recall Smyth's appealing conceptual argument that datatypes are topological spaces, cf. [50,53].

As a first approximation, a datatype X should surely be a set whose elements correspond to the data items belonging to the type. This, however, is too crude. Nothing is specified about how one can compute with data. Therefore some additional information is required that provides such information.

In fact, surprisingly little additional information is needed. In addition to the set X, one need only specify a notion of "observable" subset of X. The computational intuition is that an "observation" on X should be performed by applying a possibly time-consuming abstract procedure to individual elements of X. Such a procedure has two possibilities when applied to an element $x \in X$: either it will eventually terminate, and this is the event we observe; or it will continue forever. We say that a subset U of a datatype X is "observable" if there exists some procedure acting on elements of X that eventually terminates when applied to any element that belongs to U, but which fails to terminate when applied to elements of X that do not belong to U. Such a subset U is "observable" in the sense that, to observe if an element x is in U, one applies the procedure to x and awaits termination. If termination occurs then one knows that the element x is indeed in U. In the case of an element $x \notin U$, the procedure continues for ever and one is left twiddling one's thumbs. Thus one does not manage to ever observe the fact that x is not in U (although in the case that $X \backslash U$ is itself an observable subset such an observation would be possible by applying a different procedure to x). From this informal description, one sees that "observable" subsets are to the notion of abstract procedure what semidecidable sets are to the notion of computability.

The connection with topology is that an appealing conceptual argument shows that, in general, for any datatype X, the observable subsets of X form (the open sets of) a topology. For closure under finite intersections, given finitely many observable subsets $U_0, \ldots, U_{k-1}$, one can observe whether $x \in U_0 \cap \cdots \cap U_{k-1}$ by running each of the k tests $x \in U_0$, $\ldots$, $x \in U_{k-1}$ (either in sequence or in parallel) and waiting for all the tests to terminate. As a special case, the entire set X (the empty intersection) is observable. For closure under finite unions (including the emptyset as an empty union), one observes whether $x \in U_0 \cup \cdots \cup U_{k-1}$ by running each of the k tests $x \in U_0$, $\ldots$, $x \in U_{k-1}$ in parallel and waiting for a single test to terminate. (N.B. the tests cannot be performed in sequence because if $\alpha \in U_1 \backslash U_0$ then one cannot wait for the test $x \in U_0$ to terminate before starting the test $x \in U_1$.) More generally, one can argue that observable tests are even closed under *countable* unions. Indeed, one can test if $x \in \bigcup_{i \geq 0} U_i$ by trying each of the tests $x \in U_0$, $x \in U_1$, $\ldots$ in turn, starting each new test at a fixed time interval after the previous test (as above, one cannot wait for the previous test to terminate). As soon as any one of the tests succeeds, one concludes that $x \in \bigcup_{i \geq 0} U_i$. It is worth noting that there is no analogous procedure for observing membership of a countable intersection. In order to test if $x \in \bigcap_{i \geq 0} U_i$, one would have to perform every component test $x \in U_i$ and wait for all to terminate; but this is not possible in finite space and time. Thus there is a fundamental asymmetry between unions and intersections of observable subsets.

The above conceptual argument justifies that observable subsets should be closed under finite intersections and countable unions. Thus observable subsets *almost* form a topology. Although, it is hard to give a similarly operational justification for the remaining requirement for a topology, closure under uncountable unions, it is nonetheless a plausible idealisation of the conceptually justified closure conditions on observable subsets to actually require them to form a genuine topology. Accordingly, we henceforth make this idealised requirement on observable subsets. Note, however, that we shall obtain much better justification for it in Section 4, see the discussion after Proposition 4.6.

So far, we have that a datatype is a set together with a family of "observable" subsets forming a topology. More briefly, a datatype *is* a topological space.

Next we consider intuitive properties of functions between datatypes that can be "computed" by some abstract procedure acting as a transducer. Suppose we have two datatypes X and Y, and suppose that $f\colon X \to Y$ is a procedure turning elements of X into elements of Y. Consider any observable subset $V \subseteq Y$. Then we can define the following procedure acting on any $x \in X$: first apply f to x to obtain $f(x)$, then perform the test for $f(x) \in V$. One sees immediately that this procedure performs the test $x \in \{x \in X \mid f(x) \in V\}$. We have shown that, for any observable subset $V \subseteq Y$, the subset $f^{-1}(V) \subseteq X$ is observable; i.e., the function f is continuous.

The above argument shows that every procedure acting as a transducer from X to Y, must perform a continuous function. It thus becomes mathematically tempting to *identify* the notions of continuous and performable function. Doing this, we obtain the following "dictionary" of equivalences between computational concepts on the left and mathematical concepts on the right.

datatype	$\sim$	topological space
observable set	$\sim$	open set
transducer	$\sim$	continuous function

See [7] for extensions to this dictionary and further discussion.

The analysis presented so far has several weaknesses.

(i) No justification was given for requiring observable subsets to be closed under uncountable unions. An alternative would be to work with the weaker notion of σ-topological space, in which open sets are only required to be closed under countable unions. However, since the discrepancy between the two requirements will disappear in Section 4, we can opt for mathematical conformity safe in the knowledge that our conscience will eventually be cleared.

(ii) While we have argued that every transducer gives rise to a continuous function, no argument has been given for the converse implication. Thus the identification of transducers with continuous functions has not been justified.

(iii) The identification of datatypes with topological spaces fails fundamentally to provide a toolkit of datatype constructions for the semanticist. In particular, there is no function space construction. As is well known, the category of topological spaces is not cartesian closed.

(iv) There are many perverse topological spaces whose size or mathematical peculiarities preclude them from having any plausible connection with computation. Our "model" is vastly more inclusive than it needs to be.

In the next two sections, we shall address points (iv) and (ii) explicitly, by narrowing down the topological spaces of interest to ones for which a direct connection with computation can be argued. As a result, points (i) and (iii) will be resolved automatically, the latter in a miraculous way.

3 Physical feasibility

Computation must take place in the physical world and must therefore be physically feasible. In [38, Ch. 1], Plotkin uses an intuitive notion of "physical feasibility" to argue for the restriction to continuous functions in domain theory. In this section, we use very similar considerations to argue for a restricted class of topological spaces as the computationally relevant ones. Roughly speaking, by "physically feasible" we mean that only a finite amount of work needs to be done in order to produce any output event, such as flagging the success of an observation. We begin by presenting some important illustrative examples.

Example 3.1 (Infinite streams) The set $\mathbb{N}^\omega$ of infinite sequences of natural numbers models a datatype of infinite streams of numbers. We argue that, by considerations of physical feasibility, the *physically observable* subsets of $\mathbb{N}^\omega$ are exactly the subsets $U \subseteq \mathbb{N}^\omega$ satisfying:

$$\forall \alpha \in U.\ \exists k \geq 0.\ \{\beta \in \mathbb{N}^\omega \mid \beta\lceil_k = \alpha\lceil_k\} \subseteq U \ . \tag{1}$$

Certainly, any physically feasible observation must define a subset U satisfying (1); for, if we observe that $\alpha \in U$ after a finite amount of time, then we can have only examined finitely many positions in the infinite sequence α, hence we have no way of distinguishing α from any other β that agrees with α at the same positions. Conversely, we argue that any subset U satisfying (1) is physically observable. Because it satisfies (1), any such U is a union of "basic" subsets, each of the form

$$B_{(k,n_0,\ldots,n_{k-1})} =_{\mathrm{def}} \{\beta \in \mathbb{N}^\omega \mid \beta\lceil_k = \ n_0 \ldots n_{k-1}\} \ ,$$

for appropriate tuples $(k, n_0, \ldots, n_{k-1})$. Obviously, there are only countably many tuples $(k, n_0, \ldots, n_{k-1})$ with $B_{(k,n_0,\ldots,n_{k-1})} \subseteq U$. Thus U is a countable union of basic subsets $B_{(k,n_0,\ldots,n_{k-1})}$. Now, each basic subset $B_{(k,n_0,\ldots,n_{k-1})}$ is trivially observable, because, for any $\alpha \in \mathbb{N}^\omega$, one can test whether $\alpha \in B_{(k,n_0,\ldots,n_{k-1})}$ by looking at only a finite prefix of α. Finally, the argument given in Section 2 for justifying the closure of observable subsets under countable unions yields a physically feasible procedure (assuming unlimited time and resources) for observing membership of U. Thus U is indeed observable.

Example 3.2 (Sets of streams) Suppose we want to perform observations on streams α guaranteed to belong to a given subset $X \subseteq \mathbb{N}^\omega$. Then, by similar arguments to above, considerations of physical feasibility lead to the conclusion that a subset $U \subseteq X$ is physically observable if and only if:

$$\forall \alpha \in U.\ \exists k \geq 0.\ \{\beta \in X \mid \beta\lceil_k = \ \alpha\lceil_k\} \subseteq \ U \ . \tag{2}$$

Example 3.3 (Stream transducers) Suppose $X, Y \subseteq \mathbb{N}^\omega$ are sets of streams. We argue that a function $f \colon X \to Y$ is physically feasible, i.e., determined by some possible physical stream transducer, if and only if it satisfies:

$$\forall n \geq 0.\ \exists m \geq 0.\ \forall \beta \in X.\ \ \alpha\lceil_m = \beta\lceil_m \ \text{ implies } \ f(\alpha)\lceil_n = f(\beta)\lceil_n \ , \tag{3}$$

for all $\alpha \in X$. In words, this property states that a finite amount of output is determined by a finite amount of input. Intuitively, one would expect any physically feasible stream transducer to satisfy this property. Moreover, since any function satisfying (3) is specified by a countable table relating input prefixes to the output prefixes they determine, one can, given unlimited time and resources, produce a transducer for the function, as long as one allows the physical possibility of constructing the lookup table on a "by need" basis.

From the above arguments, one sees that the notion of physical feasibility is weaker than "computability" in the usual sense. We do not require that functions and observations are represented as finite programs, and we allow the possibility of non-effective means of construction in performing tests for countable unions and in constructing lookup tables. This is in accord with Plotkin's use of physical feasibility in [38]. His motivation is to justify the restriction to continuous functions in domain theory as the mathematical manifestation of physical feasibility, at least for particular domains. In our case, we are not (yet) working with domains; but there is nonetheless a similar correlation between physical feasibility and continuity, which we now develop.

First, observe that the observable subsets of $\mathbb{N}^\omega$, as identified in Example 3.1, form a topology; in fact they are exactly the open sets of the well-known *Baire space* topology on $\mathbb{N}^\omega$. Similarly, for a subset $X \subseteq \mathbb{N}^\omega$, the observable subsets, as identified in Example 3.2, are exactly the open subsets in the relative Baire (i.e. subspace) topology on X.

Proposition 3.4 *Suppose $X, Y \subseteq \mathbb{N}^\omega$ are sets of streams, then a function $f\colon X \to Y$ is physically feasible (i.e., satisfies property (3) of Example 3.3) if and only if it is continuous (with respect to the relative Baire topologies).*

This is a standard and straightforward result, cf. [50].

We have seen that the topology of Baire space accounts for the observable properties of infinite streams and continuity accounts for the associated physically feasible functions on streams. Our aim now is to identify a broad class of topological spaces for which there is a similar coincidence of topological concepts and computational concepts. Having already understood the relevance of Baire space, an obvious idea is to use Baire space to represent other spaces. That is, we look at spaces for which the elements are encodable as infinite streams, so that computation on elements can be performed as computation on the representing streams. Such an idea may sound unduly restrictive — why should a computational space be representable in such a simple way? Nevertheless, as we shall see, the idea turns out to be remarkably powerful.

The definitions that follow are taken from the theory of *Type Two Effectivity (TTE)*, in which Baire-space representations are used as basic structures, see [54,46,47]. First, we formulate the way in which we require elements of a topological space to be represented by streams. We begin by making a weak requirement, and then strengthen it to remedy deficiencies.

Definition 3.5 (Representation) A *representation* of a topological space X is given by a set $R \subseteq \mathbb{N}^\omega$ and a surjective continuous function $r\colon R \to X$ (with R given the relative Baire topology). If $r(\alpha) = x$ then we say that α is a *name* for x.

In this definition, the surjectivity requirement supports the idea that every element of X is represented by at least one stream. To argue for the continuity requirement, we consider how we wish to compute with a represented space X. The idea is that computation should be performed on the names of an element rather than on the elements themselves — after all, we can understand computation on sequences far better than computation on abstract mathematical entities. For example, to observe membership of a subset $U \subseteq X$, one has to make an appropriate observation on streams representing elements of X. That is, given any stream α representing $r(\alpha) \in X$, one would like to test the property $r(\alpha) \in U$ by making an appropriate observation on α. Since, by

Example 3.2, we know that the physically observable subsets of R are exactly the open sets, this leads to the following definition.

Definition 3.6 (Physically observable subset) A subset $V \subseteq X$ is said to be *physically observable* under the representation $r\colon R \to X$ if $r^{-1}(V)$ is an open subset of R.

The continuity of r can now be motivated. It ensures that every open set of X is indeed a physically observable subset.

It may be the case that the representation r gives rise to "phantom" observable subsets of X. That is, there may be physically observable subsets $V \subseteq X$ that are not open in the topology on X. In such a case, one might reasonably argue that the space that is really being represented by r is X with the finer topology given by the family of physically observable sets (which does indeed form a topology). The following definition thus ensures that a represented space includes all physically observable subsets in its topology.

Definition 3.7 (Quotient representation) A representation $r\colon R \to T$ is said to be a *quotient representation* if the function r is a topological quotient.

We may now summarise the preceding discussion thus:

> *On topological spaces with quotient representations, the physically observable subsets are exactly the open sets.*

Representations offer natural means of computing functions from a space X to another space Y, by computing with names of elements. Thus functions from X to Y can be computed by stream transducers. This allows a natural definition of physical feasibility for functions between represented spaces.

Definition 3.8 (Physically feasible function) Given spaces X, Y, representations $r\colon R \to X$ and $s\colon S \to Y$, a function $f\colon X \to Y$ is said to be *physically feasible* (from r to s) if there exists a continuous function $g\colon R \to S$ such that $f \circ r = s \circ g$.

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[d, "r"'] & S \arrow[d, "s"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

In the literature on TTE, this property is called *relative continuity*.

For general representations r and s, the continuous functions from R to S and the physically feasible functions from r to s need not coincide, indeed neither class need be included in the other. We now work towards establishing conditions under which continuity and physical feasibility coincide.

One inclusion follows from r being a quotient.

Proposition 3.9 *For a representation $r\colon R \to X$, the following are equivalent:*

(i) *r is a topological quotient.*

(ii) *For every representation $s\colon S \to Y$, every physically feasible function from r to s is continuous from X to Y.*

The proof is straightforward.

In order to obtain the converse, that every continuous function is physically feasible we require another strengthening of the notion of representation.

Definition 3.10 (Admissible representation) A representation $r\colon R \to X$ is said to be *admissible* if, for every representation $r'\colon R' \to X$ of X, it holds that the identity function on X is physically feasible from r' to r.

Intuitively, an admissible representation is one that is rich enough that it interprets every other representation. The following standard example (cf. [54]) nicely illustrates the computational relevance of admissibility.

Example 3.11 (Real numbers) The following *signed binary* representation is an admissible quotient representation $r_{\mathrm{sb}}\colon \mathbb{Z}^\omega \rightharpoonup \mathbb{R}$.

$$\mathrm{dom}(r_{\mathrm{sb}}) = \{\alpha \mid \forall i \geq 1.\ \alpha_i \in \{-1, 0, 1\}\}$$

$$r_{\mathrm{sb}}(\alpha) = \alpha_0 + \sum_{i=1}^{\infty} 2^{-i}\alpha_i \qquad\qquad \alpha \in \mathrm{dom}(r_{\mathrm{sb}})$$

On the other hand, the familiar *binary* representation $r_{\mathrm{b}}\colon \mathbb{Z}^\omega \rightharpoonup \mathbb{R}$, defined by restricting the function r_{sb} to

$$\mathrm{dom}(r_{\mathrm{b}}) = \{\alpha \mid \forall i \geq 1.\ \alpha_i \in \{0, 1\}\}$$

is a quotient representation that is not admissible. These examples account topologically for the appropriateness of the signed binary representation for exact real-number computation, and the inappropriateness of binary representation. (Any other standard base n notation has similar defects.)

An immediate consequence of the definition of admissibility is that if r and r' are both admissible representations of X then they are equivalent in the sense that the identity function is physically feasible in both directions. A sightly less immediate consequence is the desired implication between continuity and physical feasibility.

Proposition 3.12 *For a representation $s\colon S \to Y$, the following are equivalent.*

(i) *s is admissible.*

(ii) *For every representation $r\colon R \to X$, every continuous function from X to Y is a physically feasible function from r to s.*

A proof can be found in [46].

As an immediate consequence of Propositions 3.9 and 3.12 above, we obtain the coincidence of continuity and physical feasibility.

Corollary 3.13 *Given admissible quotient representations $r\colon R \to X$ and $s\colon S \to Y$, the continuous functions from X to Y coincide with the physically feasible functions from r to s.*

This result is so important, we summarise it verbally:

> *Between topological spaces with admissible quotient representations, the physically feasible functions are exactly the continuous functions.*

Accordingly, the desired coincidences between topological and computational concepts hold for topological spaces with admissible quotient representation.

4 Spaces with admissible quotient representation

We have settled on spaces with admissible quotient representations as topological spaces for which there is a coincidence between topological and computational (qua physical feasibility) notions. Of course, the restriction to spaces whose elements can be named by infinite streams is somewhat arbitrary, and one could envisage that there might possibly be other spaces for which an equivalence between topological and computational concepts could be established by other means. Nevertheless, as we shall explain in this section, the spaces with admissible quotient representation enjoy remarkable closure properties. Furthermore, one can characterise such spaces in direct topological terms, without consideration of representations. By presenting such results, the goal of this section is to establish the spaces with admissible quotient representation as the natural realm on which there is a coincidence of topological and computational concepts.

We begin by presenting two characterisations of the topological spaces with admissible quotient representation, both due to Schröder [46,47]. Working towards the first characterisation, we examine properties that follow from the the existence of an admissible quotient representation.

Recall that the *specialization order* on a topological space X is defined by: $x \sqsubseteq y$ if $x \in U$ implies $y \in U$ for all open $U \subseteq X$. In general, the specialization order is a preorder. A space X is said to be T_0 if the specialization order is a partial order.

Proposition 4.1 *If X has an admissible representation then X satisfies the*

T_0 *separation property.*

The proof is by a cardinality argument. If X were not T_0 then there would be at least $2^{2^{\aleph_0}}$ continuous functions from $\mathbb{N}^\omega$ to X. However, there are at most $2^{\aleph_0}$ functions that are physically feasible with respect to the identity representation on $\mathbb{N}^\omega$. So, by Proposition 3.12, X must be T_0.

Recall that sequence convergence in a topological space X is defined as follows: $(x_i) \to x$ if, for every open $U \subseteq X$ with $x \in U$, almost all (i.e., all but finitely many) x_i are in U. A subset $V \subseteq X$ is said to be *sequentially open* if, whenever $(x_i) \to x \in V$, it holds that almost all x_i are in V. Trivially, every open set is sequentially open. The space X is said to be *sequential* if, conversely, every sequentially open set is open. In [12], Franklin characterises the sequential spaces as the topological quotients of first countable spaces.

Proposition 4.2 *If X has a quotient representation then X is sequential.*

Immediate from Franklin's characterisation, since, for any quotient representation $r\colon R \to X$, the space R is countably based (it is a subspace of $\mathbb{N}^\omega$).

In this proof, we see that every space X with quotient representation is a topological quotient of a countably based space. It need not be the case, however, that X itself has a countable base. But it does enjoy a weaker related property. The following notion is due to Schröder [46], and is closely related to various other similarly named concepts in the topological literature, cf. [32,47,8].

Definition 4.3 (Pseudobase) A *(sequential) pseudobase* for a topological space X is a family $\mathcal{B}$ of subsets of X such that whenever $(x_i) \to x \in U$ with $U \subseteq X$ open, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$ and, moreover, almost all x_i are in B.

Importantly the subsets in a pseudobase need not be open. Indeed, a base for the topology is nothing other than a pseudobase satisfying the additional property that every $B \in \mathcal{B}$ is an open set. The reason for introducing pseudobases is because of the following characterisation due to Schröder.

Theorem 4.4 *A topological space has an admissible representation if and only if it is T_0 and has a countable pseudobase.*

For a proof see [46].

We now have all the ingredients for Schröder's characterisation of spaces with admissible quotient representation.

Theorem 4.5 *The following are equivalent for a topological space X.*

(i) *X has an admissible quotient representation.*

(ii) *X is a T_0 sequential space with countable pseudobase.*

(iii) *X is a T_0 quotient of a countably based space.*

N.B., condition (iii) says simply that X is T_0 space that can be exhibited as a topological quotient $q\colon A \to X$ for some countably based space A (without loss of generality, A can itself be assumed to be T_0). The proof of (i) $\Longleftrightarrow$ (ii) appears in [46]; and the proof of (ii) $\Longleftrightarrow$ (iii) is in [47].

Given the above characterisation, we henceforth call spaces with admissible quotient representation *qcb spaces* (T_0 *q*uotient of a *c*ountably *b*ased space).[3]

We next give an overview of some of the good topological properties enjoyed by qcb spaces.

Proposition 4.6 *If X is a qcb space then it is hereditarily Lindelöf; that is, for any family $\{U_i\}_{i \in I}$ of opens there exists a countable subfamily $J \subseteq I$ such that $\bigcup_{j \in J} U_j = \bigcup_{i \in I} U_i$.*

This property has computational significance. In Section 2, we found it impossible to give computational justification for the closure of open sets under uncountable unions in the definition of a topology. However, for qcb spaces, the fact that opens are closed under arbitrary unions does have computational justification, since uncountable unions of opens reduce to countable ones.

The next two properties are technical. The first will have an application in Section 5, and the second says that the several potentially different notions of compactness all coincide for qcb spaces. Thus some of the pathologies of general topology disappear when one restricts to qcb spaces.

Proposition 4.7 *If X is a qcb space then it is hereditarily separable; that is, for any subset $A \subseteq X$, there exists a countable $C \subseteq A$ that is dense in the subspace topology on A.*

Proposition 4.8 *If X is a qcb space then the following properties coincide for a subset $K \subseteq X$.*

(i) *K is compact; that is, for any family $\{U_i\}_{i \in I}$ of opens with $K \subseteq \bigcup_{i \in I} U_i$, there exists a finite $F \subseteq I$ with $K \subseteq \bigcup_{i \in F} U_i$*

(ii) *K is countably compact; that is, for any countable family $\{U_i\}_{i \in I}$ of opens with $K \subseteq \bigcup_{i \in I} U_i$, there exists a finite $F \subseteq I$ with $K \subseteq \bigcup_{i \in F} U_i$*

(iii) *K is sequentially compact; that is, for any sequence $(x_i)_{i \geq 0}$ of elements of K, there exists a subsequence $(x_{i_j})_{j \geq 0}$ (given by a strictly monotone function $j \mapsto i_j$) and an element $x \in K$ with $(x_{i_j})_{j \geq 0} \to x$.*

Here, the equivalence of (i) and (ii) is an immediate consequence of Proposition 4.6. The implication (iii) $\Longrightarrow$ (ii) is valid for arbitrary topological spaces. The converse implication, which is non-trivial, was communicated to us by Peter Nyikos.

[3] Our terminology mildly differs from some of the literature, where *qcb* spaces are not always assumed to be T_0.

Having considered the properties of qcb spaces individually, we now consider their collective properties. For this, we consider the category **QCB** of continuous functions between qcb spaces. This category has unexpectedly rich structure.

First, it has all countable colimits. Countable coproducts are calculated as for topological spaces. To form the coequalizer of a parallel pair $f, g\colon X \to Y$, one first constructs the quotient of Y under the coarsest equivalence relation equating $f(x)$ and $g(x)$ for every $x \in X$ (this is the coequalizer in the category **Top** of topological spaces), and then further quotients this to implement the T_0 property by identifying points that are equivalent in specialization order (this is the coequalizer in the category $\mathbf{Top_0}$ of T_0 spaces).

Dually, **QCB** also has countable limits, however these are not calculated as in **Top** (equivalently $\mathbf{Top_0}$). Indeed, one can find qcb spaces X, Y such that the topological product $X \times Y$ is not a sequential space (cf. [8, Example 5.1]). However, every topological space X has a *sequentialization*, $\mathrm{Seq}(X)$, defined on the same underlying set, with the sequentially open sets of X as its opens. The countable product of qcb spaces $(X_i)_{i\geq 0}$ is defined by:

$$\prod_{i\geq 0} X_i \;=_{\mathrm{def}}\; \mathrm{Seq}\,(\prod_{i\geq 0}^{\mathbf{Top}} X_i) \ ,$$

where the product on the right is the topological product. This is simply the product in the category **Seq** of sequential spaces. A similar issue arises in forming equalizers since a subspace of a qcb space is not necessarily itself a qcb space (again it is sequentiality that fails). Thus the equalizer of a parallel pair $f, g\colon X \to Y$ in **QCB** is constructed by sequentializing the subspace $\{x \in X \mid f(x) = g(x)\}$ of X. (This subspace is itself the equalizer in **Top**.)

Finally, we consider function spaces. Let $[X \to Y]$ be the set of all continuous functions from X to Y. We topologise this set with the topology generated by subbasic opens of the form:

$$\langle (x_i) \to x_\infty,\, V\rangle \;=_{\mathrm{def}}\; \{f \in [X \to Y] \mid \forall i \in \mathbb{N} \cup \{\infty\}.\ f(x_i) \in V\} \ ,$$

where $(x_i) \to x_\infty$ in X and $V \subseteq Y$ is open.[4] Define

$$X \Rightarrow Y \;=\; \mathrm{Seq}\,[X \to Y] \ .$$

The above definition is justified by the following surprising theorem, again due to Schröder.

[4] Such subbasic sets are a restricted form of open from the compact open topology on $[X \to Y]$. In fact, we could alternatively place the (in general finer) compact open topology on $[X \to Y]$ without affecting the discussion, cf. [8].

Proposition 4.9 *If X, Y are qcb spaces then so is $X \Rightarrow Y$, and this is an exponential in the category* **QCB**.

This is proved in [46,47]. The following theorem summarises all the structure identified above.

Theorem 4.10 *The category* **QCB** *is cartesian closed and countably complete and cocomplete.*

Having now extensively examined qcb spaces, we return to issues (i)–(iv) raised in Section 2, in which we criticised general topological spaces as a notion of datatype. We see that we have directly addressed point (iv) by restricting to spaces whose elements can be represented as streams. Furthermore, point (ii) was resolved in Section 3 via the admissibility and quotient requirements. As a result, point (i) is redundant since uncountable unions of opens are reduced to countable ones (as in the discussion below Proposition 4.6). Finally, point (iii) is addressed by Theorem 4.10. This theorem is unexpected since the restriction to spaces with admissible quotient representation is entirely motivated through considerations of physical feasibility, and cartesian closedness falls out for free without any effort being made to force it. From the results in this section, we conclude that qcb spaces provide a compelling mathematical model of the notion of datatype.

5 Topological domains

Our considerations so far have been distant from the usual concerns of domain theory. In domain theory, recursion, nontermination and partiality play prominent roles, and one starts straight away with the idea that domains should be ordered and that continuous functions should have least fixed points. In contrast, although we have argued that, through considerations of physical feasibility, datatypes should be modelled as qcb spaces, we have ignored such recursion-related issues entirely.

In this section, we place additional requirements on qcb spaces, suitable for modelling recursion. As in domain theory, this will require order-theoretic considerations. Since our spaces satisfy the T_0 separation property, they already have an intrinsic partial order, namely the specialization order $\sqsubseteq$. As in domain theory, we shall find the least fixed point of a continuous function f by taking a limit of an approximating sequence:

$$\perp \;\sqsubseteq\; f(\perp) \;\sqsubseteq\; f(f(\perp)) \;\sqsubseteq\; f(f(f(\perp))) \;\sqsubseteq \cdots \;.$$

For this, we shall, as usual, require that domains have least element and limits of ascending sequences. It is mathematically productive to address these two requirements separately.

An *ascending sequence* (or *ω-chain*) in a topological space is a sequence $(x_i)_{i\geq 0}$ with $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$.

Definition 5.1 (Topological predomain) A qcb space is said to be a *topological predomain* if every ascending sequence (x_i) has an upper bound x_∞ such that $(x_i) \to x_\infty$.

Here, we are exploiting the fact that we have a topological space to use the topological notion of sequence convergence (as defined above Proposition 4.2). For this to be a nontrivial definition, it is essential to include the requirement that x_∞ is an upper bound, because, for any ascending sequence (x_i), one always has $(x_i) \to x_k$ for every $k \geq 0$. We do not ask for x_∞ to be a least upper bound since this follows from (but is weaker than) the convergence requirement. Indeed, it is easily seen that if x is any limit of an ascending sequence (x_i) then x lies below every upper bound of (x_i). The result below is an immediate consequence.

Proposition 5.2 *If X, Y are topological predomains then:*

(i) *Every ascending sequence in X has a least upper bound.*

(ii) *Every continuous function from X to Y preserves least upper bounds of ascending sequences.*

The notion of topological predomain has been formulated by requiring suprema only for ascending sequences rather than, more generally, for directed sets. As remarked by Plotkin [38, Ch. 1], there is computational motivation for requiring suprema for ascending sequences, since such suprema are needed for finding least fixed points. In contrast, similar motivation is not easily given for requiring suprema for arbitrary directed sets. For qcb spaces, however, one does not need to motivate directed completeness; it follows from ω-chain completeness.[5] Indeed, Proposition 5.4 below establishes that every topological predomain is a dcpo (in its specialization order). The difference with respect to ordinary domain theory is that, in general, the topology is coarser than the Scott topology. These properties are captured by the following definition taken from [15] (first introduced as *d-spaces* in [55]).

Definition 5.3 (Monotone convergence space) A topological space X is a *monotone convergence space* if its specialization order is a dcpo (in particular it is T_0) and every open set is open in the Scott topology on $(X, \sqsubseteq)$.

Proposition 5.4 *A qcb space is a topological predomain if and only if it is a monotone convergence space.*

It is obvious that every qcb monotone convergence space is a topological predomain. To prove the converse, one has to show that every directed subset

[5] This generalises the situation for ω-algebraic cpos discussed in [38, Ch. 6, Exercise 1].

$D \subseteq X$ has a supremum d and that D converges to d (under net convergence). For this, one applies Proposition 4.7 to extract a countable dense subset $C \subseteq D$, using which one constructs an ascending sequence in D whose supremum is the required supremum for D. See [4, Proposition 4.7] for details.

We write **TP** for the full subcategory of **QCB** consisting of topological predomains. Usefully, this category enjoys the same richness of structure as **QCB**.

Theorem 5.5 *The category* **TP** *is a full reflective exponential ideal of* **QCB**.

It follows that **TP** is countably complete and inherits its limits from **QCB**. It is also countably cocomplete, with colimits obtained by applying the reflection functor to colimits in **QCB**. The exponential ideal property means that if X is any qcb space and Y any topological predomain then the qcb function space $X \Rightarrow Y$ is a topological predomain. In particular, the category **TP** is cartesian closed. For a proof of the Theorem 5.5, see [4, Theorem 4.8].

We now address the least element requirement on domains.

Definition 5.6 (Topological domain) A *topological domain* is a topological predomain with least element in the specialization order.

Topological domains do indeed enjoy the expected fixed-point property that we used to motivate their definition.

Theorem 5.7 *Every continuous function* $f \colon D \to D$ *on a topological domain* D *has a least fixed point* $\mathrm{lfp}(f) \in D$.

The standard proof works (on account of Proposition 5.2). We also have the expected *uniformity* property of least fixed points, as identified by Plotkin [38, Ch. 2, Exercise 30] (and independently by Eilenberg in unpublished work). As in traditional domain theory, a continuous function between topological domains is said to be *strict* if it preserves the least element.

Proposition 5.8 (Uniformity) *Given topological domains* D, E, *continuous functions* $f \colon D \to D$, $g \colon E \to E$ *and a strict continuous function* $h \colon D \to E$ *such that* $h \circ f = g \circ h$, *then* $\mathrm{lfp}(g) = h(\mathrm{lfp}(f))$.

Furthermore, as in domain theory, the property of uniformity characterises least fixed points.

We write **TD** for the category of topological domains and continuous functions. We have finally arrived at the convenient category of domains in the title of the paper. Let us begin to establish its good properties.

Theorem 5.9 *The category* **TD** *is an exponential ideal of* **QCB** *and is closed under countable products in* **QCB**.

In view of Theorem 5.5, all that needs to be verified here is that the relevant products and function spaces have least elements. This is straightforward. We have thus, in Theorems 5.9 and 5.7, established that **TD** is a cartesian closed category with fixed points.

We end this section by presenting a connection between ordinary dcppo-based domain theory and our topological domains. It is easily seen that every ω-continuous dcppo, endowed with the Scott topology, is a topological domain. (The crucial point is that it is a qcb space because the Scott topology is countably based.)[6] Thus the category ω**Cont** of ω-continuous dcppos is a full subcategory of **TD**. Although ω**Cont** is not itself cartesian closed, it is known that it has a largest full subcategory that is, namely Jung's category ω**FS** of countably based FS domains, see [27,1].

Theorem 5.10 *The inclusion of* ω**FS** *in* **TD** *preserves the cartesian closed structure and countable products.*

The above result is proved in [4, Proposition 5.2 and Theorem 5.7]. It means that function spaces of countably based FS domains in **TD** carry Scott topologies. This property does not hold in general for ω-continuous dcppos that are not FS domains. A counterexample can be found in [4, Proposition 5.3]. It is not unreasonable to have a different function space topology in such cases, since one can argue that FS domains form the largest collection of domains for which the Scott topology on function spaces is well behaved. See [4, Section 5] for further discussion.

6 Constructions on topological domains

Since **TD** is a cartesian closed category with fixed points, it follows that it does not have initial object, finite coproducts or equalizers, see [19]. As in ordinary domain theory, better category theoretic structure is possessed by the subcategory $\mathbf{TD}_\perp$ of strict continuous functions between topological domains.

Proposition 6.1 *The category* $\mathbf{TD}_\perp$ *is countably complete, with limits inherited from* **QCB**.

Since an analogous property holds for **TP** (Theorem 5.5), one just needs to show that limits of strict diagrams preserve the existence of a least element. This is straightforward.

Proposition 6.2 *The category* $\mathbf{TD}_\perp$ *has countable coproducts.*

Coproducts in $\mathbf{TD}_\perp$ are a straightforward topological generalisation of the coalesced sums of domain theory, see, e.g., [38, Ch. 3]. We exhibit the finite

[6] In fact it follows from [8, Corollary 6.11] that a continuous dcppo is a topological domain only if it is ω-continuous.

coproducts explicitly. The initial object is given by any one point space. The sum $D \oplus E$ of two topological domains has underlying set:

$$\{\mathrm{inl}(d) \mid d \in D,\ d \neq \bot_D\} \ \cup\ \{\mathrm{inr}(e) \mid e \in E,\ e \neq \bot_E\} \ \cup\ \{\bot_{D \oplus E}\}\ ,$$

using an obvious notation for the least elements of D and E. The topology on $D \oplus E$ is generated by basic opens of the form: $\{\mathrm{inl}(d) \mid d \in U\}$ where $U \subseteq D$ is open; $\{\mathrm{inr}(e) \mid e \in V\}$, where $V \subseteq E$ is open; and $D \oplus E$ itself. Thus $\bot_{D \oplus E}$ is indeed the least element of $D \oplus E$, as the notation suggests. The construction of countably infinite coproducts is similar and left to the reader. In both the finite and infinite cases, the construction does indeed yield a qcb space because it can be exhibited (in an obvious way) as a quotient of a countable sum in **QCB**. The remaining conditions for being a topological domain are routinely verified, as is the universal property of the coproduct.

Next, we consider analogues of the strict product and strict function space of domain theory [38, Ch. 3]. The binary strict product $D \otimes E$ of two topological domains has underlying set:

$$\{(d, e) \in D \times E \mid d \neq \bot_D,\ e \neq \bot_E\} \ \cup\ \{\bot_{D \otimes E}\}\ .$$

The topology has the following open sets: $W \subseteq D \otimes E \backslash \{\bot_{D \otimes E}\}$ where W is open in the **QCB** product $D \times E$ (see Section 4); and $D \otimes E$ itself. The main observation needed in showing that this indeed forms a qcb space is that $(D \otimes E) \backslash \{\bot_{D \otimes E}\}$ is an open subset of $D \times E$ and so its subspace topology is sequential. Therefore, $(D \otimes E) \backslash \{\bot_{D \otimes E}\}$, with the subspace topology, is a qcb space, from which it follows that $D \otimes E$ is too. The remaining conditions for a topological domain are easily verified.

As in domain theory, cf. [38, Ch. 3], strict product has a universal property as a classifier of bistrict continuous functions. Recall that a function of two arguments $f\colon D_1 \times D_2 \to E$ is said to be *bistrict* if it is strict in each argument separately. For example, the function $\otimes\colon D_1 \times D_2 \to D_1 \otimes D_2$ defined by:

$$\otimes(d_1, d_2) = \begin{cases} (d_1, d_2) & \text{if } d_1 \neq \bot_{D_1} \text{ and } d_2 \neq \bot_{D_2} \\ \bot_{D_1 \otimes D_2} & \text{otherwise} \end{cases}$$

is bistrict and continuous.

Proposition 6.3 *If $f\colon D_1 \times D_2 \to E$ is bistrict and continuous, for topological domains D_1, D_2, E, then there exists a unique continuous function $g\colon D_1 \otimes$*

$D_2 \to E$ such that $f = g \circ \otimes$.

$$\begin{tikzcd} D_1 \otimes D_2 \arrow[r, "g"] & E \\ D_1 \times D_2 \arrow[u, "\otimes"] \arrow[ur, "f"'] & \end{tikzcd}$$

It is also easily verified that $\otimes$ is (the action on objects of) a symmetric monoidal product on $\mathbf{TD}_\perp$ with Sierpinski space $\mathbb{S} =_{\mathrm{def}} \{\perp, \top\}$ (where $\{\top\}$ but not $\{\perp\}$ is open) as its unit; again cf. [38, Ch. 3].

The strict function space $D \Rightarrow_\perp E$ has underlying set

$$\{f \in [D \to E] \mid f \text{ strict}\} \ ,$$

and its topology is the subspace topology from $D \Rightarrow E$. In this case, $D \Rightarrow_\perp E$ is a closed subset of $D \Rightarrow E$, and hence its subspace topology is sequential; therefore $D \Rightarrow_\perp E$ is indeed a qcb space. Again, the remaining conditions for being a topological predomain are straightforward to verify, as is the proposition below.

Proposition 6.4 *Together, $\otimes$, $\mathbb{S}$ and $\Rightarrow_\perp$ provide symmetric monoidal closed structure on $\mathbf{TD}_\perp$.*

If follows from Theorem 5.10 that the strict function space $D \Rightarrow_\perp E$ between two countably based FS domains carries the Scott topology. Again, counterexamples can be found for ω-continuous dcppos that are not FS domains. (The example of [4, Proposition 5.3] also works for strict function space.)

The lifting construction of domain theory also has a topological analogue. For any topological *pre*domain D, we define $D_\perp$ to have underlying set

$$\{\lceil d \rceil \mid d \in D\} \cup \{\perp_{D_\perp}\} \ .$$

The open sets are: $\{\lceil d \rceil \mid d \in U\}$ where $U \subseteq D$ is open; and $D_\perp$ itself. This is again a qcb space, since it is trivially T_0 and sequential, and a countable pseudobase is obtained in the obvious way from one for D. As expected, lifting is left adjoint to the inclusion of $\mathbf{TD}_\perp$ in $\mathbf{TP}$.

Proposition 6.5 *If D is a topological predomain and E is a topological domain, then, for any continuous $f\colon D \to E$ there exists a unique strict contin-*

uous $g\colon D_\perp \to E$ *such that* $f = g \circ \lceil\cdot\rceil$.

$$\begin{tikzcd} D_\perp \arrow[r, "g"] & E \\ D \arrow[u, "\lceil\cdot\rceil"] \arrow[ur, "f"'] & \end{tikzcd}$$

In addition, one can easily verify that the inclusion of $\mathbf{TD}_\perp$ in $\mathbf{TP}$ is monadic, i.e., that $\mathbf{TD}_\perp$ is the category of Eilenberg-Moore algebras for the lifting monad on $\mathbf{TP}$. The moral, once again, is that the familiar structure of domain theory is present also for topological domains.

Finally in this section, we show that topological domains support the solution of recursive domain equations. This turns out to be a simple application of Smyth and Plotkin's axiomatic framework for such solutions [51]. However, we take a more modern perspective, incorporating the ideas of Freyd [13,14], as developed by Fiore in his (Plotkin-supervised) Ph.D. thesis [10].

Using the fact that, for topological domains D, E, the strict function space $D \Rightarrow_\perp E$ is again a topological domain, and applying Proposition 5.2, one easily shows that $\mathbf{TD}_\perp$ is an $\omega\mathbf{cppo}$-enriched category. Further any one point space $\mathbf{1}$ is a zero object in $\mathbf{TD}$; that is it is both initial and terminal. Moreover, for any topological domain D, the composite $D \to \mathbf{1} \to D$ of unique strict maps is (trivially) the least element of $D \Rightarrow_\perp D$, and hence lies below the identity in $D \Rightarrow_\perp D$. That is, the object $\mathbf{1}$ is an *ep-zero*, in the sense of [10], in $\mathbf{TD}_\perp$. In addition, by Proposition 6.1, $\mathbf{TD}_\perp$ has all countable limits, in particular it has limits of ω^{op}-chains of *projections*, as defined in [10]. We have now verified all the conditions needed to invoke Fiore's fundamental theorem in [10], and obtain: [7]

Proposition 6.6 $\mathbf{TD}_\perp$ *is* $\omega\mathbf{cppo}$*-parametrized algebraically compact.*

What this means is that we can solve recursive domain equations for systems of equations expressed in terms of mixed variance $\omega\mathbf{cppo}$-enriched functors of type $(\mathbf{TD}_\perp{}^{\mathrm{op}} \times \mathbf{TD}_\perp)^k \to \mathbf{TD}_\perp$. Since all the constructions on topological domains considered so far are given by functors of this form, one can solve arbitrary recursive domain equations involving such constructions. For a detailed explanation of how Proposition 6.6 leads to such conclusions, the reader is referred to the very thorough treatment in [10].

[7] Here $\omega\mathbf{cppo}$ is the category of ω-continuous functions between ω-complete pointed partial orders. The theorem applies to $\mathbf{TD}_\perp$ considered as an $\omega\mathbf{cppo}$-enriched category in the natural way.

7 Free algebras

In [36], Plotkin introduced his powerdomain construction as a means of modelling nondeterminism (hence concurrency) in domain theory. Subsequently, Henessy and Plotkin [18] characterised this construction as yielding the free domain-theoretic semilattice. More recently, Plotkin and Power [40] have advocated the idea of using general free algebras to model computational effects, refining the work of Moggi on computational monads [33].

In this section we explain how the category **TP** of topological predomains supports a wide collection of free-algebra constructions, including the usual powerdomains. Let Σ be a signature containing a countable collection of operation symbols, each with an associated arity $\leq \omega$ (note that we are allowing countably infinite arities as well as finite ones). Let $\mathcal{E}$ be a set of (in)equations over terms constructed from Σ; by which we mean that elements of $\mathcal{E}$ may have two forms: (i) $s = t$, and (ii) $s \sqsubseteq t$. Then a $(\Sigma, \mathcal{E})$-*algebra* in **TP** is a pair $(D, \{f_o\}_{o\in\Sigma})$, where each f_o is a continuous function $D^{\mathrm{arity}(o)} \to D$ (of course the power $D^{\mathrm{arity}(o)}$ is taken in the category **TP**, equivalently in **QCB**), and such that all the (in)equations in $\mathcal{E}$ are validated. In [3], Battenfeld shows that free $(\Sigma, \mathcal{E})$-algebras exist in **TP**.

Theorem 7.1 *For any topological predomain D there exists a $(\Sigma, \mathcal{E})$-algebra $(\mathcal{F}(D), \{f_o\}_{o\in\Sigma})$ with continuous function $\eta_D \colon D \to \mathcal{F}(D)$ such that, for any $(\Sigma, \mathcal{E})$-algebra $(E, \{g_o\}_{o\in\Sigma})$ and continuous $e \colon D \to E$, there is a unique continuous homomorphism $h \colon (\mathcal{F}(D), \{f_o\}_{o\in\Sigma}) \to (E, \{g_o\}_{o\in\Sigma})$ making the diagram below commute.*

$$
\begin{tikzcd}
\mathcal{F}(D) \arrow[r, "h"] & E \\
D \arrow[u, "\eta_D"] \arrow[ur, "e"'] &
\end{tikzcd}
$$

Battenfeld's construction of free algebras is carried out in three stages. First, a free algebra is constructed in the category of sequential T_0 spaces. It is possible to do this using Freyd's adjoint functor theorem. However, an explicit description is needed to show, as step two, that the free sequential algebra is actually a qcb space. Finally, the reflection functor from qcb spaces to topological predomains is applied to yield the free algebra in topological predomains. The details can be found in [3]. We remark that stages two and three crucially rely on properties of countable products in **QCB** established in [48].

Example 7.2 (Convex powerdomain) The *convex* (or *Plotkin*) powerdomain is the free algebra generated by one binary operation "or" and equations:

$$
\begin{aligned}
x \text{ or } x &= x & (4)\\
x \text{ or } y &= y \text{ or } x & (5)\\
(x \text{ or } y) \text{ or } z &= x \text{ or } (y \text{ or } z)\ . & (6)
\end{aligned}
$$

These are just the standard equations for (binary) semilattices.

Example 7.3 (Upper and lower powerdomains) The *convex* (or *Smyth*) powerdomain has the same signature and equations as the Plotkin powerdomain, and also the single inequation:

$$x \text{ or } y \sqsubseteq x\ . \tag{7}$$

The *lower* (or *Hoare*) powerdomain is obtained by replacing the inequation above with the reverse inequality:

$$x \sqsubseteq x \text{ or } y\ . \tag{8}$$

By Theorem 7.1, for any of the above inequational theories, free algebras exist in the category of topological predomains. Moreover, because the idempotency equation (4) holds in each case, it can be shown that the free algebra constructions preserve the presence of a least element. Thus one has the usual three powerdomains in the category **TD** of topological domains

Since ω**Cont** is a full subcategory of **TP**, it is interesting to compare how the powerdomains in **TP** relate to the usual domain-theoretic ones. We say that $(\Sigma, \mathcal{E})$ is a *finitary* (in)equational theory if every operation in Σ has finite arity. It is shown in [1] that the category ω**Cont** has free algebras for arbitrary finitary (in)equational theories. Battenfeld has proved the following general coincidence result.

Theorem 7.4 *If $(\Sigma, \mathcal{E})$ is a finitary theory then, for every countably based continuous dcpo D, the free $(\Sigma, \mathcal{E})$-algebra in* **TP** *carries the Scott topology and coincides with the free $(\Sigma, \mathcal{E})$-algebra in ω***Cont***.*

As is well known, Plotkin's category of *bifinite* ω-algebraic dcppos (originally called *SFP* objects [36]) is closed under the above powerdomains, and also under all the constructions on domains discussed in Sections 5 and 6. Furthermore, all such constructions are preserved by the inclusion of ω-bifinite dcppos in **TD**. Thus, nothing new is achieved by interpreting these constructions in the richer setting of topological domains. One might as well use traditional domain theory.

However, there are other free algebras of interest in semantics. One particularly important example is Jones and Plotkin's *probabilistic powerdomain*, which is used for modelling probabilistic choice [25,26]. The probabilistic powerdomain can be defined for arbitrary dcpos. Jones proved that it cuts down to the subcategory of ω-continuous dcpos [25]. However, it is not known whether the probabilistic powerdomain further restricts to any cartesian closed category of ω-continuous dcppos — see [28] for a discussion of the difficulties that arise. In practice, what this means is that, by iterating applications of function space and probabilistic powerdomain in domain theory, one may be taken outside the world of ω-continuous dcpos.

There are various approaches to obtaining the probabilistic powerdomain as a free algebra, cf. [25,17]. One possibility is to make use of a countably infinite operation to implement countable convex combinations, cf. [9]. This fits into the theory of $(\Sigma, \mathcal{E})$-algebras presented above, but the equational theory is complicated (in [9] a non-equational theory is used). An arguably preferable alternative is to instead use a single parametrized binary operation,

$$\text{choose}\colon [0,1] \times D \times D \to D \ ,$$

where, computationally, $\text{choose}(\lambda, x, y)$ reads as: choose alternative x with probability λ and otherwise choose alternative y (mathematically this amounts to a convex combination $\lambda x + (1-\lambda)y$). With such a parametrized operation, one can give elegant axioms for an appropriate equational theory, which is simply the theory of (finite) convex combinations, cf. [25]. In order to implement the right continuity constraints on algebras, it is important that the parameter space $[0,1]$ is given the Euclidean topology. In the domain-theoretic setting, one does then obtain a characterisation of the probabilistic powerdomain as a free algebra over ω-continuous dcppos, cf. [25].

In [3], Battenfeld considers a general notion of equational theory for algebras whose operations may be parametrized by countably based topological spaces. Moreover, generalising Theorem 7.1 above, [8] he shows that topological predomains have free algebras for all such parametrized equational theories. In particular, one can obtain a probabilistic powerdomain using the parametrized "choose" operation, as outlined above. Other useful examples of free parametrized algebras are also presented in [3].

With the probabilistic powerdomain, one has a computationally useful example for which a combination of constructions on topological domains need not agree with the corresponding combination in ordinary domain theory. Indeed, because of the difficulties identified in [28], it is plausible that a single application of the probabilistic powerdomain to a finite partial order might lead outside Jung's category of FS domains. Following this, a single function space construction may lead to a case in which the function space in

[8] Inequational theories are treated as examples of Sierpinski-parametrized equational theories.

topological domains does not carry the Scott topology, since Theorem 5.10 no longer applies. If so, a two step construction over a finite partial order gives a disagreement between ordinary and topological domain theory.

Since the foregoing discussion is hypothetical, we present an example in which such a disagreement can be shown to actually occur.

Example 7.5 (Midpoint algebras) Midpoint algebras (cf. [9]), have one binary operation $\oplus$ and equations:

$$\begin{aligned} x \oplus x &= x \\ x \oplus y &= y \oplus x \\ (x \oplus y) \oplus (z \oplus w) &= (x \oplus z) \oplus (y \oplus w) \ . \end{aligned}$$

These axioms capture the equational properties of the operation of taking midpoints in Euclidean space.

One can calculate the free midpoint algebra in ω**Cont** (and hence, by Theorem 7.4, in **TP**) over the four point lattice $\mathbb{S} \times \mathbb{S}$ (where $\mathbb{S}$ is Sierpinski space). This free algebra is ω-algebraic with least element. However, it is not bifinite, since the two compact elements $(\bot, \bot) \oplus (\top, \top)$ and $(\bot, \top) \oplus (\top, \bot)$ have infinitely many minimal upper bounds. One can then argue, similarly to [4, Proposition 5.3], that the function space from this free algebra to itself in **TD** does not carry the Scott topology. This example is, admittedly, not computationally motivated.[9] Nonetheless it does illustrate that combining free algebra constructions and function spaces can lead to disagreements between ordinary and topological domain theory.

The reason for emphasising such potential differences is that, when disagreement does occur, we argue that the constructions of topological domain theory are to be preferred to the ordinary domain-theoretic ones. A conceptual argument for this is that the constructions of topological domain theory support the analysis in terms of physical feasibility presented in Section 3. In Section 10 we shall argue that there are also pragmatic reasons for preferring topological domain theory. (Of course, in cases in which there is no disagreement, one can equivalently use ordinary domain theory.)

8 Computability

In this section, we discuss the way in which topological domains can be used as a basis for developing a theory of computability, thereby addressing desideratum (ii) from Section 1. In principle, such a theory of computability for topological domains should allow questions of computability and definability

[9] Although midpoint algebras do have a close connection to the probabilistic powerdomain, see [17].

to be addressed for datatypes that cannot be modelled as ω-continuous dcppos (for example, if combinations of free algebras and function spaces are used, as in the previous section). However, the theory we have at present is not yet in a sufficiently mature shape for such applications to be straightforward. Improving this situation is an interesting direction for further work.

Having abstracted away from representations and worked with "extensional" topological structure throughout Sections 4–7, in order to present our approach to computability, we now return to the more "intensional" idea of considering spaces as coming equipped with representations in the sense of Section 3. To this end, we consider the category **Rep** whose objects are quotient representations and whose morphisms are the physically feasible functions. Equivalently (and preferably), one can ignore the topology on the represented space entirely, and take the objects to be partial surjections from $\mathbb{N}^\omega$ onto sets. By the results surveyed in Section 4, the full subcategory **AdmRep** of admissible quotient representations in **Rep** is equivalent to **QCB**. Moreover, one can obviously cut down further to obtain a full subcategory **TPAdmRep** equivalent to **TP**.

The category **Rep** has a subcategory $\mathbf{Rep_{eff}}$ whose morphisms are those physically feasible functions for which there exists an associated function on names that is computable by a type two Turing machine (cf. [54], where such functions are called *relatively computable*). There is thus a natural sense in which one can identify the *effective* morphisms in **Rep**. Such considerations immediately apply also to full subcategories of **Rep** such as **AdmRep** and **TPAdmRep**. Hence, by working directly with admissible quotient representations, rather than with the spaces they represent, one has immediate access to a theory of effectivity.

This theory is, however, unsatisfactory. For the effective categories to be of any use, one needs to ensure that all constructions of interest "effectivize". For example, to obtain the cartesian closedness of the subcategory of effective maps, one needs to cut down the admissible quotient representations to those that are *effectively admissible* in the sense of [47]. When one further restricts to topological domains, additional restrictions need to be placed on the category to ensure that the domain-theoretic constructions (e.g., fixed points) also effectivize. In fact, as we now outline, such concerns can be dealt with in an automatic way by exploiting a very useful connection between topological domain theory and domain theory in "realizability models" as studied in [35,22,30,42,34].

By the description of objects of **Rep** as partial surjections, one sees that **Rep** is simply the category $\mathbf{Mod}(\mathbb{N}^\omega)$ of so-called *modest sets* on $\mathbb{N}^\omega$, cf. [29]. [10] Remarkably, the category **AdmRep** turns out to be exactly the full subcat-

[10] As is well known, $\mathbb{N}^\omega$ can be construed as a partial combinatory algebra (Kleene's second model), see, e.g., [29].

egory of *extensional* (sometimes called *regular*) objects of $\mathbf{Mod}(\mathbb{N}^\omega)$ in the sense of synthetic domain theory, cf. [22,34]. [11] Further, its full subcategory **TPAdmRep** is exactly the full subcategory of *complete* extensional objects in $\mathbf{Mod}(\mathbb{N}^\omega)$, in the sense of [30,34]. Thus **TP** is equivalent to the category of complete extensional objects in $\mathbf{Mod}(\mathbb{N}^\omega)$. Full proofs of these equivalences appear in Battenfeld's Diploma dissertation [2].

The above equivalences adapt to the effective case as follows. By mimicking the definitions of extensional and complete extensional objects in the category of $\mathbf{Rep}_{\mathbf{eff}}$ (rather than in **Rep**) one obtains full subcategories of $\mathbf{Rep}_{\mathbf{eff}}$ of *effectively extensional* and *effectively complete extensional* objects respectively. The former is exactly Schröder's category of effectively admissible representations, and hence the "correct" effective analogue of the category **QCB**. The latter is the desired effective version of the category of topological predomains. Thus, we do indeed end up with an appropriate full subcategory of $\mathbf{Rep}_{\mathbf{eff}}$ of effective maps between effectivized topological domains. However, the route followed above has not given rise to any pleasant description of what the objects of this category actually are.

In our view, the main unsatisfactory feature of the entire approach considered above is that the objects of the categories considered are representations, and thus spaces have to be encoded, often in unnatural ways, as subquotients of $\mathbb{N}^\omega$. The topology of a represented space plays no role beyond being structure that is ultimately derivable from any given representation of the space.

It would be desirable instead to have a more direct theory of *effectively presented* qcb spaces. Such an effective presentation should consist of a qcb space, together with sufficient additional information (e.g., an enumeration of a pseudobase, possibly supporting various additional operations) for it to be possible to recover an effectively admissible quotient representation from the information. Moreover, every effectively admissible quotient representation should be so recoverable (up to isomorphism) from some effectively presented qcb space. In addition, one would like a direct account of effective maps between effectively presented qcb spaces, so that the effective maps correspond exactly to those that are effective between the induced effectively admissible representations. [12] Having achieved this, one should be able to refine the approach to obtain a corresponding theory of effectively presented topological domains.

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[11] This equivalence depends on first identifying a *dominance* in $\mathbf{Mod}(\mathbb{N}^\omega)$, in the sense of [44]. For this, one can take any admissible quotient representation of Sierpinski space.

[12] An analogy might help the reader to understand what is being envisaged here. The notion of effectively presented qcb space should be considered analogous to the notion of *effectively given domain* in [38, Ch. 7]. The induced effectively admissible quotient representation is then analogous to the standard enumeration of recursive elements in an effectively given domain of [38, Ch. 7 Definition 1]. Finally, the equivalence of the two notions of effective map is analogous to the similar equivalence of [38, Ch. 7 Theorem 1].

At present, we do not know what precise form such notions of effective presentation will take. So we leave this as a challenge for future research.

Problem 8.1 *Find a theory of effectively presented qcb spaces along the lines outlined above.*

Problem 8.2 *Refine this to obtain theory of effectively presented topological domains.*

9 Polymorphism

In this section we briefly discuss a further interesting property of topological domains: they provide a model of full impredicative polymorphism. Once again, the development relies crucially on the relationship with realizability models, discussed above.

In Section 8, we stated that **QCB** is equivalent to the category of extensional objects in the category of modest sets $\mathbf{Mod}(\mathbb{N}^\omega)$, and that **TP** is equivalent to the category of complete extensional objects. It is known that the categories of extensional and complete extensional objects in any category of modest sets are (equivalent to) "small complete" categories (in the sense of [21,23]) within an ambient category of *assemblies*, cf. [30]. This alone provides sufficient structure for interpreting full impredicative polymorphism (i.e., the Girard/Reynolds second-order λ-calculus). Thus, albeit in a roundabout way, it is possible to model full impredicative polymorphism in the category **TP** of topological predomains. In fact one can even ensure that the model satisfies Reynolds' useful principle of *relational parametricity*, cf. [43].

In order to combine parametric polymorphism and domain-theoretic constructs a subtler approach is required, since full relational parametricity is inconsistent with fixed points. In his invited talk at LiCS 1993 [39], Plotkin proposed second-order intuitionistic linear type theory as a suitable framework for resolving the problem. Under this approach, linearity is used to represent strictness in domain theory, and the appropriate notion of relational parametricity accounts for the universal properties of the various domain constructors in the category of strict maps, cf. Section 6. This elegant framework for parametric polymorphism is compatible with topological domain theory. The observations made above about "small completeness" do indeed suffice to construe the category $\mathbf{TD}_\perp$ as a relationally parametric model of second-order intuitionistic linear type theory, cf. [45].

The above discussion indicates that topological domains do indeed model an appropriate polymorphic calculus for domain-theoretic constructions. However, we find the roundabout route we have taken to obtain the model far from satisfactory. As in Section 8, the drawback with the approach we have followed is that it depends crucially on considering spaces as being given via representations. Thus we pose the following two challenges, the first being a natural precursor to the second.

Problem 9.1 *Find a direct topological account of how* **QCB** *provides a (relationally parametric) model of second-order λ-calculus.*

Problem 9.2 *Find a direct topological account of how* $\mathbf{TD}_\perp$ *provides a (relationally parametric) model of second-order intuitionistic linear type theory.*

10 Summary and discussion

In this paper, we have motivated and introduced the notion of topological domain. Although topological domains are themselves dcppos, the distinguishing feature that differentiates between topological domain theory and ordinary domain theory is that topological domains are not required to carry the Scott topology. The benefit one obtains from this relaxation is that one achieves a category of domains supporting a range of constructions not available for any of the dcpo-based categories of domains. Specifically, we have shown that, in addition to the standard constructions of domain theory covered in Sections 5 and 6: topological domains support the construction of free algebras for a wide class of (in)equational theories, thereby allowing a variety of computational effects to be modelled; they have an associated theory of computability; and they model parametric polymorphism.

Notwithstanding such pragmatic considerations, the definition of topological domain also enjoys the property of having strong conceptual motivation. Essentially, we combined just two features: the idea that datatypes should be modelled by topological spaces for which the topology can be explained in terms of physical feasibility, which led to the restriction to qcb spaces; and the desire to model recursion using fixed points.

In our route to identifying qcb spaces in Sections 3 and 4, the idea of having concrete representations of elements of spaces (as streams of natural numbers) is crucial to the argument. This is intriguing because it runs contrary to a popular view of topology, that of *locale theory* [24,53], according to which elements of spaces should be disregarded in favour of considering the algebraic structure of the lattice of open sets as the primary description of a space.

It is an open question whether there is a natural localic analogue of the category of qcb spaces carrying the same useful categorical structure as **QCB**. In fact, certain interesting technical difficulties lie in the way of developing one. The standard adjunction between topological spaces and locales yields an equivalence between the full subcategories of sober spaces and spatial locales [24]. By going round the adjunction, an arbitrary space gets mapped to its *sobrification*. It turns out not to be possible to cut down this familiar situation to qcb spaces, since Gruenhage and Streicher have shown that qcb spaces are not closed under sobrification [16]. Nevertheless, it may still be possible to abstract from the lattices of opens of qcb spaces on the localic side, and relate such locales to qcb spaces by mapping a locale to its associated *replete* qcb space in the sense of [22], which is determined up to homeomorphism. One simplifying aspect of such an account would be that it would no longer be necessary to consider any notion of predomain, since replete qcb spaces are already contained in the category **TP**. However, we leave it as a question for future research whether any useful localic abstraction of the lattices of opens of qcb spaces is possible. We find it plausible that there is none, in which case the good properties of qcb spaces would seem entirely dependent on their development taking place in the setting of point-set topology.

For the traditional domain theorist, the most unpalatable aspects of topological domain theory are likely to be the descriptions of the topologies on products, function spaces and subspaces, as presented in Section 4, all of which involve sequentializations. Such descriptions seem unavoidable whenever one allows more general topologies on domains than the Scott topology. In fact, one can show that the constructions on qcb spaces are, in an empirical sense, canonical. Although there are many different approaches to obtaining "topological" (in a broad sense) cartesian closed categories, the category of **QCB** appears (perhaps unexpectedly) as a common core within all approaches. For example, in [8], it is shown that **QCB** lives as a full cartesian closed subcategory [13] of all the main cartesian closed categories of topological spaces; and, in [31], it is shown that it also lives as a full cartesian closed subcategory of Scott's category of *equilogical spaces*, introduced in [5], which is a supercategory of **Top**. [14] Such embeddings provide alternative descriptions of the products and function spaces of qcb (though no more transparent than those given here). The embeddings in cartesian closed categories of topological spaces, also allow connections between the convenient domain theory of the present paper and the subject of *convenient topology*, as presented in [6,52,41], to be established. This is developed in detail in [4].

The immediate avenues for future development concern obtaining more explicit accounts of computability and polymorphism, as outlined in Sections 8 and 9. Furthermore, the particular features we have highlighted of topological domains (points (i)–(iv) in the introduction) are by no means exhaustive as desiderata to place on a category of domains. One might, for example, also like to establish well-behavedness properties of the various functor categories used for modelling local variables and similar. More generally, one would like the category to provide as adaptable and flexible a toolkit for semantic

[13] By *cartesian closed subcategory* we mean that the inclusion preserves the cartesian closed structure.
[14] A gap in the literature that still needs filling here is to show that **QCB** is a full cartesian closed subcategory of Hyland's category of filter spaces [20].

constructions as possible. This desire is necessarily open-ended, and it will be interesting to see to what extent the "convenient" category of topological domains meets the challenge.

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