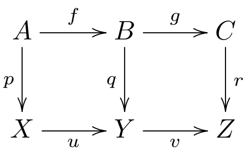
The pullback lemma in gory detail

Andrej Bauer

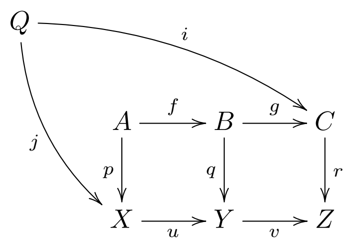
May 30, 2012

A poor soul on the internet asked for the proof of the pullback lemma, which in every book on category theory is left as an exercise. I took pity on and wrote down the proof in gory detail.

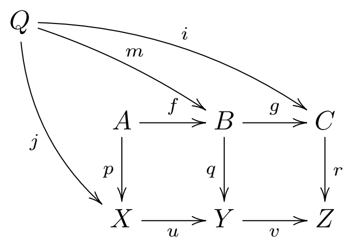
Suppose in the following diagram the two squares are pullbacks:



We would like to show that the outer rectangle is a pullback. For this purpose, consider the diagram

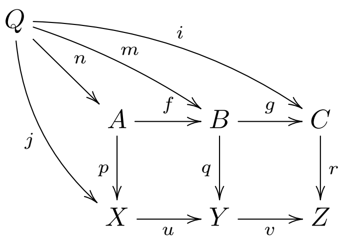


in which $r \circ i = v \circ u \circ j$. Because $v \circ (u \circ j) = r \circ i$, by the universal property of the right-hand square there exists a unique $m : Q \to B$ such that $g \circ m = i$ and $q \circ m = u \circ j$:



Now by the universal property of the left-hand pullback there exists a unique

$n : Q \to A$ such that $f \circ n = m$ and $p \circ n = j$:



We claim that n is the morphism we are looking for. Indeed, we already have $p \circ n = j$, and also

$$g \circ f \circ n = g \circ m = i.$$

It remains to show uniqueness of n. Suppose $n' : Q \to A$ satisfies $p \circ n' = j$ and $g \circ f \circ n' = i$. Let $m' = f \circ n'$, and observe that $g \circ m' = g \circ f \circ n' = i$ and $q \circ m' = q \circ f \circ n' = u \circ p \circ n' = u \circ j$, therefore by the uniqueness property of the right-hand square we get $m = m' = n' \circ f$. Now by the uniqueness property for the left-hand square we get $n = n'$, as desired.