Brief Introduction to AdS/CFT

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SISSA – PhD course

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Suggested readings

Main material:

- Maldacena, "TASI 2003 lectures on AdS / CFT", hep-th/0309246.

  Large N limit (vector and matrix theories). Conformal symmetry, symmetries of AdS, operator/state correspondence, $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ SYM. Spectrum, correlation functions, diagrams. Thermal and confining theories, transition. Plane wave limit.

- Zaffaroni, "Introduction to the AdS/CFT correspondence", http://laces.web.cern.ch/Laces/LACES09/notes/dbranes/lezioniLosanna.pdf

- Aharony, Gubser, Maldacena, Ooguri, Oz, "Large N Field Theories, String Theory and Gravity", hep-th/9905111.

  Introduction. Large N. Conformal symmetry, primaries, OPE, superconformal algebra. Conformal structures, AdS. $\mathrm{AdS}_5 \times S^5$. AdS/CFT, operator/field correspondence, correlation functions. Phases. Wilson loops. Finite temperature. More general geometries, a, c anomalies. Non-conformal deformations.

- Klebanov, "TASI lectures: Introduction to the AdS / CFT correspondence", hep-th/0009139.

  One- and two-point functions. Conifold, wrapped branes and chiral operators.

- Polchinski, "Introduction to Gauge/Gravity Duality", arXiv:1010.6134.

- Nastase, "Introduction to AdS-CFT", arXiv:0712.0689.

  A little bit of everything. Spin chains.

- Kaplan, "Lectures on AdS/CFT from the Bottom Up", http://www.pha.jhu.edu/ jaredk/AdSCFTCourseNotesPublic.pdf

Holographic renormalization:

- Skenderis, "Lecture notes on holographic renormalization", hep-th/0209067.

  Asymptotically AdS, conformal structures. Correlators, holographic renormalization.

AdS/CMT:

- Herzog, "Lectures on Holographic Superfluidity and Superconductivity", arXiv:0904.1975.

  Green's functions, transport coefficients, Ward identities. Transport coefficients from holography. Holographic model of superconductivity with computations.

- Hartnoll, "Lectures on holographic methods for condensed matter physics", arXiv:0903.3246.

- McGreevy, "Holographic duality with a view toward many-body physics", arXiv:0909.0518.

1 Introduction: What is AdS/CFT?

This course is about AdS/CFT: a surprising "correspondence" between

$$\begin{array}{ccc} \text{Theories of quantum gravity} & & \text{Non-gravitational QFTs} \\ \text{in } d+1 \text{ dimensions} & \longleftrightarrow & \text{on the } d\text{-dimensional boundary.} \end{array}$$

In its most basic but also precise examples, the correspondence is between quantum gravities on a specific background, *anti-de-Sitter* space, and conformal QFTs.

There are various indications that there should be such a correspondence. Let us mention three basic arguments.

- One argument comes from considering the large N behavior of certain QFTs. This suggests the appearance of strings, whose low energy limit is a gravitational theory.
- Another argument comes from the holographic principle. The Bekenstein bound states that the maximal entropy in a region of space is

  $$S_{\text{max}} = \frac{A}{4G_N} ,$$

  as otherwise one could lower the entropy by forming a black hole and this would violate the second law of thermodynamics.[1] This suggests that the physics can be described by a theory living on the boundary.
- A third argument is that quantum gravity does not have local gauge-invariant operators (only approximate ones), thus the observables should be at the boundary of spacetime.

Notice that if such a correspondence is true, it should necessarily relate regimes in which at least one of the two sides is strongly coupled. If both were weakly coupled, we could "see" whether the theory has almost-free gauge bosons and/or gravitons (much in the same way as we measure photons and, in principle, gravitons).

Indeed, AdS/CFT is one of the finest achievements of string theory in the last decade: one can use holography to investigate strongly-coupled quantum field theories. One crucial aspect of the correspondence is the possibility of computing quantum effects in a strongly-coupled field theory using a classical gravitational theory. This has deep consequences that go far beyond string theory.

Originally introduced to study the quantum behaviour of scale invariant theories, the correspondence has been extended to non-conformal theories, where it gives an explanation for confinement and chiral symmetry breaking. It has also been used to study non-equilibrium phenomena in strongly coupled plasmas, and applied to condensed matter systems.

[1] The Bekenstein bound [Bek81] is $S \leq 2\pi r E$, where r is the size of the system and E its energy. For given radius, the maximal possible amount of energy that can fit into a sphere is the mass of a Schwarzschild black hole, and $r_s^{d-3} \sim G_N E$. For the maximal value of the energy, the bound is given by the area of the horizon.

2 Large N limits

Certain classes of theories simplify when we take a large number of fields, or a large gauge group, or large central charges.

2.1 Vector theories

Vector theories are characterized by having fields that transform as vectors of an $SO(N)$ symmetry.

The 2d $O(N)$ model consists of N real fields n^i, with $i = 1, \dots, N$, transforming in the fundamental representation of $O(N)$, with the classical constraint that $\vec{n}^2 = 1$. In other words, this is a NLSM with target S^{N-1} and Lagrangian

$$S = \frac{1}{2g_0^2} \int d^2x \, (\partial \vec{n})^2 \qquad \text{with} \qquad \vec{n}^2 = 1 \; . \tag{2.1}$$

To understand the large N limit, we introduce a Lagrange multiplier field λ that enforces the constraint:

$$S = \frac{1}{2g_0^2} \int d^2x \left[(\partial \vec{n})^2 + \lambda(\vec{n}^2 - 1) \right] \; . \tag{2.2}$$

Since now the fields n^i have linear target and appear quadratically, they can be integrated out exactly (the Gaussian path-integral can be performed exactly):

$$\int \mathcal{D}n^i \, e^{-\frac{1}{2g_0^2} n^i \mathcal{O} n^i} = \frac{c}{\sqrt{\det \mathcal{O}}^N} = c \, e^{-\frac{N}{2} \log \det \mathcal{O}} \qquad \text{with} \qquad \mathcal{O} = -\partial_x^2 + \lambda(x) \; . \tag{2.3}$$

Thus the effective action for λ is

$$S = \frac{N}{2} \Big[\log \det(-\partial^2 + \lambda) - \frac{1}{g_0^2 N} \int d^2x \, \lambda \Big] \; . \tag{2.4}$$

We see two things. 1) In the large N limit, the theory becomes semiclassical (large action). 2) At large N, the effective coupling is $g_0^2 N$ (called the 't Hooft coupling) and we should keep it fixed as $N \to \infty$.

Since the theory is semiclassical, we can compute the VEV[2] of λ in the saddle-point approximation $\partial S / \partial \lambda = 0$. Use

$$\frac{\partial}{\partial \lambda} \log \det \mathcal{O} = \frac{1}{\det \mathcal{O}} \, \det \mathcal{O} \; \mathrm{Tr} \, \mathcal{O}^{-1} \frac{\partial \mathcal{O}}{\partial \lambda} = \mathrm{Tr} \, \mathcal{O}^{-1} \frac{\partial \mathcal{O}}{\partial \lambda} \; . \tag{2.5}$$

The saddle-point equation becomes

$$1 = N g_0^2 \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + \lambda} = \frac{N g_0^2}{4\pi} \log \left(p^2 + \lambda \right) \Big|_0^{\text{cut-off}} \tag{2.6}$$

[2]Here we just compute the VEV, so we assume that λ is constant.

We find the VEV of λ:

$$\lambda = \frac{\Lambda^2}{e^{\frac{4\pi}{Ng_0^2}} - 1} \simeq \Lambda^2 e^{-\frac{4\pi}{Ng_0^2(\Lambda)}} \tag{2.7}$$

in terms of the renormalized running coupling[3] $g_0(\Lambda)$ (the last approximation is good for $Ng_0^2(\Lambda) \ll 1$). In particular, we see that the theory is asymptotically free: imposing independence on the cut-off Λ we get

$$Ng_0^2(\Lambda) = \frac{4\pi}{\log\left(1 + \frac{\Lambda^2}{\lambda}\right)} . \tag{2.8}$$

Notice however that, at least in this approximation, Ng_0^2 does not diverge at any finite energy scale.

Going back to the action (2.2), the VEV for λ gives a positive mass to n^i. Thus the theory is gapped, the VEV for n^i is zero and the symmetry $O(N)$ is unbroken. Indeed in two dimensions one cannot break continuous symmetries spontaneously.[4]

In many respects, this theory is similar to 4d QCD: it is asymptotically free; it has a mass gap; it has a large N expansion in which the mass gap persists and the gap is non-perturbative in $g_0^2 N$.

Addendum. We can consider the version with potential:

$$S[\varphi] = -\frac{1}{2}\int d^dx \Big(\partial_\mu\vec\varphi\cdot\partial^\mu\vec\varphi + m^2\vec\varphi\cdot\vec\varphi - \frac{\lambda}{2n}(\vec\varphi\cdot\vec\varphi)^2\Big) , \tag{2.9}$$

where $\vec\varphi$ is a real vector with n components. We integrate-in the field σ:

$$S[\varphi,\sigma] = -\frac{1}{2}\int d^dx \Big(\partial_\mu\vec\varphi\cdot\partial^\mu\vec\varphi + m^2\vec\varphi\cdot\vec\varphi + \sigma\,\vec\varphi\cdot\vec\varphi + \frac{n}{2\lambda}\sigma^2\Big) . \tag{2.10}$$

Now $\vec\varphi$ can be integrated out exactly, and we have n copies of the same system:

$$S[\sigma] = -\frac{n}{2}\Big[\int d^dx\,\frac{\sigma^2}{2\lambda} + \mathrm{Tr}\log(-\partial^2 + m^2 + \sigma)\Big] . \tag{2.11}$$

2.2 4d YM at large N

Yang-Mills in 4d has no dimensionless coupling: the gauge coupling is classically marginal but quantum mechanically it is dimensionally transmuted into Λ_{QCD} which is a mass scale.

[3] This $g_0(\Lambda)$ is the curvature of the target of the NLSM: if we canonically normalize $\vec n$, then $\vec n^2 = 1/g_0^2$. Thus what we have found is the running of the curvature.

[4] In dimension $d > 2$ the result is different. The coupling is dimensionful and $1/g_0 \equiv R$ with dimension mass$^{\frac{d-2}{2}}$ is the radius of S^{N-1}. The saddle-point equation becomes $1 \simeq Ng_0^2\Lambda^{d-2}$, implying that $(Ng_0^2)^{-\frac{1}{d-2}}$ is actually the cutoff of the theory, and compatible with $\lambda = 0$. Below the cutoff the theory is an IR free NLSM (field oscillations are of order $E^{\frac{d-2}{2}} \ll R$) in agreement with symmetry breaking, while above the cutoff the effective theory is not valid.

Thus, there is no small parameter to expand on to understand the physics at energies around $\Lambda_{\rm QCD}$. However, if we consider gauge group $SU(N)$, we can play with the parameter N and take the limit $N \to \infty$.

How do we scale g as we send $N \to \infty$? In an asymptotically free theory, it is natural to keep $\Lambda_{\rm QCD}$ fixed. The one-loop beta function is

$$\frac{\partial g}{\partial \log \mu} = -\frac{11}{3} N \frac{g^3}{16\pi^2} + \mathcal{O}(g^5) \ . \tag{2.12}$$

In order to have the leading terms of the same order, we should keep fixed the 't Hooft coupling

$$\lambda = g^2 N \ . \tag{2.13}$$

Then

$$\frac{\partial \lambda}{\partial \log \mu} = -\frac{22}{3} \frac{\lambda^2}{16\pi^2} + \mathcal{O}\big(\lambda^3\big) \ . \tag{2.14}$$

Again, we are led to the 't Hooft limit.

The same computation is true if we add matter fields in the adjoint representation, at least as long as it is asymptotically free.

2.3 Matrix theories

Matrix theories are characterized by having fields that transform in the adjoint representation of, say, $U(N)$. This could be a pure $U(N)$ gauge theory, or with the addition of fields in the adjoint representation.

Consider a theory with a matter field which is a Hermitian matrix M. The Lagrangian is, schematically,

$$\mathcal{L} = \frac{1}{g^2} \operatorname{Tr} \big[(\partial M)^2 + V(M) \big] = \frac{1}{g^2} \operatorname{Tr} \big[(\partial M)^2 + c_2 M^2 + c_3 M^3 + \dots \big] \ . \tag{2.15}$$

This action is $U(N)$ invariant, as $M \to U M U^\dagger$.

This could come from a YM theory: in this case M is the gauge boson and in canonical normalization one would write

$$F \sim \partial A + g A^2 \qquad \Rightarrow \qquad \mathcal{L} \sim (\partial A)^2 + g\, \partial A\, A^2 + g^2 A^4 \ . \tag{2.16}$$

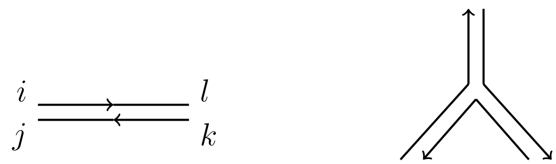
Then redefine $A = M/g$. It could also be a theory with matter fields (fermions and/or bosons) in the adjoint, as long as the interactions have the correct scaling with g.

We ask what is the large N limit. Naively,

$$\frac{1}{g^2} = \frac{N}{\lambda} \tag{2.17}$$

so one might conclude that for $N \to \infty$ with λ fixed the theory is classical, but this is tricky because also the number of components diverges.

To draw Feynman diagrams, we employ a double-line notation to keep track of the matrix indices:



For a theory of $U(N)$ or $SU(N)$ matrices, the propagators are[5]

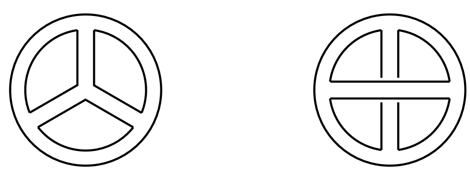
$$\langle M^i_j M^k_l \rangle_{U(N)} \propto \delta^i_l \delta^k_j \, , \qquad\qquad \langle M^i_j M^k_l \rangle_{SU(N)} \propto \delta^i_l \delta^k_j - \frac{1}{N} \delta^i_j \delta^k_l \tag{2.18}$$

respectively. Since for $SU(N)$ the second term is subleading, we can safely neglect it a leading order.

Thus given a diagram, each propagator gives a factor g^2 and each vertex a factor $1/g^2$. Each closed line represents a sum over matrix indices, therefore gives a factor of N. Thus, each diagram contributes with

$$(g^2)^{\#\text{ Prop } - \text{ }\#\text{ Vert}} N^{\#\text{ Closed lines}} \, . \tag{2.19}$$

We can transform a diagram into a two-dimensional surface, completing the closed lines with non-intersecting faces. *E.g.*:



We have

$$\#\text{ Prop } \to \#\text{ Edges} \, , \qquad\qquad \#\text{ Closed lines } \to \#\text{ Faces} \, . \tag{2.20}$$

Since the double-lines are oriented, the resulting surface is oriented. Consider first the case of connected vacuum diagrams. The resulting surface is a compact, closed, oriented surface. We can write

$$N^{\#\text{ Faces - }\#\text{ Edges + }\#\text{ Vert}} (g^2 N)^\alpha = N^{2-2h} (g^2 N)^\alpha \, , \qquad \alpha = \#\text{ Edges - }\#\text{ Vert} \, , \tag{2.21}$$

where h is the genus of the 2d surface. Thus, in a large N limit in which $N \to \infty$ with $\lambda = g^2 N$ fixed ('t Hooft limit), planar diagrams dominate while higher-genus diagrams are suppressed. The theory is not classical! But still it simplifies.

[5] The second one is derived imposing that the propagator annihilates $M^i_j = \delta^i_j$.

Notice that $\alpha \geq 0$: for instance, if all vertices are k-valent, then

$$\# \text{ Edges} = \frac{k \cdot \# \text{ Vert}}{2} \geq \# \text{ Vert} . \tag{2.22}$$

The sum of all connected vacuum planar diagrams gives

$$F_0 = N^2 f_{(0)}(g^2 N) ,$$

while the full partition function is

$$\log Z = F = \sum_{h=0}^{\infty} N^{2-2h} f_{(h)}(g^2 N) . \tag{2.23}$$

One may worry that the planar graphs give a diverging contribution to the free energy. However notice that N^2 is precisely the order of the Lagrangian (the tree level contribution).

As the 't Hooft coupling $\lambda = g^2 N$ becomes large, a large number of planar diagrams contribute and they become dense on the sphere. So we could think that they describe a discretized version of the worldsheet of some string theory. This argument is valid for any matrix theory, but it does not tell us what the worldsheet theory is. The argument tells us that we can expect a large N matrix theory to behave as a string theory, which in turn includes gravity. Moreover $\frac{1}{N}$ plays the role of the string coupling g_s (which weights genera), thus at large N we might expect string loop corrections to be negligible.

Now consider correlation functions. Consider the so-called "single-trace operators":[6] gauge-invariant operators that cannot be written as products of other gauge-invariant operators:

$$\mathcal{O} = \operatorname{Tr} P(M) . \tag{2.24}$$

To compute correlation function, we simply add them to the Lagrangian:

$$\mathcal{L} \to \mathcal{L} + \sum_j \eta_j \mathcal{O}_j , \tag{2.25}$$

so

$$\mathcal{L} = \frac{N}{\lambda} \operatorname{Tr} \Big[(\partial M)^2 + V(M) + \frac{\lambda \eta_j}{N} P_j(M) \Big] . \tag{2.26}$$

If we keep η/N fixed, we have the same large N scaling as before, thus the new partition function is dominated by planar diagrams and

$$Z = e^{N^2 f\left(\lambda, \frac{\eta_j}{N}\right) + O(1)} . \tag{2.27}$$

[6] For multi-trace operators, the connected 2-point function in general scales with some positive power of N. Therefore the operators have to be normalized dividing by some powers of N, and then the correlators are suppressed.

To compute correlation functions, we take derivatives with respect to η_j, thus

$$\begin{aligned}
\langle \mathcal{O}_j \rangle &= \frac{1}{Z}\frac{\partial Z}{\partial \eta_j} = N\, f_j + O(1/N) \\
\langle \mathcal{O}_j \mathcal{O}_k \rangle &= \frac{1}{Z}\frac{\partial^2 Z}{\partial \eta_j \partial \eta_k} = N^2 f_j f_k + f_{jk} + O(1/N^2) = \langle \mathcal{O}_j \rangle \langle \mathcal{O}_k \rangle + O(1)
\end{aligned} \tag{2.28}$$

etc... where $f_j = \partial f/\partial x_j$ etc... We see that the dominant contribution comes from disconnected diagrams (1-point functions).

If we look at connected diagrams instead, we find

$$\langle \mathcal{O}_{j_1} \dots \mathcal{O}_{j_n} \rangle_c = \frac{\partial^n F}{\partial \eta_{j_1} \dots \partial \eta_{j_n}} = N^{2-n} f_{j_1 \dots j_n} \ . \tag{2.29}$$

Restricting to connected diagrams, the operators are correctly normalized (2-point functions are of order 1) and the 3-point function goes like $1/N$. Thus, we can interpret the operators $\mathcal{O}_j$ as vertex operator insertions on the string worldsheet, if $\frac{1}{N}$ is the string coupling g_s.

The argument generalizes in many ways.

If we add matter in the fundamental representation of $U(N)$, we have propagators with a single line. This gives rise to diagrams with boundaries. This suggests that at large N one gets a string theory with open strings (and D-branes).

For $SO(N)$ and $USp(N)$ gauge group, the adjoint representation can be represented by the product of two fundamental representations, and the fundamental is real. So there is a double-line notation with no arrows, leading to non-orientable surfaces. This suggests that one could get non-orientable strings.

2.4 Extra dimension

The previous arguments suggest that certain gauge theories at large N should behave as string theories. One might think that from a D-dimensional gauge theory one gets a D-dimensional string theory on flat space, but perturbative string theory is consistent quantum mechanically only for $D = 26$ (or $D = 10$ for the superstring).

The reason is that we start with the Polyakov action

$$S \sim \frac{1}{4\pi} \int d^2\sigma \, \sqrt{\eta}\, \eta^{ab} \partial_a X \partial_b X \ . \tag{2.30}$$

This action is classically invariant under Weyl rescaling

$$\eta_{ab} \to \hat{\eta}_{ab} = e^{\phi} \eta_{ab} \ . \tag{2.31}$$

However this is not so quantum mechanically. After introducing the ghosts b, c to fix the gauge under diffeomorphisms, we are left with 2d CFT with central charge $c = D - 26$.[7]

[7]The bc system with weights $(\lambda, 0)$ and $(1-\lambda, 0)$ has $c = -3(2\lambda - 1)^2 + 1$. To fix diffeomorphisms one needs $\lambda = 2$.

Thus there is a conformal anomaly

$$T^a_a = -\frac{c}{12} R_{(2)} = \frac{26-D}{12} R_{(2)} \ . \tag{2.32}$$

The critical string lives in a number of dimensions (here $D = 26$) such that there is no conformal anomaly.

The non-critical string lives in a smaller number of dimensions. Since Weyl invariance is broken, in general a cosmological constant is produced by renormalization. Under a variation $\delta\eta_{ab} = \delta\phi\,\eta_{ab}$ the variation of the action is

$$\delta S = \int d^2\sigma\,\sqrt{\eta}\,\delta\eta^{ab}\big(T_{ab} + \text{const}\big) = \frac{26-D}{12}\int d^2\sigma\,\sqrt{\eta}\,\delta\phi\,(R_{(2)} + \mu) \ . \tag{2.33}$$

We should integrate this variation for finite ϕ. Using

$$\hat{R}_{(2)} = e^{-\phi}\big[R_{(2)} - \Box\phi\big] \ , \qquad\qquad \sqrt{\hat{\eta}} = e^{\phi}\sqrt{\eta} \ , \tag{2.34}$$

we get

$$S_{\text{eff}}(e^{\phi}\eta_{ab}) - S_{\text{eff}}(\eta_{ab}) = \frac{26-D}{48\pi}\int d^2\sigma\,\sqrt{\eta}\,\Big(\frac{1}{2}(\nabla\phi)^2 + R_{(2)}\phi + \mu e^{\phi}\Big) \tag{2.35}$$

This is called the Liouville action.

String theory in $D < 26$ is called "non-critical" string theory. We do not know how to quantize non-critical strings in general. But the message of this computation is that, quantum mechanically, the theory depends on the conformal factor, and the path-integral over ϕ is like adding a new dimension. We see the emergence of a new dimension.

What is the geometry of the resulting space? Again, it cannot be flat (unless $D = 25$). It should have D-dimensional Poincaré symmetry. Let us take $D = 4$ for simplicity. The metric should be

$$ds^2 = w(z)^2\big(dx^2_{3,1} + dz^2\big) \ . \tag{2.36}$$

Now, suppose we are dealing with a scale-invariant QFT, *i.e.* invariant under

$$x \to \lambda x \ . \tag{2.37}$$

Since string theory has a scale, the string tension, the only way that it is invariant under the rescaling of the target is that the rescaling is an isometry: $z \to \lambda z$ and $w = R/z$. We are led to a spacetime

$$ds^2 = R^2\,\frac{dx^2_{3,1} + dz^2}{z^2} \tag{2.38}$$

which is called *anti-de-Sitter*. This space has negative curvature, and indeed it is the most symmetric space with negative curvature. It is the analytic continuation of the Euclidean hyperbolic space.

These arguments suggest that the large N limit of certain D-dimensional QFTs with scale invariance, could be described by strings moving in AdS_{D+1} spacetime.

3 Conformal symmetry

Let us review some basic facts about CFTs. More details can be found in Section 2 of [AGM+00], Section 1 of [Gin88], and Section 2 of [Min98].

It is widely believed[8] (see [DKST15, DFK+16]) that unitary interacting scale-invariant theories are also invariant under the full conformal group, which is a simple group including scale invariance and Poincaré invariance. The change in the action due to a change in the metric is

$$\delta S = \int d^d x \, \sqrt{g} \, T^{\mu\nu} \delta g_{\mu\nu} \; . \tag{3.1}$$

Under an infinitesimal coordinate transformation $x^\mu \to x^\mu + \zeta^\mu(x)$, the metric changes by

$$\delta g_{\mu\nu} = \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu \; . \tag{3.2}$$

If ζ^μ generates an isometry, then the metric is left invariant: $\delta g_{\mu\nu} = 0$ and ζ^μ is called a Killing vector. The infinitesimal scale transformation

$$x^\mu \to (1 + \delta\lambda) \, x^\mu \tag{3.3}$$

gives $\delta g_{\mu\nu} = 2\delta\lambda \, g_{\mu\nu}$. A sufficient condition to have scale invariance is[9]

$$T^\mu_\mu = 0 \; . \tag{3.4}$$

Then the action is also invariant under coordinate transformations such that $\delta g_{\mu\nu} = h(x) \, g_{\mu\nu}$ with arbitrary h. Coordinate transformations of this type are called *conformal transformations.*

The conformal algebra of Minkowski space in d Lorentzian dimensions is given by the following infinitesimal transformations and their generators:

$$\delta x^\mu = \quad \begin{array}{c|ll|c} a^\mu & P_\mu & & \text{translations} \\ \omega^{\mu\nu} x_\nu & M_{\mu\nu} & (\omega^{\mu\nu} = -\omega^{\nu\mu}) & \text{Lorentz rotations} \\ \lambda \, x^\mu & D & & \text{dilations} \\ b^\mu x^2 - 2x^\mu (bx) & K_\mu & & \text{special conformal} \; . \end{array} \tag{3.5}$$

The generators are constructed (by the standard Noether procedure) from currents, and all currents are constructed with the stress tensor:

$$J^{\text{conf}}_\mu = T_{\mu\nu} \, \delta x^\nu \; , \qquad\qquad Q^{\text{conf}} = \int d^{d-1}x \, J^{\text{conf}}_0 \; . \tag{3.6}$$

In particular P_μ is conserved because $\partial^\mu T_{\mu\nu} = 0$, $M_{\mu\nu}$ because additionally $T_{[\mu\nu]} = 0$, D because $T^\mu_\mu = 0$, and then also conservation of K_μ follows.

[8] An interesting counterexample is given by free Maxwell theory in $d \neq 4$ [ESNR11].

[9] A necessary condition is that the integral of the trace is zero.

The finite form of special conformal transformations is

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2} \;. \tag{3.7}$$

In the conformal group there is also a discrete element, the inversion:[10]

$$x^\mu \to \frac{x^\mu}{x^2} \;. \tag{3.8}$$

The conformal generators satisfy the algebra

$$\begin{aligned}
[M_{\mu\nu}, D] &= 0 & [D, P_\mu] &= -iP_\mu \\
[M_{\mu\nu}, P_\rho] &= -2i\,\eta_{\rho[\mu}P_{\nu]} & [D, K_\mu] &= iK_\mu \\
[M_{\mu\nu}, K_\rho] &= -2i\,\eta_{\rho[\mu}K_{\nu]} & [P_\mu, K_\nu] &= 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D \\
[M_{\mu\nu}, M_{\rho\sigma}] &= -2i\,\eta_{\rho[\mu}M_{\nu]\sigma} - 2i\,\rho_{\sigma[\mu}M_{\rho|\nu]}
\end{aligned} \tag{3.9}$$

and the other ones vanishing. This algebra is isomorphic to the algebra of $SO(d,2)$.[11] We can put it in the standard form, with signature $(-,+,\dots,+,-)$, with the generators J_{ab} given by

$$J_{\mu\nu} = M_{\mu\nu} \;, \qquad J_{\mu d} = \frac{K_\mu - P_\mu}{2} \;, \qquad J_{\mu,d+1} = \frac{K_\mu + P_\mu}{2} \;, \qquad J_{d,d+1} = D \;. \tag{3.10}$$

If we decompose

$$SO(d,2) \to SO(d-1,1) \times SO(1,1) \tag{3.11}$$

then $M_{\mu\nu}$ generates $SO(d-1,1)$ (Lorentz group) and D generates $SO(1,1) \cong \mathbb{R}$.

A special conformal transformation with parameter b^μ maps the point $x^\mu = -b^\mu/b^2$ to infinity (alternatively, the inversion maps the origin to infinity). This suggests that the conformal group acts more nicely if we compactify the space adding the point at infinity, namely taking $S^{d-1}\times\mathbb{R}$. Indeed the maximal compact subgroup of $SO(d,2)$ is $SO(d)\times SO(2)$, and a covering of it (or its algebra)[12] acts in the obvious way on $S^{d-1}\times\mathbb{R}$. The vacuum of a CFT is invariant under all generators.

It is often useful to study the CFT in Euclidean signature on $\mathbb{R}^d$. The Euclidean conformal group is $SO(d+1,1)$. In this case the compactification of $\mathbb{R}^d$ with the point at infinity is S^d, acted upon by the maximal compact subgroup $SO(d+1)$.

If the trace of the stress tensor is zero, then the theory is also invariant under *Weyl transformations*

$$g_{\mu\nu} \to e^{\phi(x)} g_{\mu\nu} \tag{3.12}$$

[10]The composition of inversion, translations, inversion gives special conformal transformations. Then the commutator of K_μ and P_μ gives D, after removing $M_{\mu\nu}$. Thus the conformal group can be generated by adding the inversion to the Poincaré group.

[11]Thus the conformal group is some covering of $SO(d,2)$. Adding the inversion, one gets some covering of $O(d,2)$.

[12]The relevant covering of $SO(2)$ is $\mathbb{R}$, with the same algebra, and it acts on $\mathbb{R}$ by translations.

for arbitrary ϕ: these are not only coordinate transformations, but rather changes of the manifold (the curvature invariants change). In the quantum theory this symmetry has a (calculable) anomaly.

3.1 Representations, primary fields

To construct representations of the conformal group $SO(d,2)$ in $\mathbb{R}^{d-1,1}$ we can decompose into representations of

$$SO(d-1,1) \times SO(1,1) \; .$$

The first factor is the Lorentz group, while the second one is dilations D

$$x^\mu \to \lambda\, x^\mu \; . \tag{3.13}$$

D has eigenvalues $-i\Delta$, and fields or operators which are eigenfunctions transform as

$$\phi(x) \to \lambda^\Delta\, \phi(\lambda x) \; . \tag{3.14}$$

These are called *quasi-primary* fields. The operators P_μ (*i.e.* derivatives) raise the eigenvalue of D, while K_μ lower it. In unitary field theories there is a lower bound on the dimension of a field, depending on its Lorentz representation, called *unitarity bound* (see below).

Thus unitary representations are "lowest weight representations". We start with the operators with the lowest dimension, annihilated by K_μ (at $x = 0$), and in some Lorentz representation. These are called *primary operators*:

$$\text{primary:} \qquad [K_\mu, \Phi(0)] = 0 \; . \tag{3.15}$$

The representation is infinite-dimensional, and all other operators are constructed with the action of P_μ: they are called *descendants*. The action of the conformal group is

$$\begin{aligned}
[P_\mu, \Phi(x)] &= i\partial_\mu \Phi(x) \\
[M_{\mu\nu}, \Phi(x)] &= \big[i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu} \big] \Phi(x) \\
[D, \Phi(x)] &= i \big[x^\mu \partial_\mu - \Delta \big] \Phi(x) \\
[K_\mu, \Phi(x)] &= \big[i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta) - 2x^\nu \Sigma_{\mu\nu} \big] \Phi(x) \; .
\end{aligned} \tag{3.16}$$

where $\Sigma_{\mu\nu}$ are the matrices of a finite-dimensional representation of the Lorentz group, acting on the components of Φ.

Another possibility is to go in radial quantization on $S^{d-1} \times \mathbb{R}$, and then decompose into representations of

$$SO(d) \times SO(2) \; .$$

In this case the generator of $SO(2)$ is $J_{0,d+1} = (K_0 + P_0)/2$ and is called "conformal energy". This decomposition looks like the analytic continuation of the previous one.

3.2 Unitarity bounds

The conformal group in Lorentzian signature is $SO(d,2)$, and in a unitary theory the generators are Hermitian:

$$M^\dagger_{\mu\nu} = M_{\mu\nu} \;, \qquad P^\dagger_\mu = P_\mu \;, \qquad K^\dagger_\mu = K_\mu \;, \qquad D^\dagger = D \;. \tag{3.17}$$

The representation in terms of fields was given before. We are after unitary representations in Lorentzian signature, however it is useful to work in radial quantization, therefore we rotate to Euclidean signature and the conformal algebra is $\mathfrak{so}(d+1,1)$. The generators J_{ab} were defined in (3.10). To obtain $\mathfrak{so}(d+1,1)$, set

$$M'_{\mu\nu} = J_{\mu\nu} \;, \quad D' = iJ_{-1,0} \;, \quad P'_\mu = J_{\mu,-1} + iJ_{\mu,0} \;, \quad K'_\mu = J_{\mu,-1} - iJ_{\mu,0} \;. \tag{3.18}$$

These generators satisfy

$$M'^\dagger_{\mu\nu} = M'_{\mu\nu} \;, \qquad P'^\dagger_\mu = K'_\mu \;, \qquad K'^\dagger_\mu = P'_\mu \;, \qquad D'^\dagger = -D' \;. \tag{3.19}$$

These relations can be understood as follows. In radial quantization, the spacelike foliation is in terms of S^{d-1}, and it is preserved by $M'_{\mu\nu}$ which is then Hermitian. It is not preserved by P'_μ, which is not Hermitian. In Euclidean time D' is anti-Hermitian. Thus, we construct representations of $\mathfrak{so}(d+1,1)$ such that they are unitary in Lorentzian signature.

Consider the state $|\Phi\rangle$ corresponding to $\Phi(0)$:

$$\big|P'_\mu|\Phi\rangle\big|^2 \geq 0 \qquad \text{and} \qquad = 0 \quad \text{for} \quad P'_\mu|\Phi\rangle = 0 \;. \tag{3.20}$$

This implies

$$\sum_\mu \langle\Phi|K'_\mu P'^\mu|\Phi\rangle = 2\langle\Phi|\Delta - i\underbrace{M'^{\;\mu}_\mu}_{=0}|\Phi\rangle \;. \tag{3.21}$$

We obtain that $\Delta \geq 0$, and $\Delta = 0$ if and only if $D_\mu\Phi = 0$, *i.e.*, if Φ is the identity operator.

More generally, $\langle\Phi|K'_\mu P'_\nu|\Phi\rangle$ must be a semi-positive-definite matrix. One can prove that:

- Spinor: $\Delta \geq \frac{d-1}{2}$ and $\Delta = \frac{d-1}{2}$ if and only if $\partial\!\!\!/\Psi = 0$ (free fermion field).
- Vector: $\Delta \geq d-1$ and $\Delta = d-1$ if and only if $\partial^\mu J_\mu = 0$ (conserved current).
- Spin 2: $\Delta \geq d$ and $\Delta = d$ if and only if $\partial^\mu T_{\mu\nu} = 0$ (conserved stress tensor).

For a scalar operator, it is useful to go one further level up:

$$\big|P'_\mu P'^\mu|\Phi\rangle\big|^2 \geq 0 \qquad \Rightarrow \qquad \Delta\Big(\Delta - \frac{d-2}{2}\Big) \geq 0 \;. \tag{3.22}$$

Thus:

- Scalar: $\Delta \geq \frac{d-2}{2}$ and $\Delta = \frac{d-2}{2}$ if and only if $\partial^2\Phi = 0$ (free scalar field).

Any Poincaré-invariant local quantum field theory has a symmetric conserved stress tensor $T_{\mu\nu}$.[13] In a CFT, the dimension of $T_{\mu\nu}$ is fixed to be $\Delta = d$. Similarly, whenever there are continuous global symmetries, there are conserved currents J_μ with dimension $\Delta = d - 1$. The scaling dimensions of the other operators are not fixed by conformal symmetry, and receive quantum corrections. For unitary theories, unitarity bounds say that:

- For scalar fields:
$$\text{scalar} \qquad \Delta \geq \frac{d-2}{2} \tag{3.23}$$
and there is equality if and only if the field is free.

- For vector operators $\mathcal{O}_\mu$: $\Delta \geq d - 1$ and there is equality if and only if $\partial^\mu \mathcal{O}_\mu = 0$. Similarly, for spin-2 symmetric operators $\mathcal{O}_{\mu\nu}$: $\Delta \geq d$ and there is equality if and only if $\partial^\mu \mathcal{O}_{\mu\nu} = 0$.

3.3 OPE and correlation functions

Conformal symmetry strongly constrains correlation functions. The 2-point functions are completely fixed, up to a rescaling (redefinition of the operators). The 2-point functions of primary operators with different dimension vanish. For primary scalar fields of dimension Δ:
$$\langle \Phi_i(x) \Phi_j(y) \rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta}} \tag{3.24}$$
where the operators have been normalized and diagonalized, and similarly for higher spin. The 2-point functions of descendants are obtained by taking derivatives. In particular for two fields in the same conformal family:
$$\langle \phi_i(x) \phi_j(y) \rangle = \frac{c_{ij}}{|x-y|^{\Delta_i + \Delta_j}} \;, \tag{3.25}$$
with $\Delta_i - \Delta_j \in \mathbb{Z}$.

The 3-point functions of primary operators Φ_i, Φ_j, Φ_k are completely determined by "structure constants" C_{ijk}. For scalar primary fields, normalized to have canonical 2-point function:
$$\langle \Phi_i(x_1) \Phi_j(x_2) \Phi_k(x_3) \rangle = \frac{C_{ijk}}{|x_1 - x_2|^{\Delta_1 + \Delta_2 - \Delta_3} |x_2 - x_3|^{\Delta_2 + \Delta_3 - \Delta_1} |x_3 - x_1|^{\Delta_3 + \Delta_1 - \Delta_2}} \;, \tag{3.26}$$
and similarly for higher spin. The 3-point functions involving descendants are obtained by taking derivatives.

Higher point functions are functions of the conformal invariants constructed out of the x_i, not determined by the conformal symmetry alone. For instance, out of 4 points one constructs

[13] Invariance under translations and locality give, by Noether theorem, a conserved local stress tensor. Then invariance under Lorentz rotations assures that it can be improved to a symmetric tensor.

one complex "cross ratio",[14] and with more points more invariants (whose number depends on the number of points and the number of dimensions).

It turns out that the correlation functions of local operators in a CFT are all completely fixed, once we know the spectrum of primaries and their 3-point functions. This is for two reasons.

- The 3-point functions involving descendants are fixed by the 3-point functions of primaries: just act with derivatives.
- Correlators can be decomposed using the OPE.

The *operator product expansion* (OPE) is a general property of local QFTs, but it is particularly powerful in CFTs. It claims that the product of two operators at nearby points can be rewritten as a series of operators at one point only:

$$\mathcal{O}_i(x)\,\mathcal{O}_j(y) = \sum_k c_{ij}^k(x-y)\,\mathcal{O}_k(y) \tag{3.27}$$

and conformal symmetry fixes

$$c_{ij}^k(x-y) = \frac{c_{ij}^k}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}}\ . \tag{3.28}$$

Notice that the sum includes descendants and higher-spin operators! This is an operator equation, valid inside any correlation function. In CFTs, the series that one obtains for correlation functions are convergent until the circle around y hits another operator.

The coefficients of the OPE algebra are fixed by the 3-point functions of primaries.

$$\begin{aligned}\langle\Phi_i(x)\Phi_j(y)\Phi_k(z)\rangle &= \langle\Phi_i(x)\sum_\ell \frac{c_{jk}^\ell}{|y-z|^{\Delta_j+\Delta_k-\Delta_\ell}}\phi_\ell(z)\rangle \\ &= \sum_\ell \frac{c_{jk}^\ell}{|y-z|^{\Delta_j+\Delta_k-\Delta_\ell}}\,\frac{c_{i\ell}}{|x-z|^{\Delta_i+\Delta_\ell}}\end{aligned} \tag{3.29}$$

The only operators with non-vanishing 2-point function with Φ_i are Φ_i and its descendants, *i.e.* ϕ_ℓ must be in the conformal family of Φ_i. Then $\Delta_\ell = \Delta_i + \text{integer}$ and $c_{ii} = 1$ for the primary. Then, for $y \to z$:

$$\langle\Phi_i(x)\Phi_j(y)\Phi_k(z)\rangle = \frac{c_{jk}^i}{|x-z|^{2\Delta_i}|y-z|^{\Delta_j+\Delta_k-\Delta_i}} + O\Big(\frac{|y-z|}{|x-y|}\Big)\ . \tag{3.30}$$

[14] One way to construct the cross ratio is the following. Take 3 points: they lie on a plane. By a conformal transformation they can be placed at 0, 1, ∞. Now the fourth point defines a plane (which can be rotated to a canonical position), and the position z of the fourth point on that plane is the ratio.

On the other hand we can expand the 3-point function of primaries for $y \to z$:

$$\frac{C_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}|x-z|^{\Delta_i+\Delta_k-\Delta_j}|y-z|^{\Delta_j+\Delta_k-\Delta_i}} = \frac{C_{ijk}}{|x-z|^{2\Delta_i}|y-z|^{\Delta_j+\Delta_k-\Delta_i}} + O\Big(\frac{|y-z|}{|x-y|}\Big) \; . \tag{3.31}$$

It follows that the coefficients for primaries in the OPE are the same (with normalized operators) as the 3-point functions. All other coefficients can be fixed as well.

Another property of CFTs is the *state-operator correspondence*:

$$\text{states on } S^{d-1} \;=\; \text{local operators on } \mathbb{R}^d \; . \tag{3.32}$$

The correspondence is established using radial quantization: one regards time as the distance from the origin, and space as the spheres S^{d-1} around the origin. By a Weyl transformation

$$ds^2 = dr^2 + r^2 d\Omega^2_{d-1} \qquad \mathbb{R}^d \qquad \to \qquad S^{d-1}\times\mathbb{R} \qquad ds^2 = d\tau^2 + d\Omega^2_{d-1} \tag{3.33}$$

dividing by r^2 and setting $r = e^\tau$. A rescaling of r corresponds to a shift in τ, and since the Hamiltonian generates time translations, the dimension of the operator in $\mathbb{R}^d$ equals the energy of the state on S^{d-1}:

$$\Delta = E_{\text{cylinder}} \; . \tag{3.34}$$

With no insertions, the state created on S^{d-1} is the conformal vacuum $|0\rangle$. Given an operator $\mathcal{O}$, the corresponding state $|\mathcal{O}\rangle$ is the one created on S^{d-1} around $\mathcal{O}(0)$:

$$|\mathcal{O}\rangle = \mathcal{O}(0)|0\rangle \; . \tag{3.35}$$

As a functional $\Psi\big(\phi(\omega)\big)$ of field configurations on S^{d-1}, it is given by the path-integral on the ball with $\mathcal{O}$ at the origin and boundary conditions $\phi(\omega)$ on the boundary:

$$\Psi\big(\phi(\omega)\big) = \int_{\Phi=\phi \text{ on } S^{d-1}} \mathcal{D}\Phi\, \mathcal{O}(0)\, e^{-S[\Phi]} \; . \tag{3.36}$$

On the contrary, if we think of a state as a functional of field configurations, as we shrink the ball to zero size using conformal invariance we obtain a local operator. In fact, we have defined a local operator if we know how to compute its correlation functions. To compute the path-integral with insertions of $\mathcal{O}$, we cut small balls around the insertion points and we define their contribution to the path-integral as $\Psi\big(\phi(\omega)\big)$, where $\phi(\omega)$ is the field configuration on the boundary of the ball.

3.4 Superconformal algebras

The bosonic Poincaré algebra is extended, by the addition of fermionic generators, into superalgebras. If we also add conformal generators, we obtain superconformal algebras. They only exist in some dimensions — $d \leq 6$ — and for some number of supersymmetries (classified by Nahm [Nah78], see also [Min98] and [VP99]). They are constrained by the following requirements:

- it contains both the conformal algebra $\mathfrak{so}(d,2)$ and the Poincaré supersymmetry superalgebra;
- the fermionic generators are in the spinor representation of $\mathfrak{so}(d,2)$.

These requirement are quite restrictive.

We can think of a superalgebra as being generated by matrices T that act on a superspace $\mathcal{X}$:

$$T\mathcal{X} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix} . \tag{3.37}$$

The superspace is $\mathbb{Z}_2$ graded by the fermion number, and so are the homeomorphisms. Thus A, D are bosonic, while B, C are fermionic and anticommute. The structure of a superalgebra is

$$[T_b, T_b] = T_b , \qquad [T_b, T_f] = T_f , \qquad \{T_f, T_f\} = T_b . \tag{3.38}$$

Let us look at superconformal algebras.

In addition to the conformal generators P_μ, $M_{\mu\nu}$, D, K_μ and the supersymmetry generators Q_α, there are:

- generators S_α of "conformal supersymmetries", as many as the Q's, arising from the commutators of K with Q;
- R-symmetry generators forming a Lie algebra, in the anti-commutator of Q and S.

Both types of supercharges are constructed out of the supersymmetry current $S_{\mu\alpha}$. More specifically the currents for Q_α and S_α are

$$S_{\mu\alpha} \qquad \text{and} \qquad \gamma^\rho_{\alpha\beta} S^\beta_\mu x_\rho \tag{3.39}$$

respectively. Conservation follows from $\partial^\mu S_{\mu\alpha} = \gamma^\mu_{\alpha\beta} S^\beta_\mu = 0$. The second quantity is called "gamma trace".

The superconformal algebra is, schematically:

$$\begin{aligned} &[D,Q] = -\frac{i}{2} Q && [D,S] = \frac{i}{2} S && [K,Q] \simeq S && [P,S] \simeq Q \\ &\{Q,Q\} \simeq P && \{S,S\} \simeq K && \{Q,S\} \simeq M + D + R . \end{aligned} \tag{3.40}$$

In particular the R-symmetry is part of the superconformal algebra, not just an outer automorphism as for supersymmetry, and in particular it *must* be there.

In a generic superalgebra that contains $\mathfrak{so}(d,2)$, the fermionic generators are in the vector representation. The requirement that they are in the spinor representation is quite restrictive in $d \geq 3$, leading the following short list:

$$\begin{array}{llclcll} d=3: & \mathfrak{osp}(\mathcal{N}|4) & \supset & \mathfrak{so}(\mathcal{N}) \times \mathfrak{sp}(4) & \simeq & \mathfrak{so}(\mathcal{N}) \times \mathfrak{so}(3,2) & \\ d=4: & \mathfrak{su}(2,2|\mathcal{N}) & \supset & \mathfrak{su}(2,2) \times \mathfrak{u}(\mathcal{N}) & \simeq & \mathfrak{so}(4,2) \times \mathfrak{u}(\mathcal{N}) & \mathcal{N} \neq 4 \\ & \mathfrak{psu}(2,2|4) & \supset & \mathfrak{su}(2,2) \times \mathfrak{su}(4) & \simeq & \mathfrak{so}(4,2) \times \mathfrak{su}(4) & \mathcal{N} = 4 \\ d=5: & \mathfrak{f}(4) & \supset & \mathfrak{so}(5,2) \times \mathfrak{su}(2) & & & \mathcal{N} = 1 \\ d=6: & \mathfrak{osp}(6,2|\mathcal{N}) & \supset & \mathfrak{so}(6,2) \times \mathfrak{sp}(2\mathcal{N}) & & & \end{array} \tag{3.41}$$

For free field theories with no gravity, the maximal number of supersymmetries is 16. It is believed to be the same for interacting theories. In the case of SCFTs, one can rigorously prove [CDI19] that the existence of a stress tensor multiplet (containing $T_{\mu\nu}$, $S_{\mu\alpha}$ and R_μ from which the superconformal charges are constructed) and the requirement that the theory is not free, limit[15]

$$N_Q \leq 16 \tag{3.42}$$

in $d \geq 3$, where N_Q is the number of Poincaré supercharges. Therefore, the maximal number of fermionic generators in a superconformal algebra is 32. Theories with such a superconformal algebra are known in $d = 3, 4, 6$.

$$\begin{array}{llcllcl} d=3: & OSp(8|4) & \supset & SO(8)\times Sp(4) & \simeq & SO(8)\times SO(3,2) \\ d=4: & PSU(2,2|4) & \supset & SU(2,2)\times SU(4) & \simeq & SO(4,2)\times SO(6) \\ d=6: & OSp(8^*|4) & \supset & SO^*(8)\times USp(4) & \simeq & SO(6,2)\times SO(5)\ . \end{array} \tag{3.43}$$

Primary fields of the superconformal algebras are defined to be annihilated by K_μ and S_α:

$$\text{superconformal primary :} \qquad [K_\mu, \Phi(0)] = [S_\alpha, \Phi(0)] = 0\ . \tag{3.44}$$

A superconformal multiplet can include multiple conformal-primaries (primaries of the conformal group alone), obtained by acting with Q_α. There can also be special representations, called *chiral primary operators*, which are annihilated also by some of the Q's:

$$\text{chiral primary :} \qquad [K, \Phi] = [S, \Phi] = [Q', \Phi] = 0 \tag{3.45}$$

where $'$ reminds us that it is just some, not all, Q's. These representations are shorter: contain less conformal-primaries. A special property is that their dimension is fixed by the R-symmetry representation (with no quantum corrections). This follows from vanishing of some $\{Q, S\}$ anticommutators.

Example: 4d $\mathcal{N} = 1$ superconformal theories. The R-symmetry is $U(1)$. A chiral field which is a primary is also a chiral primary. Then

$$\Delta = \frac{3}{2} R\ . \tag{3.46}$$

[15] From the same requirement, one also excludes $\mathfrak{su}(2,2|4)$ as a superconformal algebra in $d = 4$, allowing $\mathfrak{psu}(2,2|4)$ instead.

4 Anti-de-Sitter space

Let us describe the space AdS and its conformal structure. AdS is a space with negative curvature, and the effect of the negative curvature is to create a "conformal boundary". To give the idea, take the Euclidean case. To compactify $\mathbb{R}^d$ it is enough to add a "point at infinity", indeed Euclidean CFTs are naturally defined on S^d. Instead, the $(d+1)$-dimensional hyperbolic space (the Euclidean version of AdS) is conformally equivalent to a disk D_{d+1}, which has a boundary S^d.

4.1 Conformal structures and Penrose diagrams

A very convenient way to understand the conformal and causal structure of a given spacetime, is to use Penrose diagrams. One performs a Weyl transformation of spacetime such that the transformed spacetime is "compact" (finite): the Penrose diagram is a diagram of the latter. A Weyl transformation preserves the signature and angles, in particular light-rays (light-like geodesics) remain at 45°, and time-like and space-like directions remain such. Thus a Penrose diagram correctly reproduces the causal structure of spacetime.[16]

Let us start with flat Minkowski spacetime, in particular $\mathbb{R}^{1,1}$:

$$ds^2 = -dt^2 + dx^2 \qquad \text{with} \qquad -\infty < t, x < +\infty \ . \tag{4.1}$$

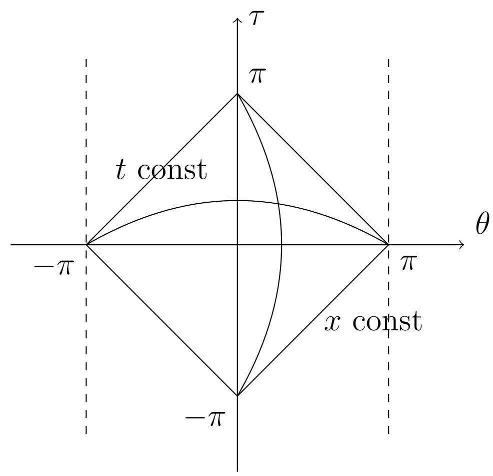
The "diagram" of this spacetime is non-compact. Perform a change of coordinates:

$$t \pm x = \tan\frac{\tau \pm \theta}{2} \qquad \text{with} \qquad |\tau \pm \theta| < \pi \ . \tag{4.2}$$

Then we can "compactify" the space with a Weyl transformation:

$$ds^2 = -dt^2 + dx^2 = \frac{-d\tau^2 + d\theta^2}{4\cos^2\frac{\tau+\theta}{2}\,\cos^2\frac{\tau-\theta}{2}} \quad \xrightarrow{\text{Weyl}} \quad -d\tau^2 + d\theta^2 \ . \tag{4.3}$$

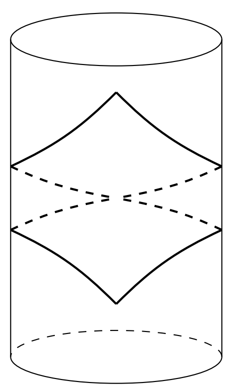
[16] For a conformal theory, the Penrose diagram is a faithful representation of spacetime because the theory is invariant under Weyl transformations. For a non-conformal theory, instead, it is a distorted representation since distances are changed.



The new coordinates (τ, θ) are well-defined at the asymptotic regions of spacetime, thus the conformal compactification can be used to give a definition of *asymptotic flatness*:

A spacetime is "asymptotically flat" if it has the same boundary structure as flat spacetime after conformal compactification.

The two corners $(\tau, \theta) = (0, \pm\pi)$ correspond to the spatial infinities $x = \pm\infty$. By identifying those two points, we can embed the image of $\mathbb{R}^{1,1}$ into the cylinder $S^1 \times \mathbb{R}$. In fact [LM75] the correlation functions of a 2d CFT on $\mathbb{R}^{1,1}$ can be analytically continued to the whole cylinder. In other words, a CFT is naturally defined on the entire cylinder $S^1 \times \mathbb{R}$, while $\mathbb{R}^{1,1}$ is just a patch of it.



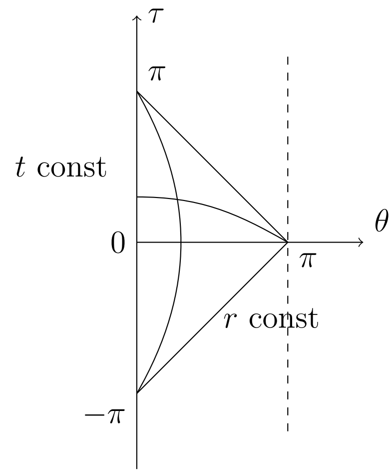
The case of a general flat Minkowski space $\mathbb{R}^{d-1,1}$ is similar. The metric is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2 = \frac{-d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega_{d-2}^2}{4\cos^2\frac{\tau+\theta}{2}\,\cos^2\frac{\tau-\theta}{2}} \tag{4.4}$$

with the change of coordinates

$$t \pm r = \tan\frac{\tau \pm \theta}{2}\ . \tag{4.5}$$

After removing the denominator by a Weyl transformation, we get half of the Penrose diagram we had before:



with $0 \leq \theta \leq \pi$. Each point represents a whole sphere S^{d-2} of radius $\sin\theta$, which shrinks at $\theta = 0, \pi$. Thus (θ, S^{d-2}) combine into S^{d-1}. This patch can be embedded into $S^{d-1} \times \mathbb{R}$.

Thus, the conformal "compactification" of $\mathbb{R}^{d-1,1}$ is a wedge, which is naturally embedded into $S^{d-1} \times \mathbb{R}$.

4.2 AdS

Anti-de-Sitter is the most symmetric Lorentzian space with (constant) negative curvature. It is a solution of Einstein equations with negative cosmological constant:

$$S = \frac{1}{16\pi G_N} \int d^{d+1}x \, \sqrt{g}\big(\mathcal{R} + \Lambda\big) \qquad \qquad (\text{take } \Lambda > 0) \tag{4.6}$$

leads to the Einstein equation

$$\mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2} g_{\mu\nu} = \frac{\Lambda}{2} g_{\mu\nu} \qquad \Rightarrow \qquad \mathcal{R} = -\frac{d+1}{d-1}\Lambda \; . \tag{4.7}$$

As a metric space, AdS_{d+1} can be described by embedding an hyperboloid into $\mathbb{R}^{d,2}$:

$$-X_{-1}^2 - X_0^2 + X_1^2 + \ldots + X_d^2 = -R^2 \qquad \text{in} \qquad \mathbb{R}^{d,2} \; . \tag{4.8}$$

Both the ambient space and the equation have $SO(d,2)$ isometry, thus the resulting space has that isometry too. The metric is the one induced by

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + \ldots + dX_d^2 \; . \tag{4.9}$$

We can solve the equation (parametrize the solutions) by

$$X_{-1} = R\cosh\rho\,\sin\tau \;, \quad X_0 = R\cosh\rho\,\cos\tau \;, \quad X_i = R\sinh\rho\,\Omega_i \quad \text{with} \quad \sum \Omega_i^2 = 1 \; . \tag{4.10}$$

The induced metric is

$$ds^2 = R^2\big(- \cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{d-1}^2\big) \ . \tag{4.11}$$

Although the ambient space had two "time directions", the embedded AdS has one and it is a standard Lorentzian spacetime.

What we have written is AdS in *global coordinates* (τ, ρ, Ω_i), where $\rho \geq 0$. However $\tau \in S^1$, and thus it seems that there are closed time-like curves (which violates causality). To solve the problem, we "unfold" τ:[17] we take the universal covering space with $\tau \in \mathbb{R}$. This is what we call global AdS.

To draw the Penrose diagram of AdS we redefine $\sinh\rho = \tan\theta$ with $\theta \in [0, \frac{\pi}{2})$:

$$ds^2 = \frac{R^2}{\cos^2\theta}\big(- d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega_{d-1}^2\big) \ . \tag{4.12}$$

After removing the denominator, we have the ball $B_d \times \mathbb{R}$ (more precisely, the metric is the one of the hemisphere times $\mathbb{R}$). This space has a boundary: $S^{d-1} \times \mathbb{R}$.

A spacetime is "asymptotically AdS" if it has the same boundary structure as AdS after conformal compactification.

Notice that the boundary of conformally compactified AdS$_{d+1}$ is equal to the conformal compactification of $\mathbb{R}^{d-1,1}$.

There is another useful parametrization of AdS$_{d+1}$:

$$X_{\mu=0,\ldots,d-1} = \frac{R}{z}\, x_\mu \ , \qquad X_{-1} = \frac{R}{2z}\Big(1 + |x_\mu|^2 + z^2 \Big) \ , \qquad X_d = \frac{R}{2z}\Big(1 - |x_\mu|^2 - z^2 \Big) \ . \tag{4.13}$$

In these coordinates the metric is

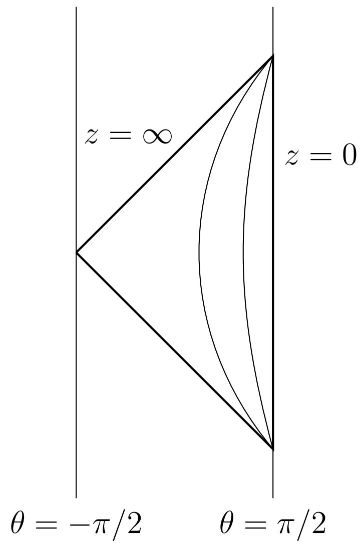
$$ds^2 = R^2\, \frac{dz^2 + d\vec{x}^2}{z^2} \tag{4.14}$$

where $\vec{x} = (x_0, x_i)$. These are called Poincaré coordinates and only cover a portion of AdS.

[17] It is possible to unfold τ from S^1 to $\mathbb{R}$ because, in this metric, the circle S^1 never shrinks. Otherwise, unfolding would introduce singularities at points where S^1 shrank.

The fundamental group of $SO^+(p,q)$ (the connected component containing the identity) is the product $\pi_1\big(SO(p)\big) \times \pi_1\big(SO(q)\big)$. In particular $SO(d,2)$ has a covering extension by $\mathbb{Z}$, which acts on the unfolded manifold. Such a covering extension has no finite-dimensional faithful representations, and so it cannot be represented as a matrix group. The case of $SO(1,2) \cong SL(2,\mathbb{R})/\mathbb{Z}_2$ is discussed in details in [Raw12].

In AdS_2:



The boundary of the Poincaré patch is $\mathbb{R}^{d-1,1}$.

The boundary of the Poincaré patch is at $z = 0$. At $z = \infty$ there is an horizon, because the Killing vector ∂_t has zero norm. It is not a singularity of the metric, and in fact the patch can be embedded into the global coordinates where there is no horizon. In a loose sense, it is an horizon because signals cannot come back (this is obvious from the Penrose diagram in global coordinates), indeed it is called an "apparent horizon".

The isometry of AdS_{d+1} is $SO(d, 2)$, but it is not manifest in the metric description.

- In global coordinates, $SO(d) \times SO(2)$ is manifest. The universal cover of $SO(2)$ is the Killing vector ∂_τ, which is the Hamiltonian on S^{d-1} in field theory.
- In the Poincaré patch, $SO(d-1, 1) \times SO(1, 1)$ is manifest. The first one is the Lorentz group of the boundary, while $SO(1, 1)$ is dilations on the boundary and it is realized as $x^\mu \to \lambda x^\mu$, $z \to \lambda z$.

5 4d $\mathcal{N} = 4$ SYM and strings on $\mathrm{AdS}_5 \times S^5$

We have argued that d-dimensional matrix QFTs at large N should/could be described by a string theory (theory of gravity) in $d+1$ dimensions. In order to find examples, it is natural to start with cases with maximal symmetry.

- On the QFT side, we consider conformal theories (the conformal group is larger than Poincaré). We have argued that this should correspond to strings in AdS_{d+1}.

  Indeed the conformal group is $SO(d,2)$, which is also the isometry group of AdS_{d+1}.

- On the QFT side, we consider supersymmetric theories. They have larger symmetry, and quantum effects are more under control. Supersymmetry is particularly crucial to have control on the string theory side.

  Thus we are led to SCFTs.

- We should start with the maximal possible number of supercharges, which is 32. Theories are known only in $d = 3, 4, 6$.

  Only in $d = 4$ the theory has Lagrangian description with fully manifest superconformal symmetry. Thus we are led to

  $$4\text{d } \mathcal{N} = 4 \text{ SYM.}$$

- The superconformal algebra is

  $$PSU(2,2|4) \;\supset\; SO(4,2) \times SO(6) \;. \tag{5.1}$$

  What could the dual string theory be? The isometry group leads to the space $\mathrm{AdS}_5 \times S^5$. Luckily, the perturbative superstring is consistent precisely in 10 dimensions, and it turns out that one particular superstring theory — type IIB — has an $\mathrm{AdS}_5 \times S^5$ vacuum solution preserving 32 (all) supercharges.

Indeed

$$4\text{d } \mathcal{N} = 4 \text{ SYM} \qquad \leftrightarrow \qquad \text{type IIB on } \mathrm{AdS}_5 \times S^5$$

is the golden (and most studied) example of AdS/CFT.

5.1 4d $\mathcal{N} = 4$ SYM

The theory has a unique multiplet: the $\mathcal{N} = 4$ vector multiplet. In components

$$V_{\mathcal{N}=4} = (A_\mu, \lambda_{i=1,\dots,4}, \phi_{I=1,\dots,6}) \tag{5.2}$$

where λ_i are Weyl spinors and ϕ_I are real scalars. The R-symmetry is

$$\text{R-symmetry:} \qquad SU(4) \cong SO(6) \ .$$

Then $\phi_I = \mathbf{6}$ is the fundamental of $SO(6)$ = antisymmetric of $SU(4)$, while $\lambda_i = \mathbf{4}$ is the fundamental of $SU(4)$ = Weyl spinor of $SO(6)$. In $\mathcal{N}=1$ notation:

$$V_{\mathcal{N}=4} = V_{\mathcal{N}=1} \text{ plus three chiral } \Phi_{1,2,3} \text{ in the adjoint.} \tag{5.3}$$

With a vector multiplet we can write a gauge theory: in $\mathcal{N}=4$ the only parameters are the group G, the gauge coupling g (if G is simple) and a theta angle θ. The Lagrangian is schematically of the form

$$\mathcal{L} = \frac{1}{g^2} \operatorname{Tr}\Big[-F_{\mu\nu}F^{\mu\nu} - D_\mu \phi_I D^\mu \phi_I + \bar\lambda^i \not{D} \lambda_i + \sum_{I,J} [\phi_I, \phi_J]^2 + \bar\lambda^i \Gamma^I \phi_I \lambda_i \Big] + \theta\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \ . \tag{5.4}$$

In $\mathcal{N}=1$ notation it is schematically

$$\mathcal{L} = \int d^4\theta \sum_i \overline{\Phi}_i e^V \Phi_i + \int d^2\theta \ \operatorname{Tr} \Big[\tau\, \mathcal{W}^\alpha \mathcal{W}_\alpha + \Phi_1 [\Phi_2, \Phi_3] \Big] + \text{h.c.} \ . \tag{5.5}$$

There τ is the complexified gauge coupling

$$\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2} \ . \tag{5.6}$$

The one-loop beta-function is given by[18]

$$\frac{\partial g}{\partial \log \mu} = - \frac{g^3}{16\pi^2} \Big(\frac{11}{3} c(\text{Adj}) - \frac{2}{3} \sum c(\text{Weyl}) - \frac{1}{6} \sum c(\text{scalars}) \Big) \ . \tag{5.7}$$

Here all fields are in the adjoint representation, so

$$\frac{11}{3} - \frac{2}{3} 4 - \frac{1}{6} 6 = 0 \ .$$

In fact the beta-function is zero to all orders in perturbation theory as well as non-perturbatively: the theory is conformal for all values of τ. In other words τ is an exactly marginal deformation. The superconformal group is $PSU(2,2|4)$.

The theory enjoys electric magnetic duality [MO77] in which

$$\tau \ \to \ -\frac{1}{\tau} \ . \tag{5.8}$$

Combining this with the invariance under $\tau \to \tau + 1$ (shift of θ angle by 2π), one gets *S-duality* $SL(2,\mathbb{Z})$ acting on τ as on the upper half-plane:[19]

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \ . \tag{5.9}$$

The 't Hooft coupling is $\lambda = g^2 N$.

[18] Here c is the quadratic Casimir of the representation, defined by $\operatorname{Tr} T^a T^b = c\, \delta^{ab}$. Notice that $c(\text{Adj})$ is usually called $C_2(G)$.

[19] The element $C = -\mathbb{1} \in SL(2,\mathbb{Z})$ does not act on τ: it corresponds to charge conjugation.

5.2 Type IIB strings on $\mathrm{AdS}_5 \times S^5$

Perturbative bosonic strings require a spacetime of dimension 26 to be consistent at the quantum level, however they have a tachyon (instability of the flat spacetime solution). Superstrings solve the problem. Consistency requires a spacetime of dimension 10, then there is no tachyon in the spectrum, therefore $\mathbb{R}^{9,1}$ is a consistent vacuum.

As in the bosonic case, the spectrum of the closed superstring around $\mathbb{R}^{9,1}$ contains the graviton plus other massless fields of spin < 2, and then an infinite tower of massive modes of masses

$$m_n^2 \sim \frac{n}{\alpha'} . \tag{5.10}$$

It turns out that the superstring gives spacetime (target) supersymmetry, thus the spectrum is organized into supermultiplets. There are two closed superstrings with 32 supercharges: called type IIA ($\mathcal{N} = (1,1)$ in 10d) and IIB ($\mathcal{N} = (2,0)$). All massless modes are in one multiplet: the graviton multiplet (two versions).

At low energies the massive string modes decouple, and one is left with an effective theory for the graviton multiplet: *supergravity*. The supergravity EOMs are obtained by requiring that the worldsheet theory remains conformal, and can be described by a Lagrangian (with a caveat). This leads to type IIA and IIB supergravity.

Of course, the supergravity approximation is valid as long as we remain "close" to $\mathbb{R}^{9,1}$, in the perturbative regime and at low energies:

$$R_s \ll \frac{1}{\alpha'} , \qquad E \ll \frac{1}{\sqrt{\alpha'}} , \qquad g_s \ll 1 . \tag{5.11}$$

It turns out that IIB supergravity admits a solution $\mathrm{AdS}_5 \times S^5$. So we focus on type IIB. The graviton multiplet contains

$$\mathcal{G} = \big(g_{\mu\nu}, \phi, B_{\mu\nu}, \chi, C_{\mu\nu}, C_{\mu\nu\rho\sigma}, \Psi_{\alpha\mu}, \psi_\alpha \big) . \tag{5.12}$$

- $g_{\mu\nu}$: the metric.
- ϕ: the dilaton.
- B: the NS 2-form with field strength $H = dB$.
- χ: a RR 0-form potential, *i.e.* an axion, with $\chi \cong \chi + 2\pi$ and "field strength" $F_1 = d\chi$.
- C_2: a RR 2-form potential, with $\widetilde{F}_3 = dC_2 - \chi H$.
- C_4: a RR 4-form potential with $\widetilde{F}_5 = dC_4 - \frac{1}{2} C_2 \wedge H + \frac{1}{2} B \wedge dC_2$ and self-dual
$$* \widetilde{F}_5 = \widetilde{F}_5 . \tag{5.13}$$
- $\Psi_{\mu\alpha}$, ψ_α: the gravitino and a chiral fermion.

As understood by Green and Schwarz, the gauge-invariant field strengths are modified because the gauge transformations are. The modified large gauge transformation of χ is

$$\chi \to \chi + 2\pi \;, \qquad\qquad C_2 \to C_2 + B \;. \tag{5.14}$$

In fact this is the element $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of $SL(2,\mathbb{Z})$. The modified gauge transformations of B and C_2 are

$$\begin{aligned} C_2 &\to C_2 + d\lambda_1 \\ C_4 &\to C_4 + \tfrac{1}{2}\lambda_1 \wedge H \end{aligned} \qquad \text{and} \qquad \begin{aligned} B &\to B + d\widetilde{\lambda}_1 \\ C_4 &\to C_4 - \tfrac{1}{2}\widetilde{\lambda}_1 \wedge dC_2 \;. \end{aligned} \tag{5.15}$$

The bosonic action in string frame is

$$\begin{aligned} S_{\rm IIB}^{\rm s.f.} = \frac{1}{2\kappa_{10}^2} \int d^{10}x\, \sqrt{-g} \Big[e^{-2\phi}\Big(R_s + 4|\partial\phi|^2 - \frac{1}{2}|H|^2 \Big) - \frac{1}{2}|F_1|^2 - \frac{1}{2}|\widetilde{F}_3|^2 - \frac{1}{4}|\widetilde{F}_5|^2 \Big] \\ - \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H \wedge dC_2 \end{aligned} \tag{5.16}$$

with $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4$. However one has to supplement the EOMs with $*\widetilde{F}_5 = \widetilde{F}_5$ (not derived from the action).

To go to Einstein frame redefine

$$e^\phi \to g_s\, e^\phi \;, \qquad g_{\mu\nu} \;\to\; e^{\phi/2} g_{\mu\nu} \;. \tag{5.17}$$

Keeping only the metric and F_5, the Einstein frame action is:

$$S_{\rm IIB} = \frac{1}{16\pi G_{\rm N}} \int d^{10}x\, \sqrt{-g} \Big[R_s - \frac{1}{4}|F_5|^2 \Big] \tag{5.18}$$

where $(2\pi^2)^{-3} G_{\rm N} = \ell_{\rm Pl}^8 = g_s^2 \alpha'^4$. There is a solution to the EOMs where[20]

$$\begin{aligned} ds^2 &= R^2 \big(ds^2_{AdS_5} + ds^2_{S^5} \big) \qquad\qquad R = (4\pi g_s N)^{1/4} \sqrt{\alpha'} \sim N^{1/4} \ell_{\rm Pl} \\ F_5 &= \tfrac{(2\pi\ell_{\rm Pl})^4}{{\rm Vol}(S^5)} \, (1+*)\, N\, d{\rm vol}_{S^5} \;. \end{aligned} \tag{5.19}$$

Dirac quantization condition imposes

$$\frac{1}{(2\pi\ell_{\rm Pl})^4} \int_{S^5} F_5 = N \in \mathbb{Z} \;. \qquad\qquad \int_{S^5} F_5 = (4\pi^2\alpha')^2 g_s\, N \;. \tag{5.20}$$

We can understand the scaling of R as the condition that the two terms in the action compete:

$$R_s \sim \frac{1}{R^2} \;, \qquad\qquad |F_5|^2 = \frac{1}{5!} (g^{\mu\nu})^5 (F_{\mu\nu\rho\sigma\tau})^2 \sim R^{-10} N^2 \ell_{\rm Pl}^8 \;. \tag{5.21}$$

IIB supergravity is invariant under $SL(2,\mathbb{R})$ transformations of[21]

$$\tau = \frac{\chi}{2\pi} + i \frac{4\pi}{g_s e^\phi} \;. \tag{5.22}$$

In the full string theory, the invariance is broken to $SL(2,\mathbb{Z})$ by instanton effects.[22]

[20] The volume of S^5 is π^3.

[21] Then (H, F_3) transform as a doublet, while F_5 is a singlet.

[22] D(-1) string instantons have an action that depends on χ.

5.3 The correspondence

We can give a physical proof of the correspondence [Mal99].

String theory contains solitonic objects called “D-branes”. They are boundary conditions for open strings. A flat Dp-brane is an $\mathbb{R}^{p,1}$ submanifold of spacetime where open strings can end. These objects exist also in the superstring, and preserve 16 out of 32 supercharges: $\frac{1}{2}$ BPS. For a single stack of N parallel flat Dp-branes, the system can be perturbatively quantized and the spectrum can be computed.

Consider a stack of N D3-branes in type IIB string theory on flat space. They span $\mathbb{R}^{3,1}$. The perturbative excitations are closed strings and open strings. Let us study the physics at energies

$$E \ll 1/\sqrt{\alpha'} \; . \tag{5.23}$$

Then only the massless modes can be excited.

- The closed strings give type IIB supergravity in 10d. We have analyzed this sector above.

- The open strings give modes localized on the branes $\mathbb{R}^{3,1}$.

  As in the case of bosonic Dp-branes, the massless open-string spectrum contains a gauge field in (p+1)-dimensions. A single brane give an Abelian $U(1)$ gauge field, while N branes give a $U(N)$ gauge field.

  Besides, there are as many scalar fields as the directions orthogonal to the brane. Such fields describe transverse oscillations of the brane. For N branes, they transform in the adjoint representation of $U(N)$. Thus, D3-branes carry 6 real scalars.

  The actions describing the dynamics of the worldvolume fields and their coupling to the bulk fields are the Dirac-Born-Infeld (DBI) and Wess-Zumino (WZ) actons. For a single Dp-brane in Einstein frame they are, schematically:

$$S_{\text{DBI+WZ}} = -\int_{\text{D}p} d^{p+1}\xi \, e^{(p-3)\phi/4} \sqrt{-\det\left(\hat{g} + e^{-\phi/2}\mathcal{F}\right)} + \int_{\text{D}p} e^{\mathcal{F}} \wedge \sum_q C_q \; . \tag{5.24}$$

  Here $\mathcal{F} = 2\pi\alpha' F + B$ while $\hat{g}$ is the induced metric and C_q are the RR potentials (q even in IIB and odd in IIA). Expanding the determinant we find[23]

$$\det\left(\hat{g} + e^{-\phi/2}\mathcal{F}\right) = \det\hat{g}\cdot\det\left(\mathbb{1} + e^{-\phi/2}\hat{g}^{-1}\mathcal{F}\right) = \det\hat{g}\cdot\left[1 + \frac{1}{2}e^{-\phi}\mathcal{F}_{ab}\mathcal{F}^{ab} + \mathcal{O}(\mathcal{F}^4)\right] .$$

[23] We use the expansion of $\det(\mathbb{1}+M) = \exp\log\det(\mathbb{1}+M) = \exp\operatorname{Tr}\log(\mathbb{1}+M)$:

$$\det(\mathbb{1}+M) = 1 + \operatorname{Tr}M + \frac{(\operatorname{Tr}M)^2 - \operatorname{Tr}M^2}{2} + \frac{(\operatorname{Tr}M)^3 - 3(\operatorname{Tr}M)(\operatorname{Tr}M^2) + 2\operatorname{Tr}M^3}{6} + \ldots \tag{5.25}$$

Here $a, b, \dots$ are worldvolume indices. Expanding the square root we find

$$\sqrt{\dots} = \sqrt{-\det \hat{g}} \left(1 + \frac{1}{4} e^{-\phi} \mathcal{F}_{ab} \mathcal{F}^{ab} + \mathcal{O}(\mathcal{F}^4)\right) .$$

Thus, at quadratic order, DBI reproduces the Maxwell kinetic term.[24] Expanding the second term (WZ) we find

$$\int_{\mathrm{D}p} \left(C_{p+1} + \mathcal{F} \wedge C_{p-1} + \frac{1}{2} \mathcal{F} \wedge \mathcal{F} \wedge C_{p-3} + \dots\right) .$$

Thus this term reproduces the electric coupling to the potential C_{p+1}, a theta term where C_{p-3} plays the role of the theta angle, as well as other terms.

For $N > 1$, the exact non-Abelian form of the DBI action is not known. However at two-derivative level its form is fixed by symmetries and supersymmetry.

The D3-branes break half of the 32 Poincaré supercharges of IIB supergravity: the only 4d two-derivative gauge theory with 16 Poincaré supercharges is $\mathcal{N} = 4$ SYM. Besides the gauge field and 6 real scalars, it contains 4 Weyl fermions. The gauge coupling is equal to the axiodilaton:

$$\frac{\theta}{2\pi} + i \frac{4\pi}{g_{\mathrm{YM}}^2} = \tau_{\mathrm{YM}} = \tau = \frac{\chi}{2\pi} + i \frac{4\pi}{g_s} . \tag{5.26}$$

In particular

$$g_{\mathrm{YM}}^2 = g_s . \tag{5.27}$$

Put more precisely, the low-energy effective action is

$$S = S_{\mathrm{bulk}} + S_{\mathrm{brane}} + S_{\mathrm{int}} . \tag{5.28}$$

S_{bulk} is IIB supergravity + higher-derivative corrections. S_{brane} is 4d $\mathcal{N} = 4$ SYM + higher derivative corrections (such as $\alpha'^2 \operatorname{Tr} F^4$ that come from DBI). S_{int} is the interaction, for instance coming from the induced metric on the brane.

As we consider processes at lower and lower energies E, the $SU(N)$ gauge interactions on the branes remain constant, because $\mathcal{N} = 4$ SYM is conformal (but higher-derivative corrections become irrelevant). The $U(1)$ part is decoupled and free. Gravitational interactions are IR free, because the dimensionless coupling is

$$\widehat{G_{\mathrm{N}}} = E^8 G_{\mathrm{N}} . \tag{5.29}$$

Thus at very low energies the 10d gravitational theory decouples from the 4d interacting SCFT and becomes free.

Suppose we have a large number N of D3-branes. From the field theory point of view the effective coupling (that truly controls the strength of the interactions) is the 't Hooft coupling

$$g_s N = g_{\mathrm{YM}}^2 N = \lambda . \tag{5.30}$$

[24] Expanding the first term in terms of the scalar fields, we reproduce their standard kinetic term (¶).

If we increase the 't Hooft coupling, the same picture is valid, but the 4d QFT is strongly coupled.

From the gravitational point of view, instead, the picture changes. The coupling of the branes to gravity is controlled by the same

$$g_s N \ .$$

Indeed disk string amplitudes are of order g_s.

For $g_s N \gg 1$ (but still $g_s \ll 1$) the backreaction of the D3-branes on the geometry cannot be neglected (the geometry is curved outside the Compton wavelength of the branes). Since D3-branes are electric-magnetic sources for F_5, the solution is

$$\begin{aligned} ds^2 &= \frac{1}{f^{1/2}} ds^2_{3,1} + f^{1/2}\big(dr^2 + r^2 d\Omega_5^2\big) \\ F_5 &= (1+*)\, d\mathrm{vol}_{3,1} \wedge df^{-1} \\ f &= 1 + \frac{R^4}{r^4} \qquad\qquad R^4 = 4\pi g_s N \alpha'^2 \ . \end{aligned} \tag{5.31}$$

Because of the warp factor, there is a redshift from the "throat" around $r = 0$ to an observer at infinity. There are two types of low energy excitations. We can have excitations of very low energy away from $r = 0$, or we can have excitations that have arbitrary energy around $r = 0$ — but have very low energy from infinity. In the IR these two types are decoupled: excitations inside the throat cannot escape because of the redshift (gravitational potential); excitations at infinity have vanishing cross-section on the brane (the cross section scales as $\sigma \sim \omega^3 R^8$).[25]

If we zoom on $r = 0$, the metric becomes

$$ds^2 \sim \frac{r^2}{R^2} ds^2_{3,1} + R^2 \frac{dr^2}{r^2} + R^2 d\Omega_5^2 = R^2 \left(\frac{ds^2_{3,1} + dz^2}{z^2} + d\Omega_5^2 \right) \tag{5.32}$$

where we have redefined $r/R = R/z$. We recognize $\mathrm{AdS}_5 \times S^5$.

Since one of the two systems is free gravity in both cases, we are led to identify the two interacting systems:[26]

$$\text{4d } \mathcal{N} = 4 \text{ SYM} \qquad \leftrightarrow \qquad \text{IIB string theory on } \mathrm{AdS}_5 \times S^5.$$

We can approximate IIB string theory by IIB supergravity as long as

$$R_s \sim \frac{1}{R^2} \sim \frac{1}{(g_s N)^{1/2} \alpha'} \ll \frac{1}{\alpha'} \ , \qquad\qquad g_s = g^2_{\mathrm{YM}} \ll 1 \ . \tag{5.33}$$

[25] The cross-section computation is summarized in Section 1.3.3 of [AGM+00]. The result, for this particular type of black branes, is that the cross section for ℓ-wave at $\omega \to 0$ is $\sigma^\ell_{\mathrm{abs}} \sim \omega^{3+4\ell} R^{8+4\ell}$. The result is specific to this black brane, for instance the s-wave cross section of Schwarzschild for $\omega \ll T_H$ approaches the horizon area.

[26] The $U(1)$ lives at the boundary, and is described by a topological theory of the B-field. See page 58 of [AGM+00].

This corresponds to[27]

$$N \gg 1 \ , \qquad\qquad \lambda = g_{\rm YM}^2 N \gg 1 \ . \tag{5.34}$$

Thus, the large N limit of the QFT at large 't Hooft coupling λ is described by classical gravity! α' corrections, *i.e.* higher derivative corrections to supergravity coming from integrating out the massive string modes, correspond to $\lambda^{-1/2}$ corrections. String corrections in g_s correspond to $1/N$ corrections.

$$\alpha' \text{ corrections} \quad \leftrightarrow \quad \lambda^{-1/2} \text{ corrections} \ , \qquad g_s \text{ corrections} \quad \leftrightarrow \quad \frac{1}{N} \text{ corrections} \ . \tag{5.35}$$

On the field theory side we should also have a free decoupled $U(1)$. This lives somehow at the boundary between AdS and the bulk. Indeed IIB supergravity on $\text{AdS}_5 \times S^5$ has a topological sector

$$S_5 = \int C_2 \wedge dB_2 \tag{5.36}$$

whose dynamical dof's are at the boundary of AdS_5, and include a free Abelian gauge field. This is called the *singleton sector*.

[27]More precisely, one takes $N \to \infty$ keeping λ fixed, possibly large. This assures that R, and thus the geometry, is fixed in the limit.

6 The holographic dictionary

In what sense the CFT is equivalent to the gravitational theory in AdS? The two look completely different, they even have different dimension.

Let us focus on the Euclidean case, where Euclidean AdS = hyperbolic plane. Moreover focus on the Poincaré patch of AdS_{d+1} (upper half-plane)

$$ds^2 = R^2 \frac{d\vec{x}^2 + dz^2}{z^2} \,, \tag{6.1}$$

whose boundary is $\mathbb{R}^d$. One could equally well consider global AdS (the ball, whose boundary is S^d).

Since AdS has a boundary, to define the gravitational theory we need to specify boundary conditions for the fields $\phi(\vec{x}, z)$. On the other hand, in a CFT important observables are the correlation functions of local operators $\mathcal{O}(\vec{x})$, whose generating functional is constructed out of sources. It is then natural to identify

$$\mathcal{Z}_{\text{bulk}}\Big[\phi(\vec{x}, z)\Big|_{z=0} = \phi_0(\vec{x})\Big] = \Big\langle e^{\int d^dx\, \phi_0(\vec{x})\, \mathcal{O}(\vec{x})} \Big\rangle_{\text{CFT}} \,. \tag{6.2}$$

On the LHS is the partition function of string theory, function of the boundary conditions $\phi_0(\vec{x})$ at the boundary $z = 0$. On the RHS is the generating functional of correlators in the CFT, functional of the sources $\phi_0(\vec{x})$. This identification requires a correspondence

$$\text{field in the bulk} \qquad \leftrightarrow \qquad \text{operator in the boundary CFT} \,. \tag{6.3}$$

A field that is the derivative of another field corresponds to a descendant in the CFT, thus we will only describe primary operators.

The partition function of string theory is a very complicated (and unknown) object.[28] When gravity is weakly coupled, we can approximate by the classical action:

$$\mathcal{Z}_{\text{bulk}} \simeq e^{-N^2 S_{\text{class}}[\phi] + O(\alpha')} + O(g_s) \tag{6.4}$$

Then the partition function is dominanted by the saddle points, *i.e.* the classical solutions to the EOMs.

Thus, in the classical limit the correspondence states that:

The classical gravity action is the generating functional
of connected correlators in the CFT.

The classical EOMs are second order: we impose Dirichelet boundary conditions on the boundary, and regularity at the horizon (in the Euclidean version). This fixes the classical solutions.

[28] The quantity $\log \mathcal{Z}_{\text{bulk}}$ can be interpreted as a string theory S-matrix element of the state ϕ_0.

Consider a scalar field ϕ of mass m^2 in AdS_{d+1}. Its action is

$$S = N^2 \int d^d x\, dz\, \sqrt{g} \Big[\frac{1}{2} \nabla_M \phi \nabla^M \phi + \frac{1}{2} m^2 \phi^2 + O(\phi^3) \Big] \,. \tag{6.5}$$

Let us first consider small quadratic fluctuations, *i.e.* we neglect the interactions and only study the linearized EOMs.

$$0 = -\nabla_M \nabla^M \phi + m^2 \phi = -\frac{1}{\sqrt{g}} \partial_M \big(\sqrt{g}\, g^{MN} \partial_N \phi \big) + m^2 \phi \,. \tag{6.6}$$

We go to Fourier space for the momentum on $\mathbb{R}^d$, $\eta^{\mu\nu} \partial_\mu \partial_\nu = -p^2$. Then

$$0 = z^{d+1} \partial_z \big(z^{1-d} \partial_z \phi \big) - p^2 z^2 \phi - m^2 R^2 \phi \,. \tag{6.7}$$

There are two independent solutions. They can be written exactly in terms of Bessel functions.[29] Let us see the asymptotic behavior at the boundary, $z \sim 0$. The term with momentum can be neglected, and the solutions are power-law:

$$\phi \,\sim\, z^{\alpha_\pm} \qquad\qquad \alpha_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2} \,. \tag{6.8}$$

The solution with α_- dominates as $z \to 0$, and the solution with α_+ always decays (while the one with α_- could diverge).

Let us assume, for now, that we impose the boundary condition on the dominant solution. We should then impose

$$\phi(x,z)\Big|_{z=\epsilon} = \epsilon^{\alpha_-}\, \phi_0^{\mathrm{ren}}(x) \,. \tag{6.9}$$

We impose the boundary condition at $z = \epsilon$, then send $\epsilon \to 0$ in such a way that the solution in the bulk has finite limit. ϕ_0^{ren} is called the "renormalized" boundary condition.

If we perform a rescaling of coordinates in the boundary theory, which is the AdS isometry

$$x \to \lambda x \,, \qquad\qquad z \to \lambda z \,, \tag{6.10}$$

the bulk field ϕ remains invariant[30] but ϕ_0^{ren} has to rescale[31] with dimension α_-. Since we identify it with the source, we conclude that the corresponding boundary operator $\mathcal{O}$ has dimension

$$\Delta = d - \alpha_- = \alpha_+ \,. \tag{6.11}$$

Thus we associate a bulk scalar field of squared mass m^2 to a boundary operator of dimension Δ.

---

[29] The two solutions are $\phi = c_1 z^{d/2} I_a(pz) + c_2 z^{d/2} K_a(pz)$ with $a = 12\sqrt{d^2 + 4m^2R^2}$.

[30] ϕ remains invariant because it is a scalar, but of course it has to be evaluated at the new location.

[31] The cutoff position gets rescaled to $\lambda z = \epsilon$, that is $z = \epsilon/\lambda$. It follows that $\phi_0^{\mathrm{ren}}(x) \to \lambda^{\alpha_-} \phi_0^{\mathrm{ren}}(\lambda x)$.

One still gets a real acceptable dimension if m^2R^2 is negative with

$$-\frac{d^2}{4} \leq m^2R^2 \ . \tag{6.12}$$

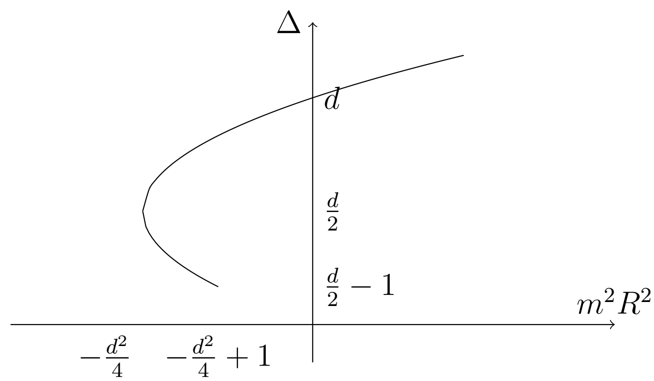
In fact particles in AdS can have negative mass squared and be stable. The bound is called *Breitenlohner-Freedman bound*. It is due to the fact that a wave-function has to decay at infinity, therefore it always has some kinetic contribution that can overwhelm a small negative potential energy.

For

$$-\frac{d^2}{4} \leq m^2R^2 \leq -\frac{d^2}{4} + 1 \tag{6.13}$$

also α_- would be an acceptable dimension (above the scalar unitarity bound $\Delta \geq \frac{d}{2} - 1$). Indeed in this range one could impose the boundary condition on the other mode, $\phi \sim z^{\alpha_+}$. In this range, double quantization is possible. Notice that this is the only way to get operators of dimension $< \frac{d}{2}$.

The picture that we get is:



$-\frac{d^2}{4} \leq m^2R^2 < 0$	$\Delta < d$	relevant	$\Big(-\frac{d^2}{4} \leq m^2R^2 \leq -\frac{d^2}{4} + 1$ double quantization $\Big)$
$m^2R^2 = 0$	$\Delta = d$	marginal	
$0 < m^2R^2$	$\Delta > d$	irrelevant	

The field $\leftrightarrow$ operator map (holographic dictionary) is not given a priori. In many cases it can be determined based on the mass/dimension, the spin, and some other quantum numbers. Some operators are easy to determine. The boundary value of the bulk metric g_{MN} is the boundary metric $g_{\mu\nu}$, which is the source for the stress tensor $T_{\mu\nu}$. Thus

$$g_{MN} \qquad \leftrightarrow \qquad T_{\mu\nu} \ . \tag{6.14}$$

An analysis of gravitational waves in AdS_{d+1} shows that the dual operator has dimension d.

Suppose we have a gauge field A_M in the bulk. The dual operator must be a vector J^μ, coupled to the source as

$$\int d^dx \, A_\mu J^\mu \ .$$

The bulk theory must be invariant under gauge transformations, $\delta A_M = D_M\lambda$, thus (in the absence of anomalies[32]) the boundary coupling should be invariant as well:

$$0 = \delta \int d^d x\, A_\mu J^\mu = -\int d^d x\, \lambda\, D_\mu J^\mu \;. \tag{6.15}$$

Thus J_μ should be a conserved current in the boundary theory. From the wave equation in AdS, one obtains that J_μ has dimension $d-1$. (¶) Thus

$$\begin{array}{ccc} \text{bulk gauge symmetry} & \leftrightarrow & \text{boundary global symmetry} \\ A_M & \leftrightarrow & J_\mu \;. \end{array} \tag{6.16}$$

The value of the dilaton at infinity is the string coupling $g_s = g_{\mathrm{YM}}^2$, thus

$$e^{-\phi_\infty} = \frac{1}{g_s} = \frac{1}{g_{\mathrm{YM}}^2} \;. \tag{6.17}$$

By taking a small variation:

$$\phi \qquad \leftrightarrow \qquad \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \;. \tag{6.18}$$

Indeed on $\mathrm{AdS}_5 \times S^5$ the dilaton is massless, and $\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$ is a marginal operator of dimension 4.

We could go on and discover that gravitinos correspond to supersymmetry currents $S_{\alpha\mu}$, Abelian p-form potentials correspond to Abelian higher-form symmetries [GKSW15]. And study the mass/dimension formula for arbitrary spin.

6.1 Correlation functions

Connected correlation functions are computed by derivatives of the classical on-shell action, and this can be done with a diagrammatic expansion. Since we use the supergravity effective action, only tree-level diagrams should be used.

6.1.1 Two-point functions

To compute two-point functions, only the part of the action quadratic in the relevant field perturbation is needed. This action is

$$S^{(2)} = \int d^d x\, dz\, \sqrt{g} \Big[\nabla_M \phi \nabla^M \phi + m^2 \phi^2 \Big] \;, \tag{6.19}$$

[32]In fact, the bulk theory should only be invariant under gauge transformations that vanish at infinity. Under gauge transformations that are non-trivial at infinity, the effective action can have an anomalous variation which is constrained, by the Wess-Zumino consistency conditions, to be a local functionals of the field strengths. This reproduces global 't Hooft anomalies on the boundary. Another argument is that the dimension of J^μ is fixed.

and the EOM is

$$\big(\Box - m^2\big)\phi = 0 \ . \tag{6.20}$$

Its solution is the propagator, and the solutions with fixed boundary conditions are called *bulk-to-boundary propagator*. We start by considering the propagator in momentum space, because it is more convenient to compute 2-point functions, but will use the one in position space for higher-point correlators.

A complete set of solutions to the linearized EOM is

$$\phi(\vec{x}, z) = e^{i\vec{p}\cdot\vec{x}}\, Z(pz) \tag{6.21}$$

where $p = |\vec{p}|$ (and we assumed $p \neq 0$). The equation is the same as before, but redefine $u = pz$:

$$\Big[u^{d+1}\partial_u\big(u^{1-d}\partial_u\big) - u^2 - m^2R^2\Big]\, Z(u) = 0 \ . \tag{6.22}$$

The two solutions are expressed in terms of Bessel functions:[33]

$$Z(u) = c_1\, u^{d/2}\, I_{\Delta-\frac{d}{2}}(u) + c_2\, u^{d/2}\, K_{\Delta-\frac{d}{2}}(u) \tag{6.26}$$

where, as before, $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2R^2}$.

We should also impose regularity in the interior of AdS (from which we get a single solution). The second solution K is selected, because the other one I is exponentially diverging in the interior of AdS ($u \to +\infty$) and it does not lead to finite-action configurations.

Imposing the boundary condition

$$\phi(\vec{x}, \epsilon) = \epsilon^{d-\Delta}\, \phi_0(\vec{x}) = \epsilon^{d-\Delta}\, e^{i\vec{p}\cdot\vec{x}} \ , \tag{6.27}$$

we get the bulk-to-boundary propagator:

$$\phi(\vec{x}, z) \equiv K_{\vec{p}}(\vec{x}, z) = \epsilon^{d-\Delta}\, \frac{z^{d/2}\, K_{\Delta-\frac{d}{2}}(pz)}{\epsilon^{d/2}\, K_{\Delta-\frac{d}{2}}(p\epsilon)}\, e^{i\vec{p}\cdot\vec{x}} \ . \tag{6.28}$$

[33] The Bessel function $I_\alpha(u)$ can be defined by a series expansion:

$$I_\alpha(u) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)}\, \Big(\frac{u}{2}\Big)^{2m+\alpha} \ . \tag{6.23}$$

The Bessel function $K_\alpha(u)$ can be defined by $K_\alpha(u) = \frac{\pi}{2}\frac{I_{-\alpha}(u) - I_\alpha(u)}{\sin(\alpha\pi)}$ for $\alpha \notin \mathbb{Z}$, and by a limit otherwise.

They have the following asymptotic behavior. At the boundary $u \to 0$:

$$I_\alpha(u) \sim \frac{1}{\Gamma(\alpha+1)}\Big(\frac{u}{2}\Big)^\alpha \ , \qquad K_\alpha(u) \sim \begin{cases} -\log\big(\frac{u}{2}\big) - \gamma & \alpha = 0 \ , \\ \frac{\Gamma(\alpha)}{2}\big(\frac{2}{u}\big)^\alpha & \alpha > 0 \ . \end{cases} \tag{6.24}$$

At the horizon $u \to +\infty$:

$$I_\alpha(u) \sim \frac{e^u}{\sqrt{2\pi u}} \ , \qquad K_\alpha(u) \sim \sqrt{\frac{\pi}{2u}}\, e^{-u} \ . \tag{6.25}$$

At this point we should compute the on-shell classical action evaluated on these configurations. To do that, we rewrite the contribution as a total derivative:

$$S = \int d^dx \int_\epsilon^\infty dz\, \sqrt{g} \Big[\frac{1}{2} \phi \big(-\Box + m^2 \big) \phi + \frac{1}{2} \nabla_M (\phi \partial^M \phi) \Big] . \tag{6.29}$$

The first term is zero on the classical solutions, while the fields can be taken to vanish at infinity in $\vec{x}$:

$$= \frac{1}{2} \int d^dx \Big[\sqrt{g}\, g^{zz} \phi \partial_z \phi \Big]_\epsilon^\infty . \tag{6.30}$$

We expand the fields into their basis:

$$\phi(\vec{x}, z) = \int d^dp\, \lambda_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}}\, \widetilde{K}_p(z) . \tag{6.31}$$

where $\widetilde{K}$ is the function of z (stripped of $e^{i\vec{p}\cdot\vec{x}}$). The integral over d^dx gives $(2\pi)^d \delta^d(\vec{p} + \vec{q})$. Thus

$$= \frac{1}{2} \int d^dp\, d^dq\, \lambda_{\vec{p}}\, \lambda_{\vec{q}}\, (2\pi)^d \delta^d(\vec{p} + \vec{q})\, \mathcal{F} \tag{6.32}$$

with

$$\mathcal{F} = - \left(\frac{R}{z} \right)^{d-1} \widetilde{K}_p \partial_z \widetilde{K}_p \Big|_{z=\epsilon} . \tag{6.33}$$

The connected 2-point function is

$$\langle \mathcal{O}(\vec{p})\, \mathcal{O}(\vec{q}) \rangle = \frac{\partial^2 S^{(2)}_{\text{on-shell}}}{\partial \lambda_{\vec{p}}\, \partial \lambda_{\vec{q}}} \Big|_{\lambda_{\vec{p}} = \lambda_{\vec{q}} = 0} = (2\pi)^d \delta^d(\vec{p} - \vec{q}) \mathcal{F} \tag{6.34}$$

We should expand $\mathcal{F}$ for $\epsilon \to 0$. To do that, we use that for $u \to 0$ (and for generic $\alpha \in \mathbb{R}$) we have

$$K_\alpha(u) = \#\, u^{-\alpha} \big(1 + \ldots + \#\, u^{2\alpha} + \ldots \big) , \tag{6.35}$$

where $\ldots$ represent the Taylor expansion in u. Working out how this propagates to $\mathcal{F}$ we find

$$\mathcal{F} = \text{analytic} + \#\, p^{2\Delta - d} + \text{subleading} . \tag{6.36}$$

The analytic terms are integer powers (possibly including a constant) of p^2, and they diverge in ϵ. Terms analytic in p^2 become contact terms once Fourier transformed to position space: they can usually be removed by local counter-terms (we will say more about this) and can be neglected. The physical pieces are the non-analytic terms (that do not vanish as $\epsilon \to 0$). Indeed, in the second term above, the powers of ϵ cancel exactly (while "subleading" means terms that vanish as $\epsilon \to 0$). Therefore

$$\langle \mathcal{O}(\vec{p})\, \mathcal{O}(\vec{q}) \rangle = \text{analytic} + \#\, \delta^d(\vec{p} - \vec{q})\, \big(\vec{p} \big)^{2\Delta - d} + \text{subleading} . \tag{6.37}$$

Performing a Fourier transform (and discarding contact terms)[34] we get

$$\langle \mathcal{O}(\vec{x})\, \mathcal{O}(\vec{y}) \rangle = \frac{\#}{|\vec{x}-\vec{y}|^{2\Delta}} \;. \tag{6.41}$$

We see that, indeed, $\mathcal{O}$ behaves as an operator of dimension Δ.

The case of $\Delta \in \mathbb{Z}$ is more subtle: the Bessel function $K_\alpha(u)$ behaves as $u^{-\alpha}(1+\dots+u^{2\alpha}\log u+\dots)$, so those are the non-analytic terms. The Fourier transform again gives the power-law behavior.[35]

6.1.2 Higher-point functions

In order to compute higher-point functions, it turns out that it is easier to work with the propagator in position space (because it leaves conformal invariance manifest).

For simplicity, we consider a set of fields ϕ_i with mass m_i interacting with a local Lagrangian $\mathcal{L}_{\text{AdS}}$. We impose boundary conditions

$$\phi_i(\vec{x}, z) \;\to\; z^{d-\Delta_i}\phi_{0,i}(\vec{x}) \qquad \text{for } z\to 0 \tag{6.43}$$

with $m_i^2 = \Delta_i(d-\Delta_i)$. The CFT connected generating function is the on-shell action evaluated on the classical solutions with prescribed boundary conditions (and regular in the interior). A connected n-point function is

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n\rangle_c = \frac{\partial^n S}{\partial\phi_{0,1}\dots\partial\phi_{0,n}}\bigg|_{\phi_{0,i}=0} \;. \tag{6.44}$$

[34] We have to perform the Fourier transforms

$$I_\beta(\vec{x}) = \int d^dp\, e^{i\vec{p}\cdot\vec{x}}\, |p|^\beta \;. \tag{6.38}$$

If $\beta=0$, then $I_0 = (2\pi)^d\delta^d(\vec{x})$. If β is a positive even integer (*i.e.* we take an analytic piece in $\vec{p}^2$), then I_β can be written as a derivative

$$I_{2n}(\vec{x}) = (-\Box)^n I_0 = (2\pi)^d(-\Box)^n\delta^d(\vec{x}) \qquad\qquad \text{for } n\in\mathbb{Z}_{\geq 0} \tag{6.39}$$

and thus it is a contact term. If β is real or complex, instead, this is not possible and indeed we get a power-law, as can be inferred from a scaling argument. The complete result is

$$I_\beta(\vec{x}) = \frac{2^{\beta+d}\pi^{d/2}\Gamma\big(\frac{\beta+d}{2}\big)}{\Gamma\big(-\frac{\beta}{2}\big)}\,\frac{1}{|x|^{\beta+d}} \qquad\qquad \text{for } \beta\notin 2\mathbb{Z}_{\geq 0}\;. \tag{6.40}$$

[35] We want to compute the Fourier transforms $\int d^dp\, e^{i\vec{p}\cdot\vec{x}}\, p^{2n}\log|p|$. We first consider the case $n=0$. A scaling argument fixes the behavior $|x|^{-d}$ up to a contact term. The contact term is $-(2\pi)^d\log\frac{|x|}{\mu}\,\delta^d(\vec{x})$. Since, as a distribution, this has to be multiplied by functions that vanish at zero, we neglect the contact term. We find

$$\int d^dp\, e^{i\vec{p}\cdot\vec{x}}\,\log|p| = -2^{d-1}\pi^{d/2}\Gamma\big(\tfrac{d}{2}\big)\,\frac{1}{|x|^d}\;. \tag{6.42}$$

Other values of n are obtained by acting with $(-\Box)^n$, and give a behavior $|x|^{-d-2n}$.

Since on-shell fields vanish with sources turned off, only terms in the action with at most n fields can contribute.

To study 1- and 2-point functions we only need the quadratic action.

To construct a classical solution with given boundary conditions, we need a Green function—the bulk-to-boundary propagator:

$$\phi(\vec{x},z) = \int d^d y \, K_\Delta(\vec{x}-\vec{y},z)\,\phi_0(\vec{y}) \tag{6.45}$$

which satisfies

$$\big(-\Box + m^2\big)K_\Delta = 0 \;, \qquad\qquad K_\Delta \to z^{d-\Delta}\delta(\vec{x}-\vec{y}) \qquad \text{for } z\to 0 \;. \tag{6.46}$$

To find the solution we notice that

$$z^\Delta$$

is a solution to the Klein-Gordon equation. If we regard $z=\infty$ as the point that compactifies the boundary of AdS from $\mathbb{R}^d$ to S^d, we can think of that solution as the one with a δ-function at the point at infinity on S^d (at all other points it vanishes). Then we move that point to $(\vec{x}=0, z=0)$ by an isometry of AdS:

$$z \to \frac{z}{z^2+\vec{x}^2} \;, \qquad\qquad \vec{x} \to \frac{\vec{x}}{z^2+\vec{x}^2} \;. \tag{6.47}$$

We obtain

$$K_\Delta(\vec{x},z) = c\left(\frac{z}{z^2+\vec{x}^2}\right)^\Delta \tag{6.48}$$

where one fixes the normalization.[36]

Now, a solution behaves at the boundary as

$$\phi(\vec{x},z) \to \phi_0(x)\big(z^{d-\Delta}+\dots\big) + \phi_1(x)\big(z^\Delta + \dots\big) \tag{6.50}$$

where $\dots$ are a series expansion in z and depend on x as well. In particular, expanding K:

$$\phi_1(x) = c\int d^d y\,\frac{\phi_0(y)}{(\vec{x}-\vec{y})^{2\Delta}} \tag{6.51}$$

[36] Let us check that we reproduce the correct boundary condition:

$$\begin{aligned}\int dx\,K(x,z)\,\phi_0(x) &= c\,z^{d-\Delta}\int dx\,\frac{z^{2\Delta-d}}{(z^2+x^2)^\Delta}\,\phi_0(x) = c\,z^{d-\Delta}\int dy\,\frac{\phi_0(yz)}{(1+y^2)^\Delta} \\ &\xrightarrow{z\to 0} c\left(\int dy\,\frac{1}{(1+y^2)^\Delta}\right) z^{d-\Delta}\phi_0(0)\;.\end{aligned} \tag{6.49}$$

One finds $c = \big(\int d^d y\,(1+y^2)^{-\Delta}\big)^{-1} = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta-\frac{d}{2})}$.

is a non-local functional of ϕ_0. Thus regularity, which is a local condition at the horizon, appears as a non-local relation at the boundary. As before, we compute the on-shell action reducing to a boundary term:

$$S = \frac{1}{2} \int d^d x \, \frac{\phi \partial_z \phi}{z^{d-1}} \Big|_{\text{boundary}} \tag{6.52}$$

Inserting ϕ we find many diverging terms, such as $\phi_0^2(x)$ or other local functions of $\phi_0(x)$ (we will see holographic renormalization). Those are contact terms that can be removed by local counter-terms. The leading non-diverging term, which is also non-local, is

$$S \sim \int d^d x \, \phi_0(x) \, \phi_1(x) \sim \int d^d x \, d^d y \, \frac{\phi_0(x) \, \phi_0(y)}{|x-y|^{2\Delta}} \, . \tag{6.53}$$

We infer the 1-point function:

$$\langle \mathcal{O}(x) \rangle = \frac{\delta S}{\delta \phi_0(x)} \Big|_{\phi_0 = 0} = \phi_1(x) \Big|_{\phi_0 = 0} \, . \tag{6.54}$$

In AdS this is zero, because with no source also $\phi_1 = 0$. This matches the fact that in CFTs the 1-point functions vanish (because are dimensionful).

However this is important: the normalizable mode at infinity is the VEV of the operator (indeed it scales with z^Δ). Thus the two modes are the source and the VEV of $\mathcal{O}$. This is important in non-conformal examples.

Then the 2-point function is

$$\langle \mathcal{O}(x) \, \mathcal{O}(y) \rangle = \frac{\delta^2 S}{\delta \phi_0(x) \, \delta \phi_0(y)} \Big|_{\phi_0 = 0} = \frac{1}{|x-y|^{2\Delta}} \, . \tag{6.55}$$

We have reproduced the 2-point functions in another way.

We have been sloppy with eliminating divergences. In fact, it turns out that this sloppy way to compute the 2-point function gives the wrong normalization (while the one in momentum space is correct). We will see how to perform holographic renormalization properly. Higher-point functions turn out correct.

To compute higher-point functions, we need interaction terms in the Lagrangian. Suppose

$$S = \int d^d x \, dz \, \sqrt{g} \Big(\frac{1}{2} \sum_i (\partial \phi_i)^2 + \frac{m_i^2}{2} \phi_i^2 + \sum_{k=3}^n \lambda_{i_1 \dots i_k} \phi_{i_1} \dots \phi_{i_k} \Big) \, . \tag{6.56}$$

We are taking canonical kinetic terms, no higher-derivative interactions, no interactions with gauge fields and the metric. All those details can be incorporated (with effort).

Now the EOMs are non-linear, but can be solved perturbatively. Suppose we have

$$\big(-\Box + m^2 \big) \phi = \lambda \phi^{k-1} \, . \tag{6.57}$$

At order λ^0 the solution is given by the bulk-to-boundary propagator:

$$\phi^{(0)}(\vec{x}, z) = \int d^d y \, K(\vec{x} - \vec{y}, z) \, \phi_0(\vec{y}) \; . \tag{6.58}$$

At order λ^1 the solution is given by the standard bulk-to-bulk propagator:

$$\phi^{(1)}(\vec{x}, z) = \lambda \int d^d x' \, dz' \, G(\vec{x} - \vec{x}', z, z') \left[\phi^{(0)}(\vec{x}', z') \right]^{k-1} , \tag{6.59}$$

where G is the Green function in the bulk (with trivial boundary conditions):

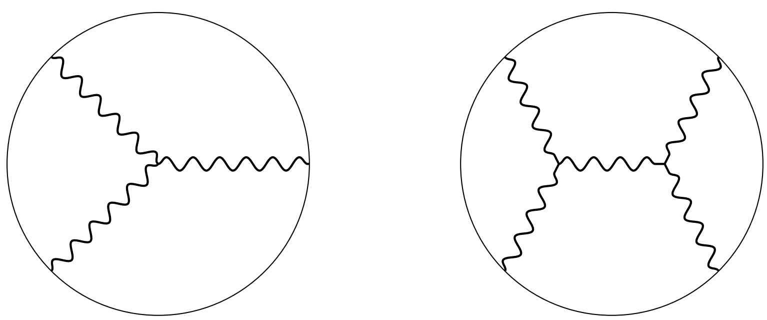
$$\big(- \Box_{(\vec{x},z)} + m^2 \big) G(\vec{x}, z, z') = \frac{1}{\sqrt{g}} \delta(\vec{x}, z) \; . \tag{6.60}$$

(This Green function vanishes at the boundary.) The explicit expressions can be found in [hep-th/0201253] eqn. (6.12) [DF02]. Then we substitute again $\phi^{(1)}$ to get $\phi^{(2)}$ at order λ^2, and so on.

Since all $\phi^{(j)}$ are functions of ϕ_0, to compute an n-point function we can stop the procedure once we have n instances of ϕ_0. The perturbative expansion ends. The resulting expansion,

$$\phi(\vec{x}, z) = \sum_{j=0} \phi^{(j)}(\vec{x}, z) \tag{6.61}$$

where $\phi^{(j)}$ is of order λ^j, should be plugged into to action, in order to obtain an expansion of the on-shell action. Then, taking derivatives with respect to ϕ_0, we get a *finite* expansion of correlations functions. We can set up a graphical Feynman representation, called *Witten diagrams*:



These are tree-level diagrams, so they are finite in number. The diagrams are tree-level because we are solving classical EOMs.[37]

For 3-point functions, we only have one graph with a cubic vertex and 3 bulk-to-boundary propagators:

$$\begin{aligned} \langle \mathcal{O}_1(\vec{x}_i) \, \mathcal{O}_2(\vec{x}_2) \, \mathcal{O}_3(\vec{x}_3) \rangle &= -\lambda \int d^{d+1}x \, \sqrt{g} \, K_{\Delta_1}(x; \vec{x}_1) \, K_{\Delta_2}(x; \vec{x}_2) \, K_{\Delta_3}(x; \vec{x}_3) \\ &= \frac{A \, \lambda}{|\vec{x}_1 - \vec{x}_2|^{\Delta_1 + \Delta_2 - \Delta_3} |\vec{x}_1 - \vec{x}_3|^{\Delta_1 + \Delta_3 - \Delta_2} |\vec{x}_2 - \vec{x}_3|^{\Delta_2 + \Delta_3 - \Delta_1}} \; . \end{aligned} \tag{6.62}$$

[37]Loop diagrams correspond to quantum effects in the bulk, and produce contributions that are suppressed by powers of $1/N^2$. We neglect quantum corrections here, but they have been studied in the literature.

The dependence on $\vec{x}_i$ is eventually fixed by conformal invariance. But to compute A one really needs to do the integral.

The computation of higher-point functions is much more complicated, for two reasons:

- There are various graphs that one has to compute.
- Some of the graphs involve standard bulk-to-bulk propagators, which however involve supergravity fields with spin, in general.

We will not look at those.[38]

6.2 Anomalies

Anomalies are an important test of AdS/CFT. 't Hooft anomalies are 1-loop exact, therefore they can be computed at strong coupling and directly compared with the gravity result.

An anomaly is the phenomenon for which a Lagrangian theory has a classical symmetry, but at the quantum level this symmetry might be spoiled.[39] Perturbative anomalies of continuous symmetries only exist in even dimensions.

There are many types of continuous anomalies:

- Gauge anomalies: a gauge symmetry is lost in the quantum theory, therefore the theory is *inconsistent* (it is not unitary).
  (In 4d: gauge-gauge-gauge triangle diagram.)
- Gauge-global (or Adler-Bell-Jackiw, ABJ) anomalies: a global symmetry is lost in the quantum theory. Ward identities are not respected, and there are no physical consequences of the symmetry. Simply the symmetry is not there.
  (In 4d: global-gauge-gauge triangle diagram.)
- 't Hooft anomalies: the global symmetry is there, correlation functions at separated points are symmetric and Ward identities are respected. However it might not be possible to choose the contact terms to be symmetric. These anomalies more easily appear when turning on a background for the global symmetry: then the current is no longer conserved.
  (In 4d: global-global-global triangle diagram.)

  Global symmetries with an 't Hooft anomaly cannot be consistently coupled to *dynamical* gauge fields.

[38] There has been some recent development, see for instance [RZ17, RZ18].

[39] It can be understood in various ways. One is that, to regularize the theory, one has to break the symmetry with the regulator, and the symmetry is not recovered as the regulator is removed. Another one is that the path-integral measure is not invariant.

- (¶) Is there a missing case?

In 4d, the non-Abelian anomaly can be expressed as the operator equation

$$(D_\mu J^\mu)_a = \frac{i}{384\pi^2} \mathcal{D}_{abc}\, \epsilon^{\mu\nu\rho\sigma} \big(F^a_{\mu\nu} F^b_{\rho\sigma} + \ldots \big) \tag{6.63}$$

which states the non-conservation of the current, in a background. This is the *consistent anomaly*, in which J_μ is defined as the variation of a generating functional with respect to its source A_μ, and which satisfies the Wess-Zumino consistency conditions. Here

$$\mathcal{D}_{abc} = 2\, \mathrm{Tr}_{\text{fermions}}\; T_a \{T_b, T_c\} \tag{6.64}$$

where the trace is over right-moving Weyl fermions. The dots stand for non-covariant terms: the consistent anomaly is not covariant (and, because of the anomaly, the current does not transform covariantly, see the clear exposition in [BZ84]). The Wess-Zumino consistency conditions follow from requiring that current correlators are reproduced by a generating functional $Z[A]$. Then the anomaly

$$\mathcal{A}(A) = \delta_\lambda \big(- \log Z[A] \big) \tag{6.65}$$

should furnish a representation of the gauge algebra:

$$[\delta_{\lambda_1}, \delta_{\lambda_2}] \big(- \log Z[A] \big) = \delta_{[\lambda_1,\lambda_2]} \big(- \log Z[A] \big) \;. \tag{6.66}$$

Solutions are expressed with the *descent formalism* (a nice review is [Bil08]). One starts in $d+2$ dimensions with a characteristic class (both closed and gauge invariant). Here

$$\alpha_6^{(0)} = \mathrm{Tr}\, F \wedge F \wedge F \;. \tag{6.67}$$

Then one applies descent:

$$\begin{aligned} \alpha_6^{(0)} &= d\alpha_5^{(0)} \;, \qquad & \alpha_5^{(0)} &= \mathrm{CS}_5 \\ \delta_\lambda \alpha_5^{(0)} &= d\alpha_4^{(1)} \;, \qquad & \alpha_4^{(1)} &= \mathrm{Tr}\, \big[\lambda\, \mathcal{A}(A) \big] \;. \end{aligned} \tag{6.68}$$

One finds

$$\alpha_5^{(0)} = \mathrm{Tr} \Big(A\, dA\, dA + \tfrac{3}{2} A^3 dA + \tfrac{3}{5} A^5 \Big) \;, \qquad \alpha_4^{(1)} = \mathrm{Tr} \Big[\lambda\, d\Big(A\, dA + \tfrac{1}{2} A^3 \Big) \Big] \;. \tag{6.69}$$

Let us remark that one can always (see [BZ84]) redefine the current by a local function of A and its derivatives,

$$J_\mu \quad \to \quad \widetilde{J}_\mu = J_\mu + X(A, dA) \;, \tag{6.70}$$

such that the new current $\widetilde{J}_\mu$ transforms covariantly and has a covariant anomaly equation.[40] This is called the *covariant anomaly*. The correlations functions of the covariant current

[40] Importantly, the consistent and covariant anomalies have different coefficient (in 4d there is a relative factor of 3). Therefore, when doing perturbative computations, it is very important to determine which current one is looking at.

cannot be obtained from a generating functional, and so the covariant current is not natural in AdS/CFT.

Let us focus on the example of 4d $\mathcal{N}=4$ SYM $\leftrightarrow$ IIB on $\mathrm{AdS}_5 \times S^5$.
In $\mathcal{N}=4$ SYM the symmetry we look at is the R-symmetry $SU(4)_R$, and the fermions are the gaugini in the adjoint of the gauge group and $\mathbf{4}$ (fundamental) of $SU(4)_R$. Therefore

$$\mathcal{D}_{abc} = (N^2-1) d_{abc} \;, \qquad\qquad d_{abc} = 2\, \mathrm{Tr}_{\text{fund}}\, T_a \{T_b, T_c\} \;. \tag{6.71}$$

How does this show up in AdS_5?

We should first compactify IIB on S^5, and obtain an effective 5d supergravity theory. This includes 5d $\mathcal{N}=8$ maximal $SO(6)$ gauged supergravity. It has $SU(4) \cong SO(6)$ gauge fields A_μ^a, coupled to the currents: $\int d^4x A_\mu^a J_a^\mu$.

There is a 5d Chern-Simons term:

$$S_5^{\text{CS}} = \frac{iN^2}{96\pi^2} \int_{\mathrm{AdS}_5} d^5x \, \sqrt{g} \left(d^{abc}\, \epsilon^{\mu\nu\rho\sigma\tau} A_\mu^a \partial_\nu A_\rho^b \partial_\sigma A_\tau^c + \dots \right) . \tag{6.72}$$

where $\dots$ contain more A and less ∂. The full expression is the one written above. Usually a CS term leads to a gauge-invariant action because its variation is a total derivative, but with a boundary this might not be true. We perform

$$A_\mu^a \;\to\; A_\mu^a + (D_\mu \lambda)^a \;. \tag{6.73}$$

The variation of the CS term is

$$\delta S_5^{\text{CS}} = -\frac{iN^2}{384\pi^2} \int_{\partial \mathrm{AdS}_5} d^4x \, d^{abc}\, \epsilon^{\mu\nu\rho\sigma} \lambda^a \big(F_{\mu\nu}^b F_{\rho\sigma}^c + \dots \big) \;. \tag{6.74}$$

The dots represent the extra terms, obtained above through descent.

If we compare with the boundary description:

$$\delta S = \int d^4x \, (D_\mu \lambda)^a J_a^\mu = - \int d^4x \, \lambda^a (D_\mu J^\mu)_a \;, \tag{6.75}$$

it reproduces the anomaly (at leading order in N). In fact, AdS/CFT explicitly realizes *anomaly inflow.* We conclude that

't Hooft anomalies are described by CS terms in the bulk.

Indeed, perturbative anomalies of continuous symmetries only exist in even dimensions and (standard) Chern-Simons terms only exist in odd dimensions.

Another type of anomaly is the conformal anomaly. To describe it, we first need holographic renormalization.

7 Holographic renormalization

A good review is by K. Skenderis [hep-th/0209067] [Ske02].

In QFT there are UV divergences that need to be removed to obtain sensible finite answers. This is renormalization. One could expect that the same problem arises in gravity doing AdS/CFT, and we saw that it is the case. Our way to deal with them was simply to discard divergences, but this is sloppy (since one might lose finite contributions). The correct way to do that (in QFT too) is to subtract divergences by means of local covariant counter-terms.

Let us start with some considerations about the metric.

Consider Euclidean AdS in global coordinates:

$$ds^2 = \frac{L^2}{\cos^2\theta}\big(dt^2 + d\theta^2 + \sin^2\theta\, d\Omega^2_{d-1}\big) \tag{7.1}$$

where $0 \leq \theta < \frac{\pi}{2}$. The metric has a second-order pole at $\theta = \frac{\pi}{2}$, where there is the boundary. Because of the pole, the bulk metric does not directly define a boundary metric. It defines a *conformal structure*.

Choose a function $z(x)$ which is positive inside AdS but has a first-order zero at the boundary. Then multiply the AdS metric by $z^2(x)$ and evaluate at the boundary: this gives a metric.

$$g_{(0)} = z^2(x)\, g_{\text{AdS}}(x)\big|_{\theta=\frac{\pi}{2}} \; . \tag{7.2}$$

Such a metric is only defined up to conformal transformations. An equally good choice would be

$$z'(x) = e^{w(x)} z(x) \tag{7.3}$$

which leads to the conformally transformed metric

$$g'_{(0)} = e^{2w} g_{(0)} \; . \tag{7.4}$$

Thus an asymptotically AdS metric only gives a boundary metric up to conformal transformations.

It has been proven in [FG85] that it is always possible to bring an asymptotically-AdS metric to the Fefferman-Graham form

$$ds^2 = \frac{1}{z^2}\big(dz^2 + g_{ij}(x,z)\big) \qquad\qquad g_{ij}(x,z) \text{ smooth at } z \to 0 \; . \tag{7.5}$$

We can expand the metric near the boundary:

$$g_{ij}(x,z) = g_{(0)ij} + z\, g_{(1)ij} + z^2 g_{(2)ij} + \dots \tag{7.6}$$

If we fix $g_{(0)}$, then some of the $g_{(k)}$ with $k > 0$ are fixed by Einstein equations. In pure gravity odd powers of z vanish up to z^d, so let us use

$$\rho = z^2 \; . \tag{7.7}$$

Then we can write

$$
\begin{aligned}
ds^2 = G_{\mu\nu}dx^\mu dx^\nu = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(x,\rho)dx^i dx^j \\
g(x,\rho) = g_{(0)} + \ldots + \rho^{d/2}g_{(d)} + \rho^{d/2}\log\rho\, h_{(d)} + \ldots\ .
\end{aligned}
\tag{7.8}
$$

The term $\rho^{d/2}\log\rho$ only appears for d even (only in that case the two Taylor expansions would otherwise overlap).

Einstein's equations can be solved order by order in ρ. The equations fix $g_{(2)}, \ldots, g_{(d-2)}$, $h_{(d)}$ (present only for d even) as well as part of $g_{(d)}$ in terms of $g_{(0)}$. In particular the equations fix the divergence and the trace of $g_{(d)}$. Those terms are fixed *algebraically* in terms of $g_{(0)}$ and its derivatives, with no need to solve any differential equation. In other words, they are local functions of $g_{(d)}$. All these terms are part of the non-normalizable mode, and $g_{(0)}$ is the source for the boundary stress tensor.

The other components are free (and fixed by regularity in the interior).[41] In other words, the other components of $g_{(d)}$ are non-local functions of $g_{(0)}$. The mode $g_{(d)}$ is the normalizable mode, corresponding to the VEV of the stress tensor.

Thus, the divergence and trace of the stress tensor are fixed local functions of the boundary metric: this the *conformal anomaly*.

We will not reproduce this difficult computation, which can be found in [hep-th/0002230] [dHSS01], and instead explain the general idea and show the example of a massive scalar.

7.1 Holographic renormalization method

Asymptotic solution. The first step is to construct the bulk solution for given (arbitrary) boundary conditions. One can use the diagrammatic method that we discussed (Witten diagrams).

Suppressing spacetime and internal indices, let the fields be $\mathcal{F}(x,\rho)$. Near the boundary there is an asymptotic expansion

$$
\mathcal{F}(x,\rho) = \rho^m\Big(f_{(0)}(x) + \rho\, f_{(2)}(x) + \ldots + \rho^\ell\big(f_{(2\ell)}(x) + \log\rho\,\tilde{f}_{(2\ell)}(x) + \ldots\big)\Big)\ . \tag{7.9}
$$

The two asymptotic solutions have behavior ρ^m and $\rho^{m+\ell}$. Here (contrary to before) we assume $\ell \in \mathbb{Z}_{\geq 0}$, then we get the logarithmic term.

Here:

- $f_{(0)}$ is the source for the dual operator, and it is fixed (it is the boundary condition).
- the EOMs fix $f_{(2)}, \ldots, f_{(2\ell-2)}, \tilde{f}_{(2\ell)}$ *algebraically* in terms of $f_{(0)}$ and its derivatives (local functions of $f_{(0)}$).

[41] This is strictly speaking true with no matter. With matter this is still true if the fields correspond to marginal or relevant operators. Fields corresponding to irrelevant operators (large mass) should have infinitesimal sources in order not to destroy AdS.

$f_{(2\ell)}$ is not fixed, because it is an independent solution. It is fixed by regularity in the interior.

Regularization. We evaluate the on-shell action on the asymptotic solution. To regularize, we cut the intergrals at $\rho = \epsilon$. By doing the integrals, we get a finite number of boundary terms that diverge as $\epsilon \to 0$:

$$S_{\text{reg}}[f_{(0)};\epsilon] = \int_{\rho=\epsilon} d^d x \, \sqrt{g_{(0)}} \Big(\epsilon^{-\nu} a_{(0)} + \epsilon^{-\nu+1} a_{(2)} + \ldots + \log\epsilon \, a_{(2\nu)} + \mathcal{O}(\epsilon^0) \Big) . \tag{7.10}$$

Here $a_{(2k)}$ are local functionals of $f_{(0)}$, and do not depend on $f_{(2\ell)}$ (which is not fixed by the boundary conditions).

Counterterms. One defines a covariant action of the fields $\mathcal{F}(x,\epsilon)$ (and their derivatives) and the induced metric $\gamma_{ij} = g_{ij}(x,\epsilon)/\epsilon$, in such a way that it reproduces the divergent terms:

$$S_{\text{ct}}[\mathcal{F}(x,\epsilon);\epsilon] = -\text{divergent terms of } S_{\text{reg}}[f_{(0)};\epsilon] \tag{7.11}$$

In practice one has to invert the asymptotic expansion and find a formula for

$$f_{(0)} = f_{(0)}\big(\mathcal{F}(x,\epsilon), \epsilon\big) \tag{7.12}$$

then plug in $a_{(2k)}(f_{(0)})$ and in S_{reg}.

For instance, in the case of a massive scalar field:

$$\Phi(x,\rho) = \rho^{\frac{d-\Delta}{2}} \Big(\phi_{(0)}(x) + \rho\, \phi_{(2)}(x) + \rho^2 \, \phi_{(4)}(x) + \ldots \Big) . \tag{7.13}$$

Solving the EOMs in AdS, order by order in ρ, one finds

$$\phi_{(2)}(x) = \frac{1}{2(2\Delta - d - 2)} \Box_0 \phi_{(0)} , \qquad \phi_{(4)}(x) = \frac{1}{4(2\Delta - d - 4)} \Box_0 \phi_{(2)} , \qquad \ldots \tag{7.14}$$

where $\Box_0 = \eta^{ij}\partial_i\partial_j$. Using $\Box_\gamma = \epsilon\,\Box_0$, up to second order we find

$$\begin{aligned} \phi_{(2)} &= \epsilon^{-\frac{d-\Delta}{2}-1} \Big(\frac{\Box_\gamma \Phi(x,\epsilon)}{2(2\Delta - d - 2)} + \mathcal{O}(\epsilon) \Big) \\ \phi_{(0)} &= \epsilon^{-\frac{d-\Delta}{2}} \left[\Phi(x,\epsilon) - \frac{\Box_\gamma \Phi(x,\epsilon)}{2(2\Delta - d - 2)} + \mathcal{O}(\epsilon) \right] . \end{aligned} \tag{7.15}$$

These terms can be used to construct the counterterm action.

Subtracted action. We define a subtracted action

$$S_{\text{sub}}\big[\mathcal{F}(x,\epsilon), \epsilon\big] = S_{\text{reg}}[f_{(0)};\epsilon] + S_{\text{ct}}\big[\mathcal{F}(x,\epsilon);\epsilon\big] . \tag{7.16}$$

This has a finite limit as $\epsilon \to 0$.

Exact 1-point functions. The 1-point function of the operator $\mathcal{O}_\mathcal{F}$ is the presence of source is defined as

$$\langle \mathcal{O}_\mathcal{F} \rangle = \lim_{\epsilon\to 0} \left(\frac{1}{\epsilon^{d/2-m}\sqrt{\gamma}} \frac{\delta S_{\text{sub}}}{\delta \mathcal{F}(x,\epsilon)} \right) \equiv \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta f_{(0)}} \,. \tag{7.17}$$

We can schematically write it as in the last expression, but we should really take the variation first and the limit after.

The limit gives

$$\langle \mathcal{O}_\mathcal{F} \rangle \sim f_{(2\ell)} + C(f_{(0)}) \,. \tag{7.18}$$

The coefficient in front of $f_{(2\ell)}$ depends on the theory, but it is scheme independent (scheme = choice of couterterms). $C(f_{(0)})$ is a local function of $f_{(0)}$, so it gives contact terms. Its exact form is scheme dependent (and usually can be removed by suitable finite counterterms).

From here one can already check Ward identities.

For instance, applying the procedure to the metric one finds the 1-point function of the stress tensor:

$$\langle T_{ij} \rangle \sim g_{(d)ij} + C(g_{(0)ij}) \,. \tag{7.19}$$

As we discussed, $g_{(d)}$ is partially fixed by the EOMs: the divergence and the trace (with sources) are fixed. Setting the sources to zero, we get that T_{ij} is conserved and we reproduce the holographic Weyl anomaly (T^i_i is function of $g_{(0)}$).

***n*-point functions.** We need the exact (as opposed to asymptotic) solutions to the EOMs with prescribed boundary conditions. Regularity in the interior fixes $f_{(2\ell)}$ as a non-local function of $f_{(0)}$.

In general the exact solution cannot be found. But to compute an n-point function we only need a perturbative expression of $f_{(2\ell)}$ to order $n-1$ in $f_{(0)}$. This is done with Witten diagrams.

Finally

$$\begin{aligned} \Big\langle \mathcal{O}(x_1)\dots\mathcal{O}(x_n) \Big\rangle &= \frac{1}{\sqrt{g_{(0)}(x_1)\dots g_{(0)}(x_n)}} \frac{\delta^n S_{\text{ren}}}{\delta f_{(0)}(x_1)\dots\delta f_{(0)}(x_n)}\Big|_{f_{(0)}=0} \\ &= \frac{1}{\sqrt{g_{(0)}(x_2)\dots g_{(0)}(x_n)}} \frac{\delta \langle \mathcal{O}(x_1)\rangle}{\delta f_{(0)}(x_2)\dots\delta f_{(0)}(x_n)}\Big|_{f_{(0)}=0} \,. \end{aligned} \tag{7.20}$$

In other words, once we have the 1-point function with sources, we use that one to compute higher-point functions.

An explicit example of all these steps with a massive scalar in AdS is given in [Ske02].

7.2 The Weyl anomaly

We can apply the previous procedure to the metric and the stress tensor.

CFTs have a Weyl anomaly: conformal invariance is broken when the theory is coupled to an external metric:

$$\langle T^\mu_\mu \rangle \neq 0 \qquad \text{if} \qquad g_{\mu\nu} \neq 0 \; . \tag{7.21}$$

In odd dimension there is no Weyl anomaly. In even dimension one can prove that the only expression which is a local invariant of the metric, with the correct dimension, and that cannot be removed by local counterterms is

$$\langle T^\mu_\mu \rangle = \#\, E_{(d)} + \#\, I_{(d)} \tag{7.22}$$

where $E_{(d)}$ is the d-dimensional Euler density[42] and $I_{(d)}$ is a conformal (Weyl) invariant. The space of conformal invariants of dimension d increases with dimension.

In $d = 2$ there are no conformal invariants of the correct dimension, and we only have the Euler density. Indeed:

$$T^\mu_\mu = -\frac{c}{12}\mathcal{R} \tag{7.23}$$

where c is the central charge.

In $d = 4$ there is one conformal invariant (of correct dimension):

$$T^\mu_\mu = -a\, E_4 - c\, I_4 \tag{7.24}$$

with

$$\begin{aligned} E_4 &= \frac{1}{16\pi^2}\big(\mathcal{R}^2_{\mu\nu\rho\sigma} - 4\mathcal{R}^2_{\mu\nu} + \mathcal{R}^2\big) \\ I_4 &= -\frac{1}{16\pi^2}\Big(\mathcal{R}^2_{\mu\nu\rho\sigma} - 2\mathcal{R}^2_{\mu\nu} + \frac{1}{3}\mathcal{R}^2\Big) = -\frac{1}{16\pi^2}\mathcal{W}^2_{\mu\nu\rho\sigma} \end{aligned} \tag{7.25}$$

and $\mathcal{W}$ is the Weyl tensor (traceless part of the Riemann tensor). Then a, c are called *4d central charges*. They can be computed in free field theory: they are function of the number of vectors, spinors and scalars. However they receive quantum corrections and are very hard to determine at strong coupling.

Holographically we can compute them. If we take Einstein gravity

$$S = \int d^5x\, \sqrt{g}\,\big(\mathcal{R} + \Lambda\big) \; , \tag{7.26}$$

the computation is done in [HS98] (try to read it!). One does find the structure above, with

$$a = c \,\sim\, R^3 \,\sim\, \Lambda^{-3/2} \; . \tag{7.27}$$

Thus, theories with a weakly-coupled holographic dual have $a = c$ at leading order.

One can generate $a \neq c$ with higher-derivative corrections, but in order to stay within weakly-coupled gravity the deviation has to be small.

---

[42] The Euler density, up to normalization, is: $E_{(2n)} = \frac{1}{2^n}\mathcal{R}_{i_1 j_1 k_1 l_1} \dots \mathcal{R}_{i_n j_n k_n l_n}\epsilon^{i_1 j_1 \dots i_n j_n}\epsilon^{k_1 l_1 \dots k_n l_n}$. Its integral on a compact manifold gives the Euler characteristic of that manifold, which is a topological invariant. In fact $\delta E/\delta g$ is a total derivative.

For instance in $d = 2$, the Euler number is $\chi = \frac{1}{4\pi}\int d^2x\sqrt{g}\,\mathcal{R} = \int d^2x\sqrt{g}\,\frac{1}{8\pi}\mathcal{R}_{\mu\nu\rho\sigma}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}$.

SUSY. In SCFTs the R-symmetry current is in the same supermultiplet as the stress tensor (supercurrent multiplet):

$$\mathcal{S} : \left(T_{\mu\nu}, J_\mu^R, S_{\mu\alpha}, \widetilde{S}_{\mu\alpha}\right) . \tag{7.28}$$

The divergence of the R-symmetry current (anomaly) is in the same multiplet as the trace of the stress tensor:

$$\mathcal{A} : \left(T^\mu_\mu, \partial^\mu J_\mu^R, \gamma^\mu S_{\mu\alpha}, \gamma^\mu \widetilde{S}_{\mu\alpha}\right) . \tag{7.29}$$

Thus, in superconformal theories (in even dimension) there is a relation between R-symmetry 't Hooft anomalies and conformal anomalies.

For instance in 2d $\mathcal{N}=2$:

$$c = 3\,\mathrm{Tr}_{\text{fermions}}\,\gamma_3 R^2 , \qquad\qquad \partial^\mu J_\mu^R = \#\Big(\mathrm{Tr}\,\gamma_3 R^2\Big)\, F_{\mu\nu}\epsilon^{\mu\nu} . \tag{7.30}$$

The 't Hooft anomaly comes from a 1-loop diagram with two sides.
In 4d $\mathcal{N}=1$:

$$a = \frac{9}{32}\,\mathrm{Tr}\,R^3 - \frac{3}{32}\,\mathrm{Tr}\,R , \qquad\qquad c = \frac{9}{32}\,\mathrm{Tr}\,R^3 - \frac{5}{32}\,\mathrm{Tr}\,R \tag{7.31}$$

and

$$\partial^\mu J_\mu^R = \#\left(\mathrm{Tr}\,R^3\right) F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} + \#\left(\mathrm{Tr}\,R\right)\,\mathrm{Tr}\,\mathcal{R}_{\mu\nu}\mathcal{R}_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} . \tag{7.32}$$

Since 't Hooft anomalies are 1-loop exact, it follows that conformal anomalies can also be determined *exactly*. Even at strong coupling.

The conformal anomalies $a = c$ from IIB supergravity on $\mathrm{AdS}_5 \times S^5$ has been computed, and they match (at leading order N^2) the ones in field theory (computable at strong coupling).

8 Wilson loops

In gauge theories, a natural observable is the Wilson loop, defined for every closed contour $\mathcal{C}$ in spacetime and representation $\mathcal{R}$ of the gauge group:

$$W_{\mathcal{R}}(\mathcal{C}) = \mathrm{Tr}_{\mathcal{R}}\ \mathrm{P}e^{i\int_{\mathcal{C}} A^a_\mu T^a dx^\mu} \ . \tag{8.1}$$

T^a are the generators in representation $\mathcal{R}$.

Path-ordered exponential. For an Abelian gauge field, the open Wilson line (where the contour $\mathcal{C}$ is taken to be open) is simply constructed as

$$W_q = \exp\left\{iq\int_{\mathcal{C}} A_\mu dx^\mu\right\} = \exp\left\{iq\int_0^1 d\tau\, A_\tau(\tau)\right\} \ . \tag{8.2}$$

Here q is the charge parametrizing representations of $U(1)$, $\tau \in [0,1]$ is a coordinate along the path, $x^\mu(\tau)$ is the path and

$$A_\tau = A_\mu \frac{dx^\mu}{d\tau} \tag{8.3}$$

is the gauge field tangent to the path. Under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\lambda$ we have

$$e^{iq\int_0^1 A_\tau d\tau} \to e^{iq\int_0^1 A_\tau d\tau + iq\int_0^1 \partial_\tau\lambda\, d\tau} = e^{iq\lambda(1)}e^{-iq\lambda(0)}W_q \tag{8.4}$$

where we have integrated by parts. We see that, even though we made a gauge transformation $\lambda(x)$ at all points in spacetime, only the endpoints of the Wilson line transform, one as a particle of charge q and the other one as of charge $-q$.

The non-Abelian case is more complicated. Under a gauge transformation $U = e^{i\lambda}$ we have

$$A_\mu \to U(A_\mu + i\partial_\mu)U^{-1} \qquad \Rightarrow \qquad \delta A_\mu = D_\mu\lambda = \partial_\mu\lambda - i[A_\mu,\lambda] \ . \tag{8.5}$$

If we define the Wilson line as the exponential of the integral, then it will have a complicated gauge transformation (and it will not be usable to construct gauge invariants). Instead we should use the path-ordered exponential:

$$\begin{aligned} W = \mathrm{P}e^{i\int_0^1 d\tau\, A_\tau} &= \mathbb{1} + \sum_{n=1}^\infty \frac{i^n}{n!}\int_0^1 d\tau_1 \cdots \int_0^1 d\tau_n\, \mathrm{P}\big[A_\tau(\tau_1)\dots A_\tau(\tau_n)\big] \\ &= \mathbb{1} + \sum_{n=1}^\infty i^n \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n\, A_\tau(\tau_1)\dots A_\tau(\tau_n) \ . \end{aligned} \tag{8.6}$$

Here P orders the operators from the last one to the first one (the operator with smallest τ is the first one to act). We can then reduce to a fundamental domain (second line) and multiply by the number $n!$ of permutations.

Let us define as I_n the quantity in the summation: $\sum_n I_n$. We compute its (infinitesimal) gauge variation:

$$\delta I_n = i^n \sum_{j=1}^{n} \int_0^1 d\tau_1 \cdots \int_0^{\tau_{n-1}} d\tau_n \, A_\tau(\tau_1) \ldots \Big(\partial_\tau \lambda - i[A_\tau, \lambda]\Big)(\tau_j) \ldots A_\tau(\tau_n) \; . \tag{8.7}$$

This gives two types of terms. The commutator gives

$$\delta I_n^{[\,,\,]} = i^{n-1} \sum_{j=1}^{n} \int_0^1 d\tau_1 \cdots \int_0^{\tau_{n-1}} d\tau_n \, A_\tau(\tau_1) \ldots [A_\tau, \lambda](\tau_j) \ldots A_\tau(\tau_n) \; . \tag{8.8}$$

The derivative gives

$$\delta I_n^{\partial} = i^n \sum_{j=1}^{n} \int_0^1 d\tau_1 \, A_\tau(\tau_1) \cdots \int_0^{\tau_{j-1}} d\tau_j \, \partial_\tau \lambda(\tau_j) \int_0^{\tau_j} d\tau_{j+1} \, A_\tau(\tau_{j+1}) \cdots \int_0^{\tau_{n-1}} d\tau_n \, A_\tau(\tau_n) \; . \tag{8.9}$$

We perform the integral in $d\tau_j$ by parts. We get:

$$\begin{aligned}
\delta I_n^{\partial} = & -i^n \sum_{j=1}^{n-1} \int_0^1 d\tau_1 \, A_\tau(\tau_1) \cdots \int_0^{\tau_{j-1}} d\tau_j \, \lambda(\tau_j) \, A_\tau(\tau_j) \int_0^{\tau_j} d\tau_{j+2} \, A_\tau(\tau_{j+2}) \ldots \\
& + i^n \sum_{j=2}^{n} \int_0^1 d\tau_1 \, A_\tau(\tau_1) \cdots \int_0^{\tau_{j-2}} d\tau_{j-1} \, A_\tau(\tau_{j-1}) \, \lambda(\tau_{j-1}) \int_0^{\tau_{j-1}} d\tau_{j+1} \, A_\tau(\tau_{j+1}) \ldots \\
& + i^n \lambda(1) \int_0^1 d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n \, A(\tau_2) \ldots A(\tau_n) \\
& - i^n \int_0^1 d\tau_1 \cdots \int_0^{\tau_{n-2}} d\tau_{n-1} \, A_\tau(\tau_1) \ldots A_\tau(\tau_{n-1}) \, \lambda(0) \; .
\end{aligned} \tag{8.10}$$

The first line comes from moving the derivative. The second line comes from the boundary term evaluated at the upper bound τ_j for $j > 1$. The third line is the case $j = 1$. For $j < n$ there is no contribution from the lower bound 0 because an integration follows, however there is such a contribution for $j = n$ and this is the fourth line. We can put the first and second line into a commutator, and write

$$\delta I_n^{\partial} = -\delta I_{n-1}^{[\,,\,]} + i \, \lambda(1) \, I_{n-1} - i \, I_{n-1} \, \lambda(0) \; . \tag{8.11}$$

Summing all terms we have

$$\delta W = i \, \lambda(1) \, W - i \, W \, \lambda(0) \; . \tag{8.12}$$

Once again, the end of the line transforms as a point particle in representation $\mathcal{R}$ while the beginning transforms as a particle in $\overline{\mathcal{R}}$. If we take a closed loop, starting and ending at x_0, we have

$$\delta W = i \big[\lambda(x_0), W\big] \qquad \Rightarrow \qquad \delta \, \mathrm{Tr} \, W = 0 \; . \tag{8.13}$$

Taking the trace, we have constructed a gauge invariant.

Physical interpretation. The Wilson loop has this interpretation. We introduce external massive sources (quarks) in representation $\mathcal{R}$. The loop represents the propagation of a quark-antiquark pair along the loop $\mathcal{C}$, from creation to annihilation, and it measures the free energy. For a rectangular Wilson loop in Euclidean space with length L and height T, we have

$$W_{\mathcal{R}}(\mathcal{C}) \simeq e^{-T\,E_I(L)} \qquad \text{(for large } T, L) \tag{8.14}$$

where E_I is the interaction energy of the pair at distance L.

The Wilson loop is a signal for confinement, if it grows as the exponential of the area inside the loop. In a confining theory the quark-antiquark pair has a binding energy that grows linearly with distance,

$$E = m_q + m_{\bar{q}} + E_I(L) \,, \qquad E_I(L) \sim \tau L \,, \tag{8.15}$$

since the interaction is given by a confined flux tube of tension τ. Therefore

$$W_{\mathcal{R}}(\mathcal{C}) \sim e^{-\tau T L} \sim e^{-\tau\,A(\mathcal{C})} \,, \tag{8.16}$$

where $A(\mathcal{C})$ is the area of the worldsheet swept by the flux tube. In this picture, the Wilson loop of pure gauge theory captures the confinement of non-dynamical external massive quarks, entirely due to gauge dynamics.

Wilson loops in AdS/CFT. We can define an analogous quantity in AdS. If we have a line source on the boundary, we can attach a string. Depending on the specific AdS/CFT realization, we will have different boundary line operators. The natural action for the string is the Nambu-Goto action

$$S = \int d^2\sigma \sqrt{\hat{g}} \,, \tag{8.17}$$

so this gives the AdS definition of an observable

$$W(\mathcal{C}) = \text{minimal area surface with boundary } \mathcal{C} \,. \tag{8.18}$$

Since the metric diverges at the boundary $z = 0$,

$$ds^2 = \frac{dx_\mu^2 + dz^2}{z^2} \,, \tag{8.19}$$

it is energetically favorable for the string to bend inside AdS.

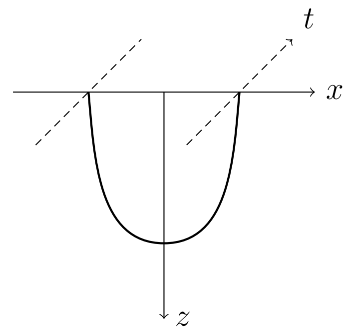
We can parametrize the string by (τ, σ), and the embedding by $X(\tau, \sigma)$. The (Euclidean) action is

$$S = \frac{1}{2\pi\alpha'} \int d\tau\, d\sigma \sqrt{\det_{ab} \partial_a X^M \partial_b X^N g_{MN}(X)} \,. \tag{8.20}$$

Let us consider a time-invariant configuration of two static sources at distance L. We have

$$t = \tau \,, \qquad x = \sigma \,, \qquad z = z(\sigma) = z(x) \,. \tag{8.21}$$

The shape is



Then the action is

$$S = \frac{T}{2\pi\alpha'} \int_{-L/2}^{L/2} dx \, \frac{\sqrt{1+(z')^2}}{z^2} \; . \tag{8.22}$$

Since the action does not depend explicitly on x, there is a conserved quantity:

$$\frac{\delta \mathcal{L}}{\delta z'} z' - \mathcal{L} = -\frac{1}{z^2\sqrt{1+(z')^2}} = \text{const} \; . \tag{8.23}$$

We can evaluate the constant at the turning point, where $x = 0$ (by symmetry), $z = z(0)$ and $z' = 0$. The constant is $-1/z(0)^2$. We thus find a differential equation

$$z' = \frac{dz}{dx} = -\sqrt{\frac{z(0)^4}{z^4} - 1} \; . \tag{8.24}$$

This can be solved (we make a change of variables $z(0)/z = y$):

$$x = \int_{z(x)}^{z(0)} \frac{dz}{\sqrt{\frac{z(0)^4}{z^4} - 1}} = z(0) \int_1^{z(0)/z(x)} \frac{dy}{y^2\sqrt{y^4-1}} \; . \tag{8.25}$$

If we go to the boundary, where $x = L/2$ and $z = 0$, we find a relation between L and the turning point:[43]

$$\frac{L}{2} = z(0) \int_1^{\infty} \frac{dy}{y^2\sqrt{y^4-1}} = \# \, z(0) \; . \tag{8.26}$$

The on-shell action is

$$S = \frac{2T}{2\pi\alpha' \, z(0)} \int_1^{\infty} \frac{y^2 dy}{\sqrt{y^4-1}} \; . \tag{8.27}$$

The integral is linearly divergent. This is because the energy includes the two infinitely massive quarks, $E = m_q + m_{\bar{q}} + E_I$. Their bare contribution is given by two straight lines at $x = \pm L/2$, which gives a linear divergence. Removing the divergence:

$$S_{\text{reg}} = \frac{2T}{2\pi\alpha' \, z(0)} \int_1^{\infty} \Big(\frac{y^2}{\sqrt{y^4-1}} - 1 \Big) dy = \# \, \frac{T}{\alpha' \, L} \; . \tag{8.28}$$

[43] The constant is $\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right) / \, \Gamma\left(\frac{1}{4}\right)$. The constant in (8.28) is $\left[4\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right) + \pi\Gamma\left(-\frac{1}{4}\right)\right] / \left[\sqrt{2}\,\Gamma\left(\frac{1}{4}\right)^3\right]$.

- The result is consistent with conformal invariance: the potential energy must scale as $\frac{1}{L}$.

  We will discuss the different behavior in confining models.

- The further apart are the quarks, the more the string penetrates in the interior of AdS. The turning point of the string is $\frac{1}{z(0)} \sim \frac{1}{L}$.

  This is consistent with the interpretation that

  $$\frac{1}{z} \sim E_{\text{process}} \tag{8.29}$$

  is an energy scale, *i.e.* processes at energy scale E take place at $\frac{1}{z} \sim E$ in AdS.

9 Non-conformal theories and confinement

AdS/CFT applies to non-conformal theories as well, and thus it can describe phenomena like confinement, mass gap, discrete spectra, symmetry breaking, RG flows, etc... This makes it extremely interesting.

We already know one way to obtain non-conformal theories. We know how to turn on sources in the CFT, and if the source is for a *relevant* operator, this will induce an RG flow to some other fixed point.

If the CFT has a moduli space (usually only for supersymmetric CFTs), then another way is to turn on a VEV for some operator without sources.

In the description of non-conformal theories, the metric has Poincaré invariance but it is not AdS:

$$ds^2 = e^{2A(z)}\big(dz^2 + dx_\mu dx^\mu\big) \tag{9.1}$$

where $e^{2A(z)}$ is called the *warp factor*. If at the boundary ($z \to 0$) the metric is asymptotically AdS, namely

$$e^{2A(z)} \;\to\; \frac{1}{z^2} \qquad \text{as } z \to 0\;, \tag{9.2}$$

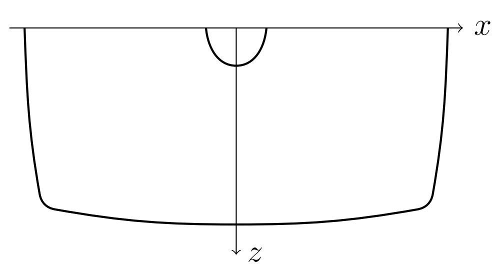
and it is everywhere regular, then we can apply the rules of AdS/CFT. The bulk fields are still associated to primary operators with definite dimension in the far UV, but not in the complete theory at finite energy scales. They are *bound states* of the gauge theory (such as mesons and glueballs). Heavier objects, like baryons, usually appear as solitonic objects like wrapped branes.

9.1 Confining theories

Let us describe the qualitative features of the gravitational dual to a confining theory.

The warp factor $e^{2A(z)}$ is bounded above zero.

This follows from the Wilson loop. When the quarks are close to each other, the suspended string is in the asymptotically AdS region and $E_I \sim 1/L$.



When the quarks are further apart, the string is essentially the sum of two vertical pieces (the masses m_q, $m_{\bar{q}}$ to be subtracted) and an horizontal piece around z_0 where $e^{2A(z_0)}$ has a minimum.

$$\text{For large } L: \qquad E(L) \simeq m_q + m_{\bar{q}} + \tau e^{2A(z_0)} L \tag{9.3}$$

where τ is the tension of the string $\quad\Rightarrow\quad$ linear confinement.

In a theory with mass gap and discrete spectrum we expect poles in the two-point functions, corresponding to physical states:

$$\langle \mathcal{O}_\phi(k)\, \mathcal{O}_\phi(-k) \rangle = \sum_i \frac{A_i}{k^2 + M_i^2} \;. \tag{9.4}$$

We are in Euclidean signature, and the poles are at $k^2 = -M_i^2$.

Let us consider a minimally-coupled scalar. At quadratic order the EOM is

$$\partial_z \big(e^{(d-1)A(z)} \partial_z \phi \big) - e^{(d-1)A(z)} k^2 \phi = e^{(d+1)A(z)} m^2 \phi \;. \tag{9.5}$$

At small z the metric is asymptotically AdS, therefore

$$z \to 0: \qquad \phi = z^{d-\Delta} \Big(A(k) + O(z) \Big) + z^{\Delta} \Big(B(k) + O(z) \Big) \tag{9.6}$$

with

$$R^2 m^2 = \Delta(\Delta - d) \;. \tag{9.7}$$

In standard quantization, A is the source and B is the VEV. In particular B is always normalizable. The corrections $O(z)$ are fixed by the EOMs, while regularity in the interior fixes B as a (non-local) function of A. By normalizing the field with the value of the source:

$$\phi_k(z) = \phi_k^0 \bigg[z^{d-\Delta} \big(1 + O(z) \big) + \frac{B(k)}{A(k)} z^{\Delta} \big(1 + O(z) \big) \bigg] \;. \tag{9.8}$$

(Here k is boundary momentum.) The two-point function has poles if and only if $A(k) = 0$:

if there are normalizable *and* regular solutions to the EOMs (*quasi-normal modes*).

There cannot be solutions for[44] $k^2 \geq 0$, but there can be for $k^2 < 0$. They correspond to the bound-states $M_i^2 = -k^2$.

We have reduced the problem to that of finding normalizable regular solutions to the EOMs. Let us redefine

$$\phi(x_\mu, z) = e^{-\frac{d-1}{2} A(z)}\, \psi(z)\, e^{i k_\mu x^\mu} \qquad \text{with} \qquad k^2 = -M^2 \;. \tag{9.9}$$

[44] Because for $k^2 \geq 0$ the Euclidean on-shell action is positive definite, while for a normalizable regular solution the on-shell action, which reduces to a boundary term, vanishes.

Then the radial equation becomes a Schrödinger-like equation:

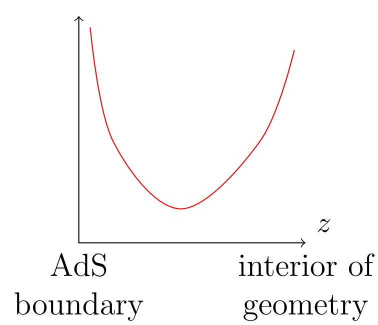
$$-\psi'' + \left(\frac{d-1}{2}A'' + \frac{(d-1)^2}{4}(A')^2 + m^2 e^{2A}\right)\psi = E\,\psi \qquad\qquad E = M^2 > 0\ . \tag{9.10}$$

We should find positive-energy solutions in a potential. The boundary conditions follow from the ones of the original problem. Normalizability

$$\int \sqrt{g}|\phi|^2 = \int e^{2A}|\psi|^2 \overset{z\to 0}{\sim} \int \frac{|\psi|^2}{z^2} \tag{9.11}$$

implies that $\psi \to 0$ at the boundary. The conditions at $z \to \infty$ are imposed by regularity.

- For $\mathrm{AdS}_{d=1}$, the potential[45] $\propto \frac{1}{z^2}$. One gets a continuous spectrum of scattering states (non-normalizable modes) starting from zero, appropriate for a conformal theory.
- The typical confining solution generates a potential that is constant or diverges for $z \to +\infty$, such as



This gives a discrete spectrum M_i^2 of glueballs above zero.

9.2 RG flows

Let us study some simple and universal properties of RG flows seen from the point of view of AdS_5. (One could repeat the discussion for AdS_{d+1}.)

Start with a local 5-dimensional gravitational theory

$$S_5 = \int d^5x\,\sqrt{-g}\left[\frac{\mathcal{R}}{4} - \frac{1}{2}G_{ab}\partial_M\varphi_a\partial^M\varphi_b - V(\varphi)\right] \tag{9.12}$$

where G_{ab} is a matrix. We consider a simple model with scalar fields. We study 4d Poincaré invariant solutions. We write the metric as

$$ds^2 = dy^2 + e^{2Y(y)}dx_\mu dx^\mu\ , \tag{9.13}$$

[45] The exact expression is $V_{\text{eff}}(z) = \left(m^2 + \frac{d^2-1}{4}\right)\frac{1}{z^2}$.

and everything is a function of y only. We recover AdS$_5$ for $Y(y) = y/R$. Then redefine $e^{y/R} = R/z$:

$$ds^2 = dy^2 + e^{2y/R} dx_\mu dx^\mu = R^2 \frac{dz^2 + dx_\mu dx^\mu}{z^2} . \tag{9.14}$$

Here R is the radius of AdS.

Exercise. Einstein's equations and the scalar EOM reduce to the following two equations:

$$3(Y')^2 - \frac{1}{2} G_{ab} \varphi'_a \varphi'_b + V = 0 , \qquad G_{ab} \varphi''_b + 4Y' G_{ab} \varphi'_b = \frac{\partial V}{\partial \varphi_a} . \tag{9.15}$$

Although not needed, one can check that they are reproduced by the effective Lagrangian

$$\mathcal{L} = e^{4Y} \left[3(Y')^2 - \frac{1}{2} G_{ab} \varphi'_a \varphi'_b - V(\varphi) \right] , \tag{9.16}$$

supplemented by the zero-energy constraint

$$0 = 3(Y')^2 - \frac{1}{2} G_{ab} \varphi'_a \varphi'_b + V . \tag{9.17}$$

An obvious solution to the two EOMs above with boundary is obtained at the critical points of the potential $V(\varphi)$, if the potential at the critical point is negative. We set all scalar fields to be constant:

$$\frac{\partial V}{\partial \varphi_a} = 0 , \qquad \varphi'_a = 0 , \qquad (Y')^2 = -\frac{V_{\text{crit}}}{3} . \tag{9.18}$$

Up to a redefinition $y \to -y$, we can take Y' positive such that the boundary is at $y = +\infty$. Then

$$Y(y) = \frac{y}{R} , \qquad \text{with} \qquad \frac{1}{R^2} = -\frac{V_{\text{crit}}}{3} . \tag{9.19}$$

We get AdS$_5$ solutions, where the radius R is controlled by the negative critical points of the potential. From our general discussion, we expect each of them to describe a (different) CFT at the boundary.

We can construct more interesting solutions.

Start from a critical point of $V(\varphi)$, say at $\varphi_a = 0$. The AdS$_5$ vacuum is dual to a CFT. Expand the action at quadratic order around $\varphi_a = 0$, then read off the masses m_a, from which we determine the dimensions via

$$R^2 m_a^2 = \Delta_a (\Delta_a - 4) \tag{9.20}$$

of the dual operators $\mathcal{O}_a$.

Now we can look for more general solutions with the same asymptotics:

$$Y(y) \to \frac{y}{R}\;, \qquad \varphi_a(y) \to 0 \qquad \text{as} \qquad y \to +\infty\;. \tag{9.21}$$

Since asymptotically the background is AdS_5, the scalars behave as

$$\varphi_a(y) \simeq A_a\, e^{(\Delta_a - 4)y} + B_a\, e^{-\Delta_a y}\;. \tag{9.22}$$

We associate

$$A_a \;\leftrightarrow\; \text{source for } \mathcal{O}_a\;, \qquad B_a \;\leftrightarrow\; \text{VEV for } \mathcal{O}_a\;. \tag{9.23}$$

The new solutions have the following interpretation:

- General solutions with $A_a \neq 0$ describe a deformation of the CFT by $\mathcal{O}_a$:

$$\mathcal{L}_{\text{CFT}} \to \mathcal{L}_{\text{CFT}} + \int d^4x\, A_a \mathcal{O}_a\;. \tag{9.24}$$

- Special solutions with $A_a = 0$ describe a different vacuum of the CFT, where

$$\langle \mathcal{O}_a \rangle \;\sim\; B_a\;. \tag{9.25}$$

In both cases, the CFT is deformed and conformal invariance is broken: this triggers a Renormalization Group (RG) flow, and the gravitational solution is dual to such flow. The fifth coordinate y is roughly identified with an energy scale in the RG flow, and the profile of $\varphi_a(y)$ and $Y(y)$ is the running of the deformation parameters:

$$\text{UV}: \quad y \to +\infty\;, \qquad \text{IR}: \quad y \to -\infty \quad \text{or it can stop before}\;. \tag{9.26}$$

In general the solutions are singular in the IR.

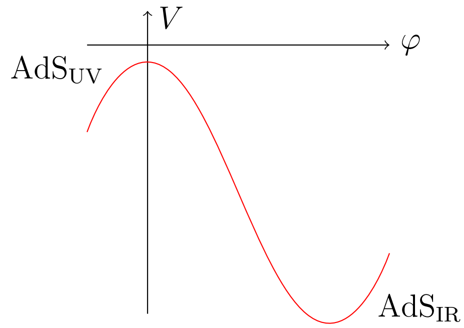
Particularly interesting is the case that the CFT is perturbed by a *relevant* operator $\mathcal{O}_a$, and the RG flow leads to another fixed point. In our supergravity description this happens when $V(\varphi)$ has another critical point. The gravity description is a kink solution that interpolates between the two critical points. The asymptotic is:

$$\begin{aligned} &Y \to \frac{y}{R_{\text{UV}}}\;, \qquad \varphi_a \to 0 \qquad &&\text{for} \qquad y \to +\infty \\ &Y \to \frac{y}{R_{\text{IR}}}\;, \qquad \varphi_a \to \varphi_a^{\text{IR}} \qquad &&\text{for} \qquad y \to -\infty\;. \end{aligned} \tag{9.27}$$

Recall that

$$\begin{aligned} &\text{Irrelevant operators } (\Delta > 4): \qquad && R^2m^2 > 0 \\ &\text{Marginal operators } (\Delta = 4): && R^2m^2 = 0 \\ &\text{Relevant operators } (\Delta < 4): && -4 \leq R^2m^2 < 0\;. \end{aligned} \tag{9.28}$$

To have an RG flow that starts from $\mathrm{CFT}_{\mathrm{UV}}$ we need a *relevant* operator: its mass must be negative, thus $V(0)$ is a *maximum*. Yet there is no tachyon, because of the BF bound![46] To hit $\mathrm{CFT}_{\mathrm{IR}}$, the deformation must be irrelevant there and $V(\varphi_{\mathrm{IR}})$ must be a minimum:



The search for gravity solutions usually simplifies if *supersymmetry* is present. If the potential V can be written in terms of a superpotential W as

$$V = \frac{1}{8}(G^{-1})^{ab}\frac{\partial W}{\partial \varphi_a}\frac{\partial W}{\partial \varphi_b} - \frac{1}{3}W^2 \,, \tag{9.29}$$

then the 2^{nd} order EOMs can be reduced to 1^{st} order BPS equations.

Exercise. Check that the solutions to

$$\varphi_a' = \frac{1}{2}(G^{-1})^{ab}\frac{\partial W}{\partial \varphi_b} \,, \qquad\qquad Y' = -\frac{1}{3}W \tag{9.30}$$

also solve the EOMs.

Indeed various gauged-supergravity supersymmetric solutions follow from a superpotential (although not all).

The existence of a holographic dual to an RG flow has striking consequences for CFTs at strong coupling. For instance, one can easily prove a c-theorem for 4d theories with an AdS dual.

There are 2 central charges a, c in the 4d superconformal algebra, that control the conformal anomalies. It was conjectured by Cardy [Car88] and later proven by Komargodski-Schwimmer [KS11] that a is always decreasing along (unitary) RG flows from one CFT to another.[47] This has remained as a conjecture for a long time, but it is easy to prove holographically.

[46] The vacuum cannot spontaneously decay. It does decay only after the boundary conditions are modified such that the non-normalizable mode corresponding to the relevant deformation is turned on.

[47] On the other hand, one can show in counter-examples that any other combination of a and c does not satisfy such a theorem [Car88]. KS have also constructed a function along the flow which is monotonic and agrees with a at the end-points. This is usually called a c-function. However, as opposed to Zamolodchikov's c-function in 2d, the one of KS in 4d is not such that the RG flow is its gradient flow.

In AdS: $a = c \sim R^3 \sim \Lambda^{-3/2}$. We can construct a monotonically decreasing c-function

$$c(y) = \#(Y')^{-3} , \tag{9.31}$$

(here # is positive) which reproduces the central charges at fixed points. Its derivative is

$$c'(y) = -3\,\#\,Y''(Y')^{-4} , \tag{9.32}$$

so the sign is fixed by Y''. From the EOMs one gets

$$Y'' = -\frac{2}{3} G_{ab} \varphi'_a \varphi'_b . \tag{9.33}$$

Since G_{ab} appears in the kinetic term, such a matrix should be positive definite in a consistent theory, in order to avoid ghosts. The holographic c-theorem follows.

More generally, without resorting to a specific Lagrangian, the EOMs of 5d gravity coupled to matter are

$$\frac{1}{4}\Big(\mathcal{R}_{MN} - \frac{1}{2} g_{MN} \mathcal{R} \Big) = \frac{1}{2} T_{MN} , \qquad\qquad T_{MN} \equiv -\frac{2}{\sqrt{g}} \frac{\delta \sqrt{g}\, \mathcal{L}}{\delta g^{MN}} \tag{9.34}$$

where T_{MN} is the 5d stress tensor. Evaluated on the general 4d-Poincaré invariant metric they imply (check all this!)

$$\frac{3}{2} Y'' = T^0_0 - T^y_y . \tag{9.35}$$

One of the weakest of the classical energy conditions is the "null energy condition":

$$\xi^M \xi^N T_{MN} \geq 0 \qquad\qquad \forall\, \xi \text{ null vector} \tag{9.36}$$

and it is expected to classically hold in all physically-relevant gravity solutions (for a fluid, it means $p + \rho \geq 0$).[48] On 4d-Poincaré invariant solutions it implies

$$-T^0_0 + T^y_y \geq 0 , \tag{9.37}$$

and then the c-theorem follows.

[48] In the context of quantum theories, it is easy to construct counter-examples even in flat space. The condition that has been proven to be satisfied by all CFTs in flat space is the "averaged null energy condition" [FLPW16, HKT17], which is integrated along a null direction. Classically, the ANEC is necessary to avoid time machines and violations of the second law of thermodynamics.

10 Theories at finite temperature

A simple way to obtain non-conformal theories is to turn on a temperature. This also breaks supersymmetry. Let us see how theories at finite temperature appear in AdS/CFT.

10.1 From 4d $\mathcal{N}=4$ SYM to 3d YM

Let us study 4d $\mathcal{N}=4$ SYM with gauge group $SU(N)$ at finite temperature T. This can be obtained by first going to Euclidean signature, and then compactifying the Euclidean time to a circle of length $\beta = \frac{1}{T}$.

The reason is the following. The path-integral on a "strip" of length β with fixed boundary conditions $\Phi_{i,f}$ computes the propagation from an initial state $|\Phi_i\rangle$ to a final state $|\Phi_f\rangle$:

$$\int_{\Phi_i}^{\Phi_f} \mathcal{D}\varphi\, e^{-S} = \langle \Phi_f | e^{-\beta H} | \Phi_i \rangle \tag{10.1}$$

because H is the generator of time translations. The path-integral on a circle is obtained by identifying $\Phi_i = \Phi_f$ and integrating over them. This produces a trace:

$$\int_{\text{periodic on } S^1} \mathcal{D}\varphi \quad e^{-S} = \operatorname{Tr}\, e^{-\beta H} \ . \tag{10.2}$$

If we insert operators (and compute correlators) on the cylinder, we compute $\operatorname{Tr} e^{-\beta H}\mathcal{O}$. But $e^{-\beta H}$ (up to normalization) is the density matrix of a thermal state, and so we are computing matrix elements of $\mathcal{O}$ in a thermal state.

The fermions have anti-periodic boundary conditions along S^1:[49]

$$\psi(y) = -\psi(y+\beta)\ , \qquad \psi = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k\, e^{2\pi i k y/\beta} \qquad \Rightarrow \qquad m^2_{\psi_k} = 4\pi^2 \frac{k^2}{\beta^2} > 0\ . \tag{10.3}$$

Thus all fermionic modes get a mass at tree level. Conformal invariance is broken by the compactification (the temperature set a scale, and it also breaks Lorentz) and supersymmetry is broken by the boundary conditions. Then the scalars get a mass at one-loop (loops below the fermionic mass are no longer canceled).

For $\beta \to 0$ all fermionic and scalar modes get a mass and decouple: one is left with pure YM in three dimensions. From

$$\frac{1}{g_4^2}\int d^4x\, F_{\mu\nu}^2 = \frac{\beta}{g_4^2}\int d^3x\, F_{\mu\nu}^2 \qquad \Rightarrow \qquad \frac{1}{g_3^2} = \frac{\beta}{g_4^2}\ . \tag{10.4}$$

To have a smooth IR physics, we should send

$$\beta \to 0\ , \qquad g_4 \to 0 \qquad \text{with} \qquad g_3 \text{ fixed}\ . \tag{10.5}$$

[49] That is because non-vanishing correlators have an even number of fermions. If we take one and bring it all along the circle, it crosses an odd number of fermions and so it changes sign.

In this limit we obtain a non-supersymmetric and non-conformal YM theory in 3d: it confines, it has a mass gap and a discrete spectrum of massive glueballs.

The model can be studied using a weakly-coupled gravity dual. The starting point is a system of N Euclidean D3-branes on $\mathbb{R}^3 \times S^1$. In the Lorentzian version they form a *black three-brane*: a sort of Schwarzschild black hole that extends along $\mathbb{R}^3$ and is charged under F_5. The near-horizon metric is

$$ds^2 = R^2 \left\{ \frac{dz^2}{z^2 \Big(1 - \frac{z^4}{z_0^4} \Big)} + \frac{1}{z^2} \left[\Big(1 - \frac{z^4}{z_0^4} \Big) dt^2 + dx_{1,2,3}^2 \right] + d\Omega_5^2 \right\} \tag{10.6}$$

with $R^4 = 4\pi g_s N \alpha'^2$ (there is also a flux for F_5 that we do not write). Lorentz invariance along the boundary directions is broken by temperature. At the boundary we still have $\mathrm{AdS}_5 \times S^5$ asymptotically. But at $z = z_0$ the Lorentzian metric has a true horizon. The Euclidean metric is simply capped at $z = z_0$.

We can expand around $z = z_0$:

$$z = z_0 (1 - \rho^2) \; . \tag{10.7}$$

Then expanding around $\rho = 0$:

$$ds^2 \simeq R^2 \left\{ d\rho^2 + \frac{4\rho^2}{z_0^2} dt^2 + \frac{1}{z_0^2} dx_{1,2,3}^2 + d\Omega_5^2 \right\} \; . \tag{10.8}$$

The metric looks like $\mathbb{R}^2$ $(\times \mathbb{R}^3 \times S^5)$ in polar coordinates around $\rho = 0$, but it has a conical singularity unless t is a periodic variable on a circle of length

$$\beta = \pi z_0 \qquad\qquad \Rightarrow \qquad T = \frac{1}{\pi z_0 R} \; . \tag{10.9}$$

This is a way to compute the temperature of the black hole (or a black brane as here).

This space is simply connected, thus there is a unique possible spin structure, in which the fermions are antiperiodic on S^1. A translation of β along S^1 is a standard 2π rotation in $\mathbb{R}^2$ around $\rho = 0$, and the spin-$\frac{1}{2}$ wavefunction changes sign under such a rotation.

We see that the black three-brane solution has all correct features to be the gravitational dual to 4d $\mathcal{N} = 4$ SYM on $\mathbb{R}^3 \times S^1$. Let us see what we can infer.

Confinement. The warp factor attains a minimum at the horizon (for strings along $\mathbb{R}^3$):

$$\text{At } z = z_0: \qquad\qquad e^{2A(z_0)} = \frac{1}{z_0^2} \; . \tag{10.10}$$

Then the theory has stable strings with tension

$$\tau = e^{2A(z_0)} / 2\pi\alpha' = \frac{\pi}{2} \sqrt{4\pi g_4^2 N} \; T^2 \; . \tag{10.11}$$

The fact that it goes like $\lambda^{1/2}$ is a hallmark of strong coupling.

Glueball spectrum. The masses of bound states can be extracted from the behavior of correlation functions of gauge-invariant operators at large distances:[50]

$$\langle \mathcal{O}(\vec{x})\, \mathcal{O}(0) \rangle \sim \sum_i A_i \, \frac{e^{-M_i x}}{x^{(d-1)/2}} \;. \tag{10.13}$$

By a Fourier transform, in momentum space this is

$$\langle \mathcal{O}(\vec{k})\, \mathcal{O}(-\vec{k}) \rangle \sim \sum_i \frac{A_i}{k^2 + M_i^2} \;. \tag{10.14}$$

Thus we can extract M_i from the normalizable solutions to the EOMs.

If we consider zero-modes on S^5, we can decompose the field into modes with fixed momentum $\vec{k}$ along $\mathbb{R}^3$ and n along S^1:

$$\phi(z,t,x) = \phi(z)\, e^{int/\beta}\, e^{i\vec{k}\cdot\vec{x}} \;. \tag{10.15}$$

For simplicity, let us further restrict to zero-modes on S^1 as well: $n = 0$. Then, factoring out the volumes of S^5, $\mathbb{R}^3$ and S^1, the action

$$S = \int dz\, dt\, d^3x\, d\Omega_5 \, \sqrt{g} \Big(\partial_M \phi \partial^M \phi + m^2 \phi^2 \Big) \tag{10.16}$$

becomes

$$S \sim \int_0^{z_0} \frac{dz}{z^5} \bigg[z^2 \Big(1 - \frac{z^4}{z_0^4} \Big) (\partial_z \phi)^2 + k^2 z^2 \phi^2 + m^2 \phi^2 \bigg] \;. \tag{10.17}$$

The EOM from this action is

$$-\partial_z \bigg(\frac{1}{z^3} \Big(1 - \frac{z^4}{z_0^4} \Big) \partial_z \phi \bigg) + \frac{k^2}{z^3} \phi + \frac{m^2}{z^5} \phi = 0 \;. \tag{10.18}$$

To bring it to the form of a Schrödinger equation, first we change radial coordinate to

$$\tau(z) = \int_0^z \frac{dy}{\sqrt{1 - \frac{y^4}{z_0^4}}} \qquad \to \qquad d\tau = \Big(1 - \frac{z^4}{z_0^4} \Big)^{-\frac{1}{2}} dz \;. \tag{10.19}$$

Then we redefine the field as

$$\phi = e^{-A/2} \psi \qquad\qquad \text{with} \qquad e^{A(\tau)} = \frac{1}{z^3} \sqrt{1 - \frac{z^4}{z_0^4}} \;. \tag{10.20}$$

This gives the equation

$$-\psi'' + \bigg(\frac{A''}{2} + \Big(\frac{A'}{2} \Big)^2 + \frac{m^2}{z^2} \bigg) \psi = E\, \psi \qquad\qquad E = -k^2 = M^2 \tag{10.21}$$

where everything is a function of τ.

[50] The Green's function of $-\Box\, G(\vec{x}) = \delta^d(\vec{x})$ is proportional to $1/x^{d-2}$. Instead the Green's function of

$$\big(-\Box + M^2 \big) G(\vec{x}) = \delta^d(\vec{x}) \qquad \text{is proportional to} \qquad \frac{1}{x^{(d-2)/2}}\, K_{\frac{d-2}{2}}(kx) \sim e^{-Mx}\, \frac{1}{x^{(d-1)/2}} \;. \tag{10.12}$$

Exercise. Study qualitatively the spectrum (for $n = 0$).
One finds

$$\frac{A''}{2} + \left(\frac{A'}{2}\right)^2 = \frac{15 - 18z^4/z_0^4 - z^8/z_0^8}{4z^2\big(1 - z^4/z_0^4\big)} \,.$$

10.2 Thermal phase transitions

The holographic prescription at large N and large (but fixed) $g_{YM}^2 N$ involves extremizing the classical gravity action subject to asymptotic boundary conditions. This is the saddle-point approximation to the path-integral of gravity.[51]

There can be more than one saddle point: this is a general feature of boundary value problems in differential equations. In this case we are supposed to sum

$$e^{-S_{\text{gravity}}}$$

over the various classical configurations. The solution that globally minimizes S_{gravity} dominates the saddle-point approximation. When there are two or more competing solutions, *e.g.*

$$Z = e^{-S_1} + e^{-S_2} + \dots \,, \tag{10.22}$$

there can be *phase transitions*.

Let us study an example in AdS_5. We take the standard action

$$S = -\frac{1}{16\pi G_{\text{N}}} \int d^5x \,\sqrt{g}\,\Big(\mathcal{R} + \frac{12}{R^2}\Big) \,. \tag{10.23}$$

One can embed the (Euclidean) Schwarzschild black hole in AdS_5:

$$ds^2 = f\, dt^2 + \frac{1}{f}\, dr^2 + r^2 d\Omega_3^2 \,, \qquad\qquad f = 1 + \frac{r^2}{R^2} - \frac{\mu}{r^2} \,. \tag{10.24}$$

Assuming $\mu > 0$, it follows that f has a positive root $r^2 = r_+^2$ and a negative root $r^2 = -r_-^2$ (with $r_-^2 - r_+^2 = R^2 > 0$):

$$f = \frac{(r^2 - r_+^2)(r^2 + r_-^2)}{r^2 R^2} \,, \tag{10.25}$$

then the Euclidean solution is defined for $r \geq r_+$. By the same argument as before,[52] the geometry is smooth at $r = r_+$ iff

$$t \cong t + \beta \qquad \text{with} \qquad \beta = \frac{1}{T} = \frac{2\pi R^2 r_+}{2r_+^2 + R^2} \,. \tag{10.26}$$

[51] The path-integral of gravity is ill-defined because gravity is non-renormalizable. However string amplitudes render on-shell quantities well-defined.

[52] Define $r - r_+ = \rho^2$, and use $r_+^2 + r_-^2 = 2r_+^2 + R^2$.

We can think of this as fixing $\mu = r_+^2 r_-^2 = r_+^2(R^2 + r_+^2) > 0$ in terms of the temperature T.

Let us call this space

$$X_2 = \text{Euclidean Schwarzschild BH in AdS}_5 \,.$$

This space has topology $D_2 \times S^3$, with boundary $S^1 \times S^3$. The latter is the relevant space for a QFT on S^3 at finite temperature.

This space is simply connected, and so it has a unique spin structure in which the fermions are anti-periodic in t.

There is another (Euclidean) solution with the same boundary conditions:

$$ds^2 = f\,dt^2 + \frac{1}{f}\,dr^2 + r^2 d\Omega_3^2 \,, \qquad\qquad f = 1 + \frac{r^2}{R^2} \,. \tag{10.27}$$

(The same as before, but with $\mu = 0$). In fact this is global AdS_5,[53] but with compactification of the Euclidean time:

$$t \cong t + \beta \tag{10.28}$$

with any β. This space is called *thermal AdS*. We call it

$$X_1 = \text{Thermal AdS} \,.$$

This space has a topology completely different from the previous one: $S^1 \times D_4$, but with same boundary $S^1 \times S^3$.

This space is not simply connected, so it admits two spin structures: periodic or anti-periodic fermions along S^1. With anti-periodic boundary conditions it describes AdS at finite temperature, *i.e.* a thermal gas of gravitons in AdS. With periodic boundary conditions, instead, supersymmetry remains unbroken and the path-integral computes

$$\operatorname{Tr}(-1)^F\, e^{-\beta H} \ldots$$

which is called a Witten index.

If we impose periodic (supersymmetric) conditions on S^1 at the boundary, *i.e.* we compute $\operatorname{Tr}(-1)^F e^{-\beta H}$ in the QFT, only the space X_1 is relevant (has the correct boundary conditions). But if we impose anti-periodic (thermal) conditions, *i.e.* we study the QFT on

[53] With the coordinate change $r = R\tan\theta$ and $t = R\tau$, one gets

$$ds^2 = \frac{R^2}{\cos^2\theta}\big(d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega_3^2\big) \,.$$

On the other hand, with the coordinate change $r = R\sinh\rho$ and $t = R\tau$, one gets

$$ds^2 = R^2\big(\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2\big) \,.$$

S^3 at finite temperature, both X_1 and X_2 are relevant.[54] Both are saddle-point contributions to the path-integral, and the dominant one is the one with smaller on-shell action.

This problem has been studied by Hawking and Page [HP83] first, as a question in 4d quantum gravity, and later adapted by Witten [Wit98] to AdS/CFT.

One computes the on-shell actions. Both $S[X_1]$ and $S[X_2]$ are divergent and should be regularized, however $S[X_2]-S[X_1]$ is finite (one sets a cutoff $r \leq R_0$ and then takes $R_0 \to \infty$). One imposes that X_1 has the same S^1 radius as X_2. One finds

$$S[X_2] - S[X_1] = \frac{\pi^2 r_+^3 (R^2 - r_+^2)}{4G_N(2r_+^2 + R^2)} \tag{10.29}$$

This expression changes sign at $r_+ = R$, therefore there is a phase transition between thermal AdS_5 and a large black hole.

- $r_+ > R$, *i.e.* $T > \dfrac{3}{2\pi R}$: large black hole, X_2 Confinement
- $r_+ < R$, *i.e.* $T < \dfrac{3}{2\pi R}$: thermal AdS_5, X_1 Deconfinement

On the boundary, this phase transition is interpreted as a *confinement/de-confinement* transition, from the behavior of the Wilson loop. If T is roughly smaller than the scale $\frac{1}{R}$ set by the sphere, we don't see T and the physics is that of a conformal theory. If T is larger than $\frac{1}{R}$, we do see the temperature, scalars and fermions get massive and we are left with 3d YM on S^3 which confines.

[54] The BH solution has a minimal temperature $T_{\min} = \sqrt{2}/\pi R$ (attained at $2r_+^2 = R^2$), and for all larger values of the temperature there are two solutions: the small BH and the large BH. Thus for $T < T_{\min}$ only X_1 can contribute, while for $T > T_{\min}$ there are X_1, X_2^{small} and X_2^{large}. It turns out that X_2^{small} has larger action and so is always thermodynamically unfavored [HP83].

11 Example: the $\frac{1}{2}$-BPS spectrum of $\mathcal{N} = 4$ SYM

We discussed how to compute the spectrum of primary operators using AdS/CFT. Let us apply it to IIB string theory on $\mathrm{AdS}_5 \times S^5$.

11.1 Field theory side

In general, the dimension of composite operators is renormalized:

$$\Delta_{\mathcal{O}} = \text{canonical dim.} + \gamma_{\mathcal{O}}(g_{\mathrm{YM}}) \,, \qquad\qquad \gamma_{\mathcal{O}} = \frac{\partial \log Z_{\mathcal{O}}}{\partial \log \Lambda} \tag{11.1}$$

where $\gamma_{\mathcal{O}}$ is the anomalous dimension, $Z_{\mathcal{O}}$ is the wavefunction renormalization of $\mathcal{O}$.

Some operators are protected by the (super)conformal algebra and their dimension is not renormalized. *E.g.*: conserved currents, and then their superpartners. The multiplets of conserved currents are a special case of *short multiplets*: they contain less states than a generic multiplet, saturate a unitarity bound, and the dimension is fixed by Lorentz and R-symmetry quantum numbers.

- In $\mathcal{N} = 4$ SYM there are various conserved currents:[55] the stress tensor $T_{\mu\nu}$, the supersymmetry currents $S^a_{\mu\alpha}$, the $SU(4)$ R-symmetry currents $J^a_{b\,\mu}$. They all belong to the same supermultiplet:

  $$\Big(\operatorname{Tr} \phi_i \phi_j - \text{trace} \,,\; \dots \,,\; J^a_{b\,\mu} \,,\; S^a_{\mu\alpha} \,,\; T_{\mu\nu} \Big)$$

  whose lowest component is a scalar in the $\mathbf{20}$.[56] The dimension of these fields is not renormalized.

- The other short multiplets of $\mathcal{N} = 4$ SYM are a generalization of the chiral multiplets in $\mathcal{N} = 1$.

  In $\mathcal{N} = 1$ a multiplet is chiral when it is annihilated by half of the Q's (say $\overline{Q}$). The corresponding superfield depends only on θ (and not on $\bar{\theta}$) therefore it contains less operators. A chiral multiplet satisfies the unitarity bound

  $$\Delta = \frac{3}{2} R \,. \tag{11.2}$$

  In particular Δ and R can both be renormalized, but their ratio is not.

[55] The charges P_μ, $M_{\mu\nu}$, D and K_μ are constructed with $T_{\mu\nu}$. The supercharges Q^a_α and S^a_α are constructed with $S^a_{\mu\alpha}$. Conservation of all charges follows from $\partial^\mu T_{\mu\nu} = T_{[\mu\nu]} = T^\mu_\mu = 0$ and $\partial^\mu S_{\mu\alpha} = \gamma^\mu_{\alpha\beta} S^\beta_\mu = 0$.

[56] The multiplet contains more scalars, and then the spin $\frac{1}{2}$ operators. In particular there are the $\Delta = 3$ scalars $\operatorname{Tr} \lambda^a \lambda^b + \phi^3$ in the $\mathbf{10}$ (which is a symmetric bi-spinor, or an antisymmetric self-dual rank-3 tensor), and the $\Delta = 4$ scalar $\mathcal{L} = \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} + \dots$.

In $\mathcal{N} = 4$ a (single-trace) chiral multiplet has maximal spin 2, and a scalar lowest component transforming in the symmetric traceless representation of rank k of $SO(6)$. With Dynkin diagrams:[57]

$$\operatorname{Tr} \phi_{(i_1} \ldots \phi_{i_k)} - \text{traces} \qquad\qquad [0, k, 0] \text{ of } SU(4) \qquad\qquad \underbrace{\begin{array}{|c|c|c|c|}\hline & & \cdots & \\ \hline & & \cdots & \\ \hline \end{array}}_{k} \;.$$

The dimension of the lowest state is fixed to $\Delta = k$, and since it is related to the dimension of an $SU(4)$ representation, it cannot be renormalized.

We call these chiral multiplets A_k.

The multiplet A_2 corresponds to the supercurrent multiplet. All other single-trace chiral multiplets of $\mathcal{N} = 4$ SYM are one of the A_k. All single-trace operators not in one of the A_k are renormalized.

Notice that depending on the gauge group the spectrum changes. For $SU(N)$ we have all $k \geq 2$. Only $U(1)$ can have $k = 1$ (and that is a free field). For $SO(N)$ we have all even k.

Example. Consider the quadratic operators made from the six scalars of $\mathcal{N} = 4$ SYM:

$$\operatorname{Tr} \phi_i \phi_j \;.$$

These are automatically symmetric (because of the trace), but fall in two $SU(4)$ representations:

$$\begin{array}{lll} \mathbf{1} & \sum_i \operatorname{Tr} \phi_i \phi_i & \Delta = 2 + \mathcal{O}(g_{\text{YM}}) \\ \mathbf{20} & \operatorname{Tr} \phi_i \phi_j - \frac{\delta_{ij}}{6} \sum_k \operatorname{Tr} \phi_k \phi_k & \Delta = 2 \;. \end{array} \tag{11.3}$$

The first one is the lowest component of a long, unprotected multiplet: *Konishi multiplet*. Its dimension is renormalized (at one-loop and beyond). The second one is the lowest component of A_2, and its dimension is not renormalized.

We can understand why only symmetric traceless combinations are in $\mathcal{N} = 4$ chiral multiplets using an $\mathcal{N} = 1$ subalgebra. On the one hand, the symmetric traceless combinations include, in $\mathcal{N} = 1$ notation, the operators[58]

$$\operatorname{Tr} \Phi_{(I_1} \ldots \Phi_{I_k)} \qquad\qquad I_i = 1, 2, 3$$

[57] For $SU(N)$, the representation $[d_1, \ldots, d_{N-1}]$ has a Young diagram with d_j columns of height j. The chiral multiplets with lowest component in the $[p, k, p]$ play a role in multi-trace operators.

[58] When written in $\mathcal{N} = 1$ notation, the symmetric traceless combinations contain more operators. For instance, they contain the anti-chiral operators $\operatorname{Tr} \overline{\Phi}_{(I_1} \ldots \overline{\Phi}_{I_k)}$ which are protected. They also contain operators written in terms of both chiral and anti-chiral fields. These, from the point of view of $\mathcal{N} = 1$, do not look protected, however they are because they sit in the same $\mathcal{N} = 4$ multiplet.

with indices symmetrized. These are $\mathcal{N}=1$ chiral. The superpotential of $\mathcal{N}=4$ SYM

$$W = \operatorname{Tr}\Phi_1[\Phi_2,\Phi_3] \tag{11.4}$$

fixes their R-charge, and consequently their dimension, to be

$$R[\Phi_I] = 2/3 \qquad\qquad \Rightarrow \qquad\qquad \Delta = k\;, \tag{11.5}$$

because $\Delta = \frac{3}{2}R$ for chiral primaries.

On the other hand, if we consider the operators $\operatorname{Tr}\Phi_{I_1}\ldots\Phi_{I_k}$, only those that are completely symmetrized are chiral primaries. This is because the EOMs (or the F-term relations) imply

$$[\Phi_I,\Phi_J] \sim \epsilon_{IJK} D^2 \overline{\Phi}_K\;. \tag{11.6}$$

Therefore, operators that are not completely symmetrized are superconformal descendants, and not primaries. Now, if we write the symmetrized operators $\operatorname{Tr}\Phi_{(I_1}\ldots\Phi_{I_k)}$ in terms of real scalars ϕ_i, we discover that they are automatically traceless. Indeed, take for concreteness

$$\begin{aligned} \Phi_1\Phi_2 &= \phi_1\phi_3 - \phi_2\phi_4 + i\phi_1\phi_4 + i\phi_2\phi_3 \\ \Phi_1\Phi_1 &= \phi_1\phi_1 - \phi_2\phi_2 + 2i\phi_{(1}\phi_{2)}\;. \end{aligned} \tag{11.7}$$

In the first line each operator on the RHS is traceless (it gives zero when contracted with δ^{ij}); in the second line, both the real and the imaginary parts are traceless.

11.2 Gravity side

To read off the spectrum from gravity, we need the effective 5d supergravity theory in AdS_5, that follows from the 10d IIB supergravity on $\mathrm{AdS}_5\times S^5$ from KK reduction.

In the standard KK reduction on S^1 of radius R, one expands all fields in Fourier modes on S^1:

$$\phi(x_\mu,y) = \sum\nolimits_k \phi_k(x_\mu)\, e^{iky/R} \qquad\qquad k\in\mathbb{Z}\;. \tag{11.8}$$

Then one plugs into the EOMs:

$$-\Box_{d+1}\,\phi(x_\mu,y) = \big(-\Box_d - \partial_y^2\big)\,\phi(x_\mu,y) = \sum_k e^{iky/R}\left(-\Box_d + \frac{k^2}{R^2}\right)\phi_k(x_\mu)\;. \tag{11.9}$$

Thus one gets an infinite tower or KK modes, with square masses k^2/R^2. (One can then write an action that reproduces the EOMs).

We should do a similar KK reduction on S^5. The bosonic massless modes of 10d type IIB string theory are

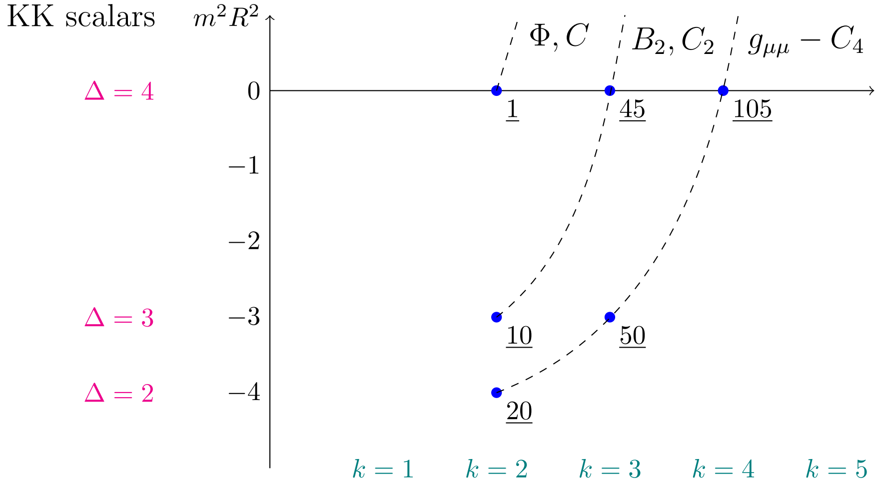
$$\big(g_{\mu\nu}\,,\,\phi\,,\,B_{\mu\nu}\,,\,C\,,\,C_{\mu\nu}\,,\,C^+_{\mu\nu\rho\sigma}\big)\;. \tag{11.10}$$

We should diagonalize the Laplacian on S^5 using spherical harmonics. For a scalar:

$$\phi(x,y) = \sum\nolimits_I \phi_I(x)\,Y_I(y) \tag{11.11}$$

where Y_I are the eigenfunctions of the scalar Laplacian on S^5.

The full reduction has been done in [KRvN85, GM85]. The result is the following.



We plot only the scalar modes, and for dimensions up to $\Delta = 4$. For each mode we indicate the $SU(4)$ representation.[59] The mass/dimension is on the vertical axis. Modes in the same vertical line are scalars in the same supermultiplet A'_k. Modes connected by a dashed line are different KK modes of the same 10d field.

The lowest scalar in each multiplet is in the k-fold symmetric traceless representation of $SO(6)$, namely

$$[0, k, 0] \qquad \text{with } m^2 = \frac{k(k-4)}{R^2} \quad \leftrightarrow \quad \Delta = k \; .$$

The next scalar is in representation

$$[2, k-2, 0] \qquad \text{with } \Delta = k + 1 \; .$$

The next scalar is in representation

$$[0, k-2, 0] \qquad \text{with } \Delta = k + 2 \; .$$

The multiplet contains spins up to 2 (because no 10d field has higher spin).

Let us make some remarks:

- In the standard KK reduction the massive modes in each tower are separated from the massless modes by a gap of order $1/R$. Thus one can decouple massive modes by taking the limit $R \to 0$.

  Due to the curvature, in AdS there is no separation: all KK modes have a mass of the same order as the zero-modes. In fact, we even have fields with different masses in the same supermultiplet!

[59] $\underline{10}$ is a self-dual antisymmetric 3-tensor equal to a symmetric 2-spinor, while $\underline{20}$ is the symmetric traceless 2-tensor.

- Even if we cannot decouple the multiplets A'_k with $k \geq 3$ by taking the radius of S^5 small, we can write an effective action that reproduces the EOMs for A'_2. This action is 5d $\mathcal{N} = 8$ $SO(6)$ maximal gauged supergravity.

  A'_2 is coupled to $A'_{k\geq 3}$, however if we set the latter to zero, they are not sourced by A'_2. In practice, the fields $A'_{k\geq 3}$ never appear linearly in the Lagrangian.[60] The reduction to A'_2 is called a *consistent truncation*: every solution of 5d $\mathcal{N} = 8$ gauged SUGRA can be uplifted to a 10d solution of IIB SUGRA.

11.3 Comparison

There is a perfect match between short multiplets of 4d $\mathcal{N} = 4$ SYM and multiplets in AdS$_5$ upon KK reduction of IIB SUGRA on S^5. In both cases, there is one short multiplet A_k of the superconformal algebra for each $k \geq 2$.

The multiplet A'_2 contains the graviton, the gravitino and the $SO(6)$ gauge fields: it corresponds to the supercurrent multiplet in FT (gauge symmetries in bulk = global symmetries at boundary).

The multiplets A'_k correspond to the FT multiplets A_k with lowest component

$$\mathrm{Tr}\, \phi_{(i_1} \ldots \phi_{i_k)} - \text{traces} \;.$$

The scalars in figure are identified as follows:

$SU(4)$ rep		operator	mult./dim.
20	$\ydiagram{2,2}$	$\mathrm{Tr}\, \phi_i \phi_j - \text{trace}$	A_2 $\Delta = 2$
10	$\ydiagram{2}$	$\mathrm{Tr}\, \lambda_a \lambda_b + \phi^3$	A_2 $\Delta = 3$
1		$F_{\mu\nu}(F^{\mu\nu} + i\widetilde{F}^{\mu\nu})$	A_2 $\Delta = 4$
50	$\ydiagram{3,3}$	$\mathrm{Tr}\, \phi_{(i}\phi_j\phi_{k)} - \text{traces}$	A_3 $\Delta = 3$
45	$\ydiagram{3,1}$	$\mathrm{Tr}\, \lambda_a \lambda_b \phi_i + \phi^4$	A_3 $\Delta = 4$
105	$\ydiagram{4,4}$	$\mathrm{Tr}\, \phi_{(i}\phi_j\phi_k\phi_{l)} - \text{traces}$	A_4 $\Delta = 4$

In the bulk there are also string modes. In the supergravity approximation their masses go like

$$m^2 = \frac{\Delta(\Delta - 4)}{R^2} \sim \frac{1}{\alpha'} = \frac{\sqrt{4\pi\lambda}}{R^2} \qquad \Rightarrow \qquad \Delta \sim \lambda^{1/4} \;. \tag{11.12}$$

We thus have the following prediction about $\mathcal{N} = 4$ SYM: at strong coupling, all unprotected multiplets (and in particular all operators with spin > 2) have divergent dimensions that go like[61] $\lambda^{1/4}$. The only operators with finite dimension are those in the protected chiral multiplets A_k.

[60] One way for this to happen is if the (classical) gravitational theory has a global symmetry, and the fields $A'_{k\geq 3}$ are charged while A'_2 is neutral.

[61] This is true for operators of fixed length as $N \to \infty$. One could also consider operators whose length increases with N, like giant gravitons.

12 A richer example: the conifold

In this lecture we want to study another exact example of AdS/CFT, with only $\mathcal{N} = 1$ supersymmetry. This is a generalization of type IIB on $\mathrm{AdS}_5 \times S^5$, in which the manifold S^5 is substituted by another one, $T^{1,1}$, called the "base of the conifold". The physics is much richer: anomalies, dimensional transmutation, confinement, chiral symmetry breaking, domain walls separating inequivalent vacua appear. Good reviews are [Kle00, HKO02].

The starting point of Maldacena's proof is D3-branes on flat space, namely on

$$\mathbb{R}^{3,1} \times \mathbb{R}^6 .$$

We can construct other examples by taking a more general geometry:

$$\mathbb{R}^{3,1} \times \mathcal{M}_6 ,$$

in other words the D3-branes are at a point of $\mathcal{M}_6$. In order to have a stable vacuum, we'd better preserve some supersymmetry: the minimal one is $\mathcal{N} = 1$ in 4d. The condition for SUSY[62] is that on $\mathcal{M}_6$ there is a covariantly constant spinor ζ, because the gravitino variation is

$$\delta_\zeta \Psi_\mu = \nabla_\mu \zeta = 0 . \tag{12.1}$$

Manifolds $\mathcal{M}_6$ satisfying the condition are very special: they are called *Calabi-Yau manifolds.*

- They are complex and Kähler: the tensors

$$J_{ij} = \zeta^\dagger \Gamma_{ij} \zeta , \qquad\qquad J^i_{\ j} = g^{ik} J_{kj} \tag{12.2}$$

  are a (closed) Kähler form and a (integrable) complex structure.

- The metric is Ricci-flat:[63]

$$R_{ij} = 0 . \tag{12.3}$$

- They have trivial canonical bundle: the complex tensor

$$\Omega_{ijk} = \zeta^\mathsf{T} \Gamma_{ijk} \zeta \tag{12.4}$$

  is closed and covariantly-constant.

- The holonomy group is reduced from $SO(6)$ to $SU(3)$: a chiral spinor is in the **4** of $SU(4)$, and it breaks it to $SU(3)$.

[62] We assume that we do not turn on fluxes, besides the F_5 generated by the D3-branes. Then the dilatino variation automatically vanishes, $\delta\lambda = 0$.

[63] From $[\nabla_i, \nabla_j]\zeta = 0$ one gets $R_{ij}{}^{kl}\Gamma_{kl}\zeta = 0$. Then use $\Gamma^j\Gamma^{kl} = \Gamma^{jkl} + \delta^{jk}\Gamma^l - \delta^{jl}\Gamma^k$ and the Bianchi identity $R_{i[jkl]} = 0$ to get $R_{ij}\Gamma^j\zeta = 0$, finally use that the gamma matrices are independent.

In Maldacena's argument we take a near-horizon limit to the branes, *i.e.* we focus around the branes. If the D3-branes sit at a smooth point on $\mathcal{M}_6$, the near-horizon leads to the same $\mathrm{AdS}_5 \times S^5$ as before.

In order to get something new, we should place the D3-branes at a singular point on $\mathcal{M}_6$. Focusing, the singularity looks like a *conical singularity*:

$$ds^2(\mathcal{M}_6) = dr^2 + r^2\, ds^2(X_5)\ . \tag{12.5}$$

The SUSY condition for X_5 is that it is Sasaki-Einstein (in particular positively curved).

The near-horizon geometry to D3-branes at a CY singularity is

$$\mathrm{AdS}_5 \times X_5$$

with N units of 5-form flux on AdS_5 and X_5.

12.1 Conifold geometry and SCFT

The conifold (a conical Calabi-Yau threefold) is described by one complex equation in four variables:

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\ . \tag{12.6}$$

The equation is invariant under rescaling, thus the geometry is a cone. The base is called $T^{1,1}$, which is a coset space:

$$T^{1,1} = \frac{SU(2)\times SU(2)}{U(1)} \tag{12.7}$$

and has $SU(2)\times SU(2)\times U(1)$ isometry. It is obtained by intersecting the equation with

$$\sum_{a=1}^{4} |z_a|^2 = 1\ , \tag{12.8}$$

which has $SO(4)\times U(1)$ invariance. Since there is a unique Abelian isometry, that is identified with the superconformal R-symmetry $U(1)_R$.

The Ricci-flat metric on the conifold is known (a rare fact) because of the large isometry. It follows from the metric on $T^{1,1}$:

$$ds^2(T^{1,1}) = \frac{1}{6}\sum_{i=1}^{2}\big(d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2\big) + \frac{1}{9}\Big(d\psi - \sum_{i=1}^{2}\cos\theta_i\, d\varphi_i\Big)\ . \tag{12.9}$$

The first term represents $S^2\times S^2$, while the second one with $\psi \cong \psi + 4\pi$ represents a $U(1)$ bundle over it. Therefore

$$T^{1,1} = U(1) \text{ bundle over } S^2\times S^2 \ \cong\ S^2\times S^3\ . \tag{12.10}$$

The fact that the topology is $S^2 \times S^3$ is a geometric fact.[64]

After putting N D3-branes at the tip of the conifold and taking the near-horizon limit, we get the following 10d solution of IIB supergravity:[65]

$$
\begin{aligned}
& ds^2 = R^2\big(ds^2(\mathrm{AdS}_5) + ds^2(X_5)\big) && R^4 = 4\pi g_s N \alpha'^2 \, \frac{\mathrm{Vol}(S^5)}{\mathrm{Vol}(X_5)} \\
& F_5 = \tfrac{(4\pi^2\alpha')^2 g_s}{\mathrm{Vol}(X_5)} \, (1 + *) \, N \, d\mathrm{vol}_{X_5} \; . &&
\end{aligned}
\tag{12.11}
$$

The dual field theory has been identified by Klebanov and Witten [KW98]. To identify the dual field theory we use the following argument. In general, the theory on the D3-branes is a gauge theory with matter fields that parametrize their motion in the orthogonal directions. Recall that for a single D3-brane on flat space, we have a $U(1)$ theory with three neutral complex scalars $\Phi_{1,2,3}$ parametrizing the orthogonal $\mathbb{C}^3$. For N branes we have a $U(N)$ theory, the neutral scalars become adjoint, and interactions are fixed by SUSY. The $U(1)$ and the traces of Φ_i decouple in the IR.

In the case of the conifold we can rewrite the equation as follows:

$$
Z = \sum_{a=1}^4 i\sigma_a z_a = \begin{pmatrix} z_4 + iz_3 & iz_1 + z_2 \\ iz_1 - z_2 & z_4 - iz_3 \end{pmatrix} \; , \qquad\qquad \det Z = 0 \tag{12.12}
$$

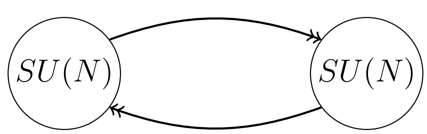
where $\sigma_{a=1,2,3}$ are the Pauli matrices and $\sigma_4 = -i\mathbb{1}$. The equation is solved by imposing that Z_{ij}, as a matrix, is the product of two vectors:

$$
Z_{ij} = A_i B_j \tag{12.13}
$$

with unconstrained $A_{i=1,2}$, $B_{j=1,2}$. If we parametrize the conifold by A_i, B_j there is no equation, but there is a redundancy:

$$
(A_i, B_j) \;\to\; (e^{i\theta} A_i, e^{-i\theta} B_j) \tag{12.14}
$$

which is implemented by a $U(1)$ gauge symmetry. Led by the $\mathcal{N} = 4$ example, we should expect another decoupled $U(1)$. Indeed we can propose the following theory:



$$
W = \epsilon^{ij}\epsilon^{kl} \, \mathrm{Tr}\, A_i B_k A_j B_l \; .
$$

This is a "quiver diagram": nodes are gauge groups, and arrows are chiral multiplets in the bifundamental representation: fundamental with respect to the group at the tail, antifundamental under the group at the head.

[64] Oriented S^3 bundles over S^2 are topologically parametrized by $\pi_1\big(SO(4)\big) = \mathbb{Z}_2$.

[65] The volume of S^5 is π^3, the volume of $T^{1,1}$ is $16\pi^3/27$.

For a single D3-brane we have a $U(1) \times U(1)$ gauge theory with 4 chiral multiplets A_i, B_j, where A_i have charges $(1, -1)$ while B_j have charges $(-1, 1)$.

For multiple D3-branes the groups are $U(N)$. However one $U(1)$ is decoupled and free, while the other one becomes free in the IR. We thus have gauge group $SU(N) \times SU(N)$. This theory indeed has $SU(2) \times SU(2) \times U(1)_B \times U(1)_R$ symmetry, where $U(1)_B$ is a baryonic symmetry that gives charge $+1$ to A_i and -1 to B_j. Led by the $\mathcal{N} = 4$ example, we should also expect a superpotential.

Cancelation of the $U(1)_R$ anomaly fixes the dimensions of A_i, B_j at the fixed point. We impose

$$\operatorname*{Tr}_{\text{fermions}} T^a_{SU(N)}\, T^b_{SU(N)}\, R = 0 \,. \tag{12.15}$$

We use that $\operatorname{Tr} T^a T^b$ equals N for the adjoint representation, and $\frac{1}{2}$ for the (anti)fundamental. Calling R_A the R-charge of A_i, B_j:

$$N + \frac{1}{2}(R_A - 1)4N = 0 \qquad \Rightarrow \qquad R_A = \frac{1}{2} \,, \qquad \Delta = \frac{3}{4} \,.$$

The chiral multiplets do not have canonical dimension, thus the fixed point is necessarily strongly-coupled. There is a unique superpotential which is compatible with the symmetries:

$$W = \epsilon^{ij} \epsilon^{kl} \operatorname{Tr} A_i B_k A_j B_l = \operatorname{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1) \,. \tag{12.16}$$

In fact, this superpotential is necessary to give a 2d line of exactly marginal deformations, as observed in SUGRA.

How does $U(1)_B$ appear in supergravity? The internal manifold has topology $S^2 \times S^3$, therefore the KK reduction of C_4 gives a gauge field in AdS$_5$. This is the bulk gauge field dual to the baryonic current.

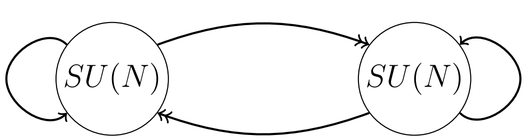
12.2 RG flow to $T^{1,1}$

As a nice check, we can reproduce the conifold field theory from an RG flow.

Start with N D3-branes on the orbifold geometry

$$\mathbb{R}^{3,1} \times \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C} \,.$$

The near-horizon geometry is AdS$_5 \times S^5/\mathbb{Z}_2$. Since the original geometry is an orbifold of flat space, the perturbative open string is well-defined and one can compute the spectrum exactly as one does for flat D3-branes. The theory on the D3-branes has 4d $\mathcal{N} = 2$ supersymmetry and it is



$$W = \operatorname{Tr}\left(\Phi \sum_{i=1}^{2} A_i B_i - \widetilde{\Phi} \sum_{i=1}^{2} B_i A_i\right) .$$

We can add a relevant deformation:

$$W_{\text{def}} = \frac{m}{2} \operatorname{Tr} \left(\Phi^2 - \widetilde{\Phi}^2 \right) . \tag{12.17}$$

The fields Φ, $\widetilde{\Phi}$ are massive and can be integrated out. One obtains precisely the conifold theory.

We can understand that geometrically. The perturbation W_{def} is odd under the exchange of the two gauge groups, and it is odd under $\mathbb{Z}_2$. Thus it corresponds to a twisted mode. Such twisted mode corresponds to a resolution of the orbifold singularity into the conifold. The full RG flow has been constructed in supergravity.

It is interesting to examine how the central charges change under this RG flow. Recall that, al leading order in N:

$$a \simeq c \simeq \frac{9}{32} \operatorname*{Tr}_{\text{fermions}} R^3 . \tag{12.18}$$

In the $\mathcal{N} = 2$ orbifold theory we have

$$c \simeq \frac{9}{32} \left(2N^2 + 6 \Big(-\frac{1}{3} \Big)^3 \right) = \frac{N^2}{2} . \tag{12.19}$$

In the $\mathcal{N} = 1$ KW theory we have

$$c \simeq \frac{9}{32} \left(2N^2 + 4 \Big(-\frac{1}{2} \Big)^3 \right) = \frac{27}{64} N^2 . \tag{12.20}$$

Therefore

$$\frac{c_{\text{IR}}}{c_{\text{UV}}} = \frac{27}{32} . \tag{12.21}$$

According to the c-theorem, this is smaller than 1.

We can compare with the holographic computation. We have seen that the central charge is proportional to the Newton constant in AdS_5, which is inversely proportional to the radius of X_5:

$$\frac{c_{\text{IR}}}{c_{\text{UV}}} = \frac{\text{Vol}(S^5/\mathbb{Z}_2)}{\text{Vol}(T^{1,1})} = \frac{27}{32} . \tag{12.22}$$

12.3 Spectrum of chiral primaries

The analysis of single-trace chiral primaries for the conifold theory is very similar to the one for $\mathcal{N} = 4$ SYM.

On the gravity side, we should KK reduce IIB supergravity on $T^{1,1}$ to get an effective theory in AdS_5. The lowest scalar in each multiplet comes from a mixture of modes of the warp factor $g_{\mu\mu}$ and C_4. The wavefunctions are in the representation

$$\left(\frac{k}{2}, \frac{k}{2} \right)_k$$

of the isometry group $SU(2) \times SU(2) \times U(1)_R$ — and are neutral under $U(1)_B$. The mass of the scalars in AdS$_5$ is

$$m^2 R^2 = \frac{3}{4} k(3k-8) \,, \tag{12.23}$$

therefore the dimensions of the dual operators follow from

$$\Delta_\pm = 2 \pm \left| \frac{3}{2} k - 2 \right| \,. \tag{12.24}$$

For $k \geq 2$ there is only one possible quantization, and the dimension must be

$$\Delta_+ = \frac{3}{2} k \,. \tag{12.25}$$

However for $k = 1$ both quantizations are allowed: $\Delta_+ = \frac{5}{2}$, $\Delta_- = \frac{3}{2}$. As we will see in a moment in FT, the one compatible with supersymmetry is the unusual one:

$$k = 1 \;: \qquad\qquad \Delta_- = \frac{3}{2} \,. \tag{12.26}$$

Thus we have here an example in which the alternative quantization is chosen: the boundary conditions fix the subleading mode, as opposed to the leading mode.

In the field theory, the single-trace chiral primaries are

$$\mathcal{O} = \mathrm{Tr}\left(A_{i_1} B_{j_1} \dots A_{i_k} B_{j_k} \right) \qquad\qquad k \geq 1 \,.$$

From the superpotential $W = \mathrm{Tr}\left(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1 \right)$ one obtains the F-term relations

$$A_{[i}\, B_{|j|}\, A_{k]} = 0 \,, \qquad\qquad B_{[j}\, A_{|i|}\, B_{l]} = 0 \tag{12.27}$$

antisymmetrized in $[ik]$ and $[jl]$ respectively. It follows that the single-trace chiral primaries are fully symmetrized in $(i_1 \dots i_k)$ and $(j_1 \dots j_k)$ separately, and thus are in representation $\left(\frac{k}{2}, \frac{k}{2}\right)$ of $SU(2)^2$. The R-symmetry of A, B is fixed by the superpotential:

$$R_A = \frac{1}{2} \tag{12.28}$$

and it cannot be renormalized. It follows that

$$R[\mathcal{O}] = k \,, \qquad\qquad \Delta[\mathcal{O}] = \frac{3}{2} k \,. \tag{12.29}$$

They precisely match the spectrum computed in supergravity.

There is an important difference between the S^5 and $T^{1,1}$ case. In the S^5 case, all modes in the KK reduction to AdS$_5$ sit in short (and thus protected) multiplets. In this sense, we are not learning something new about the spectrum. In the case of $T^{1,1}$, many of the KK modes on AdS$_5$ and in long *unprotected* multiplets. AdS/CFT makes the prediction that their dimensions remain finite in the large λ limit, and it gives a way to compute spectrum and correlators.

12.4 Dibaryons as wrapped D3-branes

No KK mode in AdS_5 is charged under $U(1)_B$, because the baryonic current comes from C_4 and the SUGRA modes are not charged under C_4.

In the field theory, what operators are charged under $U(1)_B$? For instance the "dybaryons"

$$\mathcal{B}_{i_1 \dots i_N} = (A_{i_1})^{\alpha_1}{}_{\beta_1} \, \dots \, (A_{i_N})^{\alpha_N}{}_{\beta_N} \, \epsilon_{\alpha_1 \dots \alpha_N} \, \epsilon^{\beta_1 \dots \beta_N} \; . \tag{12.30}$$

This object is, by construction, totally symmetric in $(i_1 \dots i_N)$, and thus it is in representation

$$\Big(\frac{N}{2}, 0\Big)_{\frac{N}{2}, N} \qquad \text{of} \qquad SU(2)^2 \times U(1)_R \times U(1)_B \; .$$

In particular these are $N+1$ operators, and are chiral primaries.

There is of course a second dibaryon operator $\widetilde{\mathcal{B}}_{j_1 \dots j_N}$ constructed out of B_j, in representation

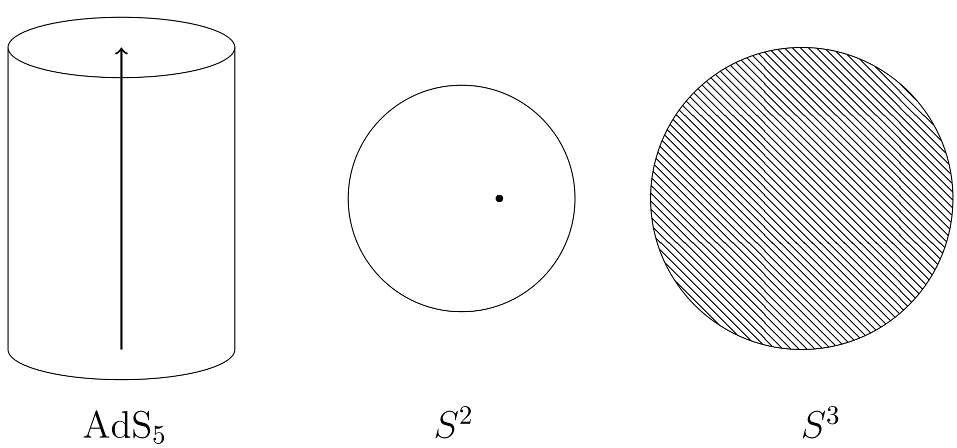
$$\Big(0, \frac{N}{2}\Big)_{\frac{N}{2}, -N} \; .$$

The dimension of these operators is

$$\Delta = \frac{3}{4} N \; , \tag{12.31}$$

in particular it is large as $N \to \infty$. These operators are not described by the KK reduction of 10d massless string modes. They are also not described by massive perturbative string modes, since for those the dimension scales as $\Delta \sim \lambda^{1/4}$. They are described by *solitonic* string states, in fact D-branes.

Since $T^{1,1} \cong S^2 \times S^3$, we can wrap a D3-brane on S^3 to get a particle in AdS_5.



A D3-brane is charged under C_4, thus the particle is charged under $U(1)_B$. It turns out that a D3-brane on a minimal-surface S^3 is also supersymmetric, thus the particle will be described by a chiral field. Such a chiral field, $\Phi_{\mathcal{B}}$, is then singled-out to be the bulk field dual to the dibaryon operator.

A minimal area S^3 in $T^{1,1}$ is given by

$$(\theta_1, \varphi_1) = \text{fixed} \qquad\qquad \forall\, \theta_2, \varphi_2, \psi \; .$$

Its volume is

$$\mathrm{Vol}(S^3) = \frac{8}{9}\pi^2 R^3 \; . \tag{12.32}$$

Since the D3-brane has a tension, the mass of the particle is

$$m_{\mathrm{D3}} = \mathrm{Vol}(S^3)\, \frac{\sqrt{\pi}}{\kappa} \qquad\qquad \kappa = \sqrt{8\pi G_{\mathrm{N}}} = 8\pi^{7/2} g_s \alpha'^2 \; . \tag{12.33}$$

For the precise coefficients see Polchinski's book or [Kle00]. Finally we find the dimension of the operator dual to $\Phi_{\mathcal{B}}$:

$$R^2 m_{\mathrm{D3}}^2 = \frac{9}{16} N^2 \; , \qquad\qquad \Delta = 2 + \sqrt{2 + R^2 m^2} = \frac{3}{4} N + \mathcal{O}(1) \; . \tag{12.34}$$

We reproduce the dimension of $\mathcal{B}$, at leading order in N.

Classically, we have one such field $\Phi_{\mathcal{B}}$ for each value of (θ_1, φ_1). At the quantum level, we have a moduli space $\mathbb{CP}^1$ that we should quantize, in the sense that we should find wavefunctions on this space. Reducing on S^3, we find a point particle on S^2 immersed into N units of magnetic flux (from the reduction of F_5). The quantization of this system, more precisely of the fermionic oscillations, leads to $N+1$ Landau levels: the vacuum is degenerate with $N+1$ states. Such states form an $(N+1)$-dimensional representation of the rotation group $SU(2)$, *i.e.* of spin $\frac{N}{2}$.

Thus we reproduce the fact that there are $N+1$ such operators, transforming in the spin $\frac{N}{2}$ representation of the first $SU(2)$.

12.5 Fractional branes and the Klebanov-Tseytlin solution

Another interesting object to introduce is M D5-branes wrapped on S^2 of $T^{1,1}$. This object is a domain wall in AdS_5, that separates two phases with a different 5d effective theory.

It is not obvious to understand what happens on the other side of the wall, and to make things simple we use some intuition. We take the baryonic particle coming from a D3-brane on S^3 and follow it as it crosses the wall. When a D3 and M D5's cross, by "brane-creation effect" [DFK97] M D1-strings are created between the two. As a result, the dibaryon operator has M free gauge index (attached to an external heavy quark). For instance

$$(A_{i_1})^{\alpha_1}_{\beta_1} \,\ldots\, (A_{i_{N+M}})^{\alpha_{N+M}}_{\beta_{N+M}}\; \epsilon_{\alpha_1 \ldots \alpha_N}\; \epsilon^{\beta_1 \ldots \beta_{N+M}} \; .$$

This is possible if the gauge group has become

$$SU(N) \times SU(N+M) \; .$$

There are more refined arguments to draw this conclusion.

The wall is not BPS and is not really stable in AdS_5: its energy scales as r^4, thus the wall wants to "fall inside AdS" towards the horizon at $r = 0$.

The stable configuration is without the wall (the wall has disappeared behind the horizon), but with a magnetic flux on S^3 of the RR 3-form F_3 which is the remnant of its passage.

Another way to think about the system is before taking the near-horizon limit: at the tip of the conifold we can place N D3-branes, as well as M D5-branes that wrap the vanishing S^2 cycle at the tip and thus "look like 3-branes" — these are called *fractional branes*.

Thus we are after a supergravity solution with

$$\frac{1}{4\pi\alpha'}\int_{S^3} F_3 = M \;, \qquad\qquad \frac{1}{(4\pi\alpha')^2}\int_{T^{1,1}} \overset{?}{=} N \;. \tag{12.35}$$

We are studying a SUSY theory, and we are looking for SUSY ground states. Thus we can solve the BPS equations

$$\begin{cases} \delta\Psi_{\mu\alpha} = 0 & \qquad \text{gravitino} \\ \delta\lambda_\alpha = 0 & \qquad \text{dilatino}\;. \end{cases} \tag{12.36}$$

These equations imply

$$H_3 = g_s *_6 F_3 \;, \tag{12.37}$$

therefore the solution should have a non-trivial NS 3-form flux H_3 as well. The Bianchi identity for F_5 is

$$dF_5 = H_3 \wedge F_3 \neq 0 \qquad\qquad \text{because} \qquad F_5 = dC_4 + B_2 \wedge F_3 \;. \tag{12.38}$$

It follows that the number N defined above is *not* constant! It is a function of r.

These considerations lead to the following supergravity solution, constructed by Klebanov and Tseytlin [KT00]. The metric is a "warped product" of $\mathbb{R}^{3,1}$ and the conifold:

$$ds^2_{10} = h(r)^{-1/2} dx^2_{3,1} + h(r)^{1/2}\big(dr^2 + r^2 ds^2_{T^{1,1}}\big) \tag{12.39}$$

where we recall

$$ds^2_{T^{1,1}} = \frac{1}{6}\sum\nolimits_{i=1,2}\Big(d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2\Big) + \frac{1}{9}\Big(d\psi - \sum\nolimits_{i=1,2}\cos\theta_i\, d\varphi_i\Big)^2 \;. \tag{12.40}$$

To write the fluxes, we introduce two forms proportional to the volume forms on S^2 and S^3:[66]

$$\omega_2 = \frac{1}{2}\big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 - \sin\theta_2\, d\theta_2 \wedge d\varphi_2\big) \propto d\mathrm{vol}_{S^2} \;, \qquad \omega_3 = d\psi \wedge \omega_2 \propto d\mathrm{vol}_{S^3} \;. \tag{12.41}$$

[66] A representative for S^2 is $\theta_1 = \theta_2$, $\varphi_1 = -\varphi_2$. A representative for S^3 is $(\theta_2, \varphi_2) =$ fixed.

They also satisfy $\omega_2 \wedge \omega_3 = 54\, d\mathrm{vol}_{T^{1,1}}$. Then

$$F_3 = \alpha' \frac{M}{2} \omega_3 \;, \qquad\qquad H_3 = g_s \alpha' \frac{3M}{2} \frac{dr}{r} \wedge \omega_2 \;. \tag{12.42}$$

The running 5-form flux is

$$F_5 = (1+*)\, 27\pi\alpha'^2 N_{\text{eff}}(r)\, d\mathrm{vol}_{T^{1,1}} \;, \qquad N_{\text{eff}}(r) = N + \frac{3}{2\pi} g_s M^2 \log \frac{r}{r_0} \;. \tag{12.43}$$

Finally the warp factor is given by

$$h(r) = \frac{L^4}{r^4} \log \frac{r}{r_s} \qquad \text{for some } r_s \;, \qquad L^2 = \frac{9 g_s M \alpha'}{2\sqrt{2}} \;. \tag{12.44}$$

This supergravity solution encodes an incredibly rich physics. Let us explore some aspects. An excellent review on Seiberg duality and the physics of the KW theory is by Strassler [Str05].

Exact $\boldsymbol{\beta}$-function. In 4d supersymmetric $\mathcal{N} = 1$ gauge theories there is an "exact" expression for the gauge β-function, if we use a holomorphic scheme $\mathcal{L} = \frac{1}{g^2} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu}$. This is called the NSVZ beta function [NSVZ83, NSVZ86]:[67]

$$\beta_{\text{NSVZ}} \Big[\frac{8\pi^2}{g^2} \Big] = 3 C_2(G) - \sum_{\text{chirals}} C(\mathfrak{R})\, (1-\gamma) \;. \tag{12.46}$$

If we neglect γ, this is the standard one-loop beta function. Instead

$$\gamma = \frac{\partial \log Z_\Phi}{\partial \log \mu} \qquad\qquad \Delta[\Phi] = 1 + \frac{1}{2} \gamma_\Phi \tag{12.47}$$

is the anomalous dimension of the chiral field Φ. This formula expresses the exact beta function, as a function of the unknown anomalous dimensions which receive contributions to all orders.

Applied to the KW theory with gauge group $SU(N+M) \times SU(N)$:

$$\begin{aligned} \frac{\partial}{\partial \log \mu} \frac{8\pi^2}{g_1^2} &= 3(N+M) - 2N(1-\gamma) \\ \frac{\partial}{\partial \log \mu} \frac{8\pi^2}{g_2^2} &= 3N - 2(N+M)(1-\gamma) \;. \end{aligned} \tag{12.48}$$

[67] The standard one-loop beta function is

$$\beta(g) = -\frac{g^3}{16\pi^2} \bigg(\frac{11}{3} C_2(G) - \frac{2}{3} \sum_{\text{Weyl}} C(\mathfrak{R}_{\text{W}}) - \frac{1}{6} \sum_{\text{scalar}_{\mathbb{R}}} C(\mathfrak{R}_{\text{s}}) \bigg) \;. \tag{12.45}$$

Here $C(\mathfrak{R})$ is the Dynkin index, or quadratic Casimir, of the representation $\mathfrak{R}$: $\operatorname{Tr} T^a T^b = C(\mathfrak{R})\, \delta^{ab}$. Instead $C_2(G) = C(\text{adj})$ is the dual Coxeter number.

We need to determine γ. For $M = 0$ the theory is conformal and $\Delta[A, B] = \frac{3}{4}$, then $\gamma = -\frac{1}{2}$. It follows that γ has an expansion in $\frac{M}{N}$. Since the theory is invariant under

$$M \to -M \ , \qquad\qquad N \to N + M \tag{12.49}$$

but this changes sign to the first order term, it follows that

$$\gamma = -\frac{1}{2} + \mathcal{O}\big(\tfrac{M}{N}\big)^2 \ . \tag{12.50}$$

We compute

$$\tfrac{\partial}{\partial \log\mu}\big(\tfrac{8\pi^2}{g_1^2} + \tfrac{8\pi^2}{g_2^2}\big) = \mathcal{O}\big(M \cdot \tfrac{M}{N}\big) \ , \qquad\qquad \tfrac{\partial}{\partial \log\mu}\big(\tfrac{8\pi^2}{g_1^2} - \tfrac{8\pi^2}{g_2^2}\big) = 6M\left(1 + \mathcal{O}\big(\tfrac{M}{N}\big)^2\right) \ . \tag{12.51}$$

We can compute the same runnings in gravity. The gauge couplings are identified through

$$\frac{8\pi^2}{g_1^2} + \frac{8\pi^2}{g_2^2} = \frac{2\pi}{g_s e^\phi} \ , \qquad \frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = \frac{2}{g_s e^\phi}\left(\frac{1}{2\pi\alpha'}\int_{S^2} B_2 - \pi \pmod{2\pi}\right) \ . \tag{12.52}$$

These formulae can be understood as follow. The sum (or average) of the inverse gauge couplings is the inverse gauge coupling on a D3-brane; this makes sense if we go on the Coulomb branch. The sum of the two expressions gives

$$\frac{8\pi^2}{g_1^2} = \frac{1}{2\pi\alpha' g_s e^\phi}\int_{S^2} B_2 \ .$$

This is the 4d inverse coupling that one obtains for a D5-brane on a 2-cycle, from the DBI action

$$\frac{1}{g_s}\int_{\mathbb{R}^{3,1}\times S^2} e^{-\phi}\sqrt{\det(g + B_2 + F_2)}$$

when the 2-cycle is vanishing.

In the SUGRA solution the dilaton is constant, $e^\phi = 1$, and this reproduces the vanishing of $\beta_1 + \beta_2$. To compute $\beta_1 - \beta_2$, we write

$$B_2 = g_s\alpha' \, \frac{3M}{2}\, \omega_2 \log r \tag{12.53}$$

as a potential for F_3, and identify the scale $\log\mu$ with $\log r$. This reproduces exactly

$$\beta_1 - \beta_2 = 6M \tag{12.54}$$

at leading order in N.

The chiral anomaly. For $M \neq 0$, $U(1)_R$ becomes anomalous.[68] Very roughly, the non-trivial β-function contributes to the trace anomaly $\langle T^\mu_\mu \rangle$, and by supersymmetry this is related to the chiral anomaly $\partial^\mu J^R_\mu$.

In field theory the chiral anomaly is one-loop exact and given by

$$\partial^\mu J^R_\mu = \frac{1}{16\pi^2}\,\mathcal{A}_{Rab}\,F^a_{\mu\nu}\widetilde{F}^{b\mu\nu} \qquad\qquad \widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{12.55}$$

with the anomaly $\mathcal{A}$ following from triangle diagrams:

$$\mathcal{A}_{Rab} = \mathrm{Tr}_{\text{fermions}}\; R\,T^a\,T^b\ . \tag{12.56}$$

The simple computation in the $SU(N+M)\times SU(N)$ KW theory gives

$$\partial^\mu J^R_\mu = \frac{M}{16\pi^2}\Big(F^a_{\mu\nu}\widetilde{F}^{a\mu\nu} - G^a_{\mu\nu}\widetilde{G}^{a\mu\nu}\Big)\ , \tag{12.57}$$

where F and G are the field strengths for the two groups, respectively.

From Noether's theorem, if we perform an R-symmetry rotation by $e^{i\epsilon}$ and then a "gauge" transformation $A^R_\mu \to A^R_\mu + \partial_\mu\epsilon$ for an external gauge field coupled to the R-symmetry, the action changes by

$$\delta S = \int d^4x\, J^\mu_R\,\delta A^R_\mu = -\int d^4x\,\epsilon\,\partial^\mu J^R_\mu\ . \tag{12.58}$$

Because of the anomaly, this is a shift of the theta angles in the theory, and therefore it is not a symmetry. The θ-angle terms are

$$S \;\supset\; \int d^4x\left(\frac{\theta_1}{32\pi^2}F^a_{\mu\nu}\widetilde{F}^{a\mu\nu} + \frac{\theta_2}{32\pi^2}G^a_{\mu\nu}\widetilde{G}^{a\mu\nu}\right)\ , \tag{12.59}$$

thus an R-symmetry rotation induces a shift of the θ-angles

$$\theta_1 \to \theta_1 + 2M\epsilon\ , \qquad\qquad \theta_2 \to \theta_2 - 2M\epsilon \tag{12.60}$$

and it is not a symmetry. However, since $\theta_i \cong \theta_i + 2\pi$, a residual discrete subgroup of $U(1)_R$ remains unbroken:

$$U(1)_R \quad\to\quad \mathbb{Z}_{2M}\ .$$

How does this anomaly appear in supergravity? Although F_3 is invariant under $U(1)_R$ rotations, its RR potential C_2 cannot be. We can choose, for instance,

$$F_3 = \alpha'\frac{M}{2}\omega_3 \qquad \Rightarrow \qquad C_2 = \alpha'\frac{M}{2}\,\psi\,\omega_2\ . \tag{12.61}$$

[68] We stress that this is an R-gauge-gauge anomaly that spoils the symmetry, not a 't Hooft R-R-R anomaly as we computed in $\mathcal{N}=4$ SYM.

Of course C_2 is not a gauge invariant (it is a gauge potential), however

$$\frac{1}{2\pi\alpha'}\int_{S^2} C_2 = M\,\psi \tag{12.62}$$

is a gauge-invariant (recall $\delta C_2 = d\lambda_1$), but it is not invariant under shifts of ψ. Such a gauge invariant is an axion, with period 2π, thus

$$\psi \;\to\; \psi + \frac{4\pi}{2M} \tag{12.63}$$

is a symmetry, and since $\psi \cong \psi + 4\pi$ we conclude that there is a residual $\mathbb{Z}_{2M}$ discrete symmetry.

In fact we can see the parallel with FT even better. The axion is the bulk mode dual to the operator sourced by the difference of the theta angles, while the other RR axion is dual to the operator sourced by the sum:[69]

$$\frac{1}{\pi\alpha'}\int_{S^2} C_2 = \theta_1 - \theta_2 \;, \qquad\qquad C_0 \sim \theta_1 + \theta_2 \;. \tag{12.65}$$

We see that a shift $\delta\psi = 2\epsilon$ leaves $\theta_1 + \theta_2$ fixed, while shifting

$$\theta_1 - \theta_2 \;\to\; \theta_1 - \theta_2 + 4M\epsilon \;, \tag{12.66}$$

as in the FT calculation.

The picture we gave is in 10 dimensions, and the anomaly appears as a spontaneous breaking in the internal directions: the solution is not invariant. How does that appear in AdS_5? Global symmetries on the boundary are gauge symmetries in the bulk, and of course a gauge symmetry cannot be anomalous. Indeed, what happens is that, because of the spontaneous breaking, the gauge field get massive by Higgs mechanism in the bulk. Thus

$$\text{Anomalous symmetry on the boundary} \qquad \Leftrightarrow \qquad \text{Massive vector in the bulk} \;\;.$$

The computation can be found in [HKO02]. Since the vector is massive, the dual current operator gets an anomalous dimension $\Delta > 3$. AdS/CFT allows to compute such a dimension:

$$\Delta(J_\mu) = 2 + \sqrt{1 + m^2R^2} \;\simeq\; 3 + \frac{(g_sM)^2}{\pi g_sN} \;. \tag{12.67}$$

[69]Notice that these modes are constant in AdS_5, consistently with the fact that the dual operators, $F^a_{\mu\nu}\widetilde{F}^{a\mu\nu} \pm G^a_{\mu\nu}\widetilde{G}^{a\mu\nu}$ have dimension 4. The operator map follows from the WZ D-brane action

$$S_{\mathrm{WZ}} = \int_{\mathrm{D}p} e^{2\pi\alpha'(\mathcal{F}_2 + B_2)} \sum\nolimits_{\mathrm{RR}} C_{\mathrm{RR}} \;. \tag{12.64}$$

Cascading RG flow. The $SU(N+M) \times SU(N)$ theory is non-conformal. It has a very peculiar RG flow, that can be understood in terms of a "cascade" of Seiberg dualities.

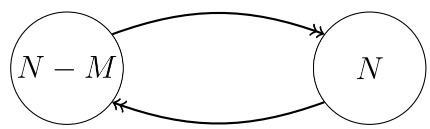
We have computed the β-functions (both in FT and SUGRA) at leading order:

$$\frac{\partial}{\partial \log \mu} \frac{8\pi^2}{g_1^2} = 3M \;, \qquad\qquad \frac{\partial}{\partial \log \mu} \frac{8\pi^2}{g_2^2} = -3M \;. \tag{12.68}$$

Thus the gauge group $SU(N+M)$ goes towards strong coupling in the IR, while $SU(N)$ goes towards weak coupling. At a scale

$$\Lambda = \mu \, e^{-\frac{1}{3M} \frac{8\pi^2}{g_1^2(\mu)}} \tag{12.69}$$

the group $SU(N+M)$ is strongly coupled. Since $N_c = N+M$ and $N_f = 2N$, we can use a Seiberg-dual description: $\widetilde{N}_c = N_f - N_c = N - M$. Working out the details (and integrating out massive fields) we obtain



and the conifold superpotential is reproduced. This is the same theory has before, but with

$$N \;\to\; N - M \;.$$

Now the role of the gauge groups gets exchanged: $SU(N)$ goes towards strong coupling and $SU(N-M)$ towards weak coupling, until we perform another duality on $SU(N) \to SU(N-2M)$, and we keep going. In fact this process continues indefinitely in the UV.[70]

The "running" of the gauge ranks appears in supergravity as the fact that

$$\int_{T^{1,1}} F_5 \text{ is not constant} \;,$$

and in particular

$$N_{\text{eff}}(r) = N + \frac{3}{2\pi} g_s M^2 \log \frac{r}{r_0} \;. \tag{12.70}$$

The precise coefficients, expressing the "RG distance" between one duality and the next, can be successfully compared with field theory.[71]

[70]This peculiar RG flow is a clever way that the theory has to remain strongly-coupled at all energies, even though the couplings run and so one would expect them to reach weak coupling at some point (either UV or IR). This would necessarily make the supergravity approximation break down.

[71]More precisely, the integer ranks should be compare with the Page charge $\int_{T^{1,1}} dC_4$, which is integer and it jumps discontinuously.

12.6 Chiral symmetry breaking, confinement, and the Klebanov-Strassler solution

It is clear that the cascading RG flow can continue indefinitely in the UV, but not in the IR. In supergravity, there is a naked singularity in the IR (small r) where either $N_{\text{eff}}(r)$ or $h(r)$ become negative. This signals that new physics is needed in the IR.

Luckily, it turns out that the resolution is possible within the supergravity approximation!

Let us first understand the physics (explained in depth in [Str05]). Suppose that towards the end of the cascade we reach

$$SU(2M) \times SU(M) \,.$$

Now $SU(2M)$ goes to strong coupling. However this time $N_f = N_c$, and there is no Seiberg duality in this case. Instead, the theory confines, mesons $\mathcal{M}$ and baryons $\mathcal{B}$, $\widetilde{\mathcal{B}}$ become the fundamental fields, but they are subjected to a quantum deformed constraint

$$\det \mathcal{M} - \mathcal{B}\widetilde{\mathcal{B}} = \Lambda^{2N_c} \,. \tag{12.71}$$

The constraint can be imposed by a Lagrange multiplier X in the superpotential. The mesons are $\mathcal{M}_{ji} = B_j A_i$. Thus the superpotential is

$$W = -\operatorname{Tr}\left(\mathcal{M}_{11}\mathcal{M}_{22} - \mathcal{M}_{12}\mathcal{M}_{21}\right) + X\bigg(\det\begin{pmatrix}\mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22}\end{pmatrix} - \mathcal{B}\widetilde{\mathcal{B}} - \Lambda^{4M}\bigg) \,. \tag{12.72}$$

Because of the first term, the mesons are massive and can be integrated out. The baryons get a VEV, $\mathcal{B}\widetilde{\mathcal{B}} = -\Lambda^{4M}$, and leave a flat direction. Since the baryons are neutral under $SU(M)$, we are left with pure $SU(M)$ SYM. This theory has gaugino condensation, chiral symmetry breaking

$$\mathbb{Z}_{2M} \quad \to \quad \mathbb{Z}_2 \,,$$

M inequivalent vacua and confinement.

Thus, the IR physics we were missing is *chiral symmetry breaking*. One can include such an ingredient by substituting the conifold by the *deformed conifold*:[72]

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \varepsilon^2 \,. \tag{12.73}$$

The parameter ε breaks the R-symmetry, that rotates the variables z_a, to $\mathbb{Z}_2$. The deformed conifold is still Calabi-Yau. It is not a cone, rather it is a smooth three-fold with a finite-size S^3 at the tip:

figure

The Klebanov-Strassler solution has metric

$$ds_{10}^2 = h(\tau)^{-1/2} dx_{3,1}^2 + h(\tau)^{1/2} ds_6^2(\text{def conifold}) \,, \tag{12.74}$$

[72] A better way to motivate the deformed conifold is to study the moduli space of $SU(M+1) \times SU(1)$.

3-form flux F_3 on S^3, H_3 along $dr \wedge d\mathrm{vol}_{S^2}$, and it asymptotes the KT solution far from the tip. However it is everywhere smooth, and according to our discussion on confinement, it gives confinement of flux tubes with a discrete spectrum of glueballs.

13 Response functions and AdS/CMT

We will discuss real-time correlators in AdS/CFT, transport coefficients, and how AdS/CFT can be applied to systems oriented towards condensed matter problems. Good reviews are [Her09, Har09].

13.1 Spectral functions and transport coefficients

Suppose we have a system described by the (possibly time-dependent) Hamiltonian $\widetilde{H}(t)$. We perturb the system by external sources $\phi_i(t,x)$ coupled to a set of operators $\mathcal{O}_i(t,x)$. The Hamiltonian is modified by a term

$$\delta H = -\int dt\, d^{d-1}x\, \phi_i(t,x)\, \mathcal{O}_i(t,x) \;. \tag{13.1}$$

Notice that by x we mean the spatial components. Under the assumption that we keep fixed the states in the far past, we ask what is the effect on the expectation values $\langle \mathcal{O}_j(t,x)\rangle$.

To answer, it is convenient to do the computation in the Schrödinger picture. The evolution of states is controlled by the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \widetilde{H}(t)\,|\psi(t)\rangle \;, \tag{13.2}$$

where we use $\widetilde{\ }$ for operators in the Schrödinger picture. This is solved by the unitary time-evolution operator

$$U(t,t_0) = \mathrm{T}\, e^{-i\int_{t_0}^{t}\widetilde{H}(t')\,dt'} \qquad \text{such that} \qquad \frac{\partial}{\partial t}U(t,t_0) = -i\,\widetilde{H}(t)\,U(t,t_0) \;. \tag{13.3}$$

The expectation value of an operator $\mathcal{O}(t,x)_i$ is

$$\langle \mathcal{O}_i(t,x)\rangle = \mathrm{Tr}\,\rho(t)\,\widetilde{\mathcal{O}}_i(t,x) = \mathrm{Tr}\,U(t,t_0)\,\rho_0\,U(t,t_0)^{-1}\,\widetilde{\mathcal{O}}_i(t,x) = \mathrm{Tr}\,\rho_0\,\mathcal{O}_i(t,x) \;, \tag{13.4}$$

where ρ is a density matrix (evolving with $i\partial_t\rho = [\widetilde{H},\rho]$) and ρ_0 is the density matrix at the far-past time t_0. Operators in the Heisemberg picture are related to those in the Schrödinger picture by the unitary transformation

$$\mathcal{O}(t,x) = U(t)^{-1}\,\widetilde{\mathcal{O}}(t,x)\,U(t) \;. \tag{13.5}$$

The Heisenberg picture is the one used in QFT.[73]

[73] In the Schrödinger picture states evolve with time, while operators $\widetilde{\mathcal{O}}$ do not (but can have an explicit dependence on time, like $\widetilde{H}$). In the Heisenberg picture, instead, states remain constant while operators $\mathcal{O}$ evolve. Things get a bit confusing, though, if the Hamiltonian depends explicitly on time, since $\widetilde{H}(t) \neq H(t)$ but the evolution operator is constructed with $\widetilde{H}$.

Now we separate

$$\widetilde{H}(t) = \widetilde{H}_0(t) + \delta\widetilde{H}(t) \ .$$

The evolution operator for $\widetilde{H}$ is (as can be checked by computing ∂_t)

$$U(t) = U_0 \,\cdot\, \mathrm{T}\exp\Big\{ -i \int^t U_0(t')^{-1}\, \delta\widetilde{H}(t')\, U_0(t')\, dt' \Big\} = U_0 \,\cdot\, \mathrm{T}\, e^{-i\int^t \delta H(t')\, dt'} \ , \tag{13.6}$$

where $U_0(t, t_0)$ is the time evolution of $\widetilde{H}_0$. Expanding at first order we find

$$\begin{aligned} \delta\langle \mathcal{O}_i(t,x)\rangle &= -i \,\mathrm{Tr}\, \rho_0 \int^t dt' \, \big[U_0(t)^{-1}\, \widetilde{\mathcal{O}}_i(t,x)\, U_0(t) \,,\, \delta H(t') \big] \\ &= i \int^t dt' \int d^{d-1}x' \,\Big\langle \big[\mathcal{O}_i(t,x) \,,\, \mathcal{O}_j(t',x') \big] \Big\rangle\, \phi_j(t',x') \ . \end{aligned} \tag{13.7}$$

Notice that time integration is only over $t' < t$.

Thus, the causal effect of the perturbation is controlled by the *retarded Green's function*

$$G_R^{ij}(t,x,\, t',x') = i\,\theta(t-t') \Big\langle \big[\mathcal{O}_i(t,x) \,,\, \mathcal{O}_j(t',x') \big] \Big\rangle \ , \tag{13.8}$$

where θ is the Heaviside step function. Using translational invariance, in momentum space we have

$$G_R^{ij}(\omega,k) = \int dt\, d^{d-1}x\, e^{i\omega t - ikx}\, i\theta(t) \Big\langle \big[\mathcal{O}_i(t,x) \,,\, \mathcal{O}_j(0,0) \big] \Big\rangle \ . \tag{13.9}$$

Then the causal effect of a perturbation is simply

$$\delta\langle \mathcal{O}_i(\omega,k)\rangle = G_R^{ij}(\omega,k)\, \phi_j(\omega_k) + O(\phi^2) \ . \tag{13.10}$$

Retarded Green's functions have interesting properties.

- If the operators $\{\mathcal{O}_i\}$ are Hermitian, it easily follows

$$G_R^{ij}(t,x)^* = G_R^{ij}(t,x) \ , \qquad\qquad G_R^{ij}(\omega,k)^* = G_R^{ij}(-\omega,-k) \ . \tag{13.11}$$

- If the system is invariant under time reversal, and the operators $\{\mathcal{O}_i\}$ transform as $T\mathcal{O}_i(t,x)T^{-1} = \epsilon_i \mathcal{O}_i(-t,x)$ with $\epsilon_i = \pm 1$, then

$$G_R^{ij}(t,x)^* = \epsilon_i\, \epsilon_j\, G_R^{ji}(t,-x) \ . \tag{13.12}$$

  This follows because T is an anti-unitary operator:

$$\Big\langle \big[\mathcal{O}_i(t,x), \mathcal{O}_j(0,0) \big] \Big\rangle = \Big\langle T \big[\mathcal{O}_i(t,x), \mathcal{O}_j(0,0) \big] T^{-1} \Big\rangle^* = -\epsilon_i\epsilon_j \Big\langle \big[\mathcal{O}_j(t,-x), \mathcal{O}_i(0,0) \big] \Big\rangle \ .$$

  Combined with Hermiticity of $\{\mathcal{O}_i\}$ one gets

$$G_R^{ij}(\omega,k) = \epsilon_i\, \epsilon_j\, G_R^{ji}(\omega,-k) \ . \tag{13.13}$$

  If time reversal is broken, for instance by a magnetic field B, the property is still true except that one of the two sides is evaluated in the flipped background (*e.g.* $B \to -B$). At zero momentum k the relation is called Onsager relation.

- Retarded Green's functions are causal, in the sense that a perturbation only affects later times and so $G_R^{ij}(t,x)$ vanishes for $t<0$. Consider

$$G_R(t,k) = \int d\omega\, e^{-i\omega t}\, G_R(\omega,k) \ .$$

For $t<0$ we can close the contour in the upper half-plane, and the fact that we get zero means that $G_R(\omega,k)$ is analytic (no poles) in the upper half-plane.

From this fact, if moreover $G_R(\omega)$ vanishes for $|\omega|\to\infty$, one obtaines the Kramers-Kronig relations[74]

$$\mathbb{R}\mathrm{e}\, G_R(\omega) = P\int_{-\infty}^{\infty} \frac{d\omega'}{\pi}\, \frac{\mathbb{I}\mathrm{m}\, G_R(\omega)}{\omega'-\omega} \ , \qquad \mathbb{I}\mathrm{m}\, G_R(\omega) = -P\int_{-\infty}^{\infty} \frac{d\omega'}{\pi}\, \frac{\mathbb{R}\mathrm{e}\, G_R(\omega)}{\omega'-\omega} \ . \tag{13.15}$$

- Retarded Green's function satisfy positivity properties. One can show that

$$-\, i\omega\big(G_R^{ij}(\omega) - G_R^{ji}(\omega)^*\big) \ \text{is def} \ \geq 0 \ . \tag{13.16}$$

The anti-Hermitian part of the retarded Green's function is called the *spectral function*.

If there is a unique operator involved, the spectral function satsfies

$$\omega\ \mathbb{I}\mathrm{m}\, G_R(\omega) \geq 0 \ . \tag{13.17}$$

Retarded Green's functions are directly related to *transport coefficients*. Take the example of Ohm's law, which defines the electric conductivity. It states that for an electric field that is constant in space ($k=0$) but oscillating in time with frequency ω, the spatial part of the charge current response is given by

$$J_i(\omega) = \sigma_{ij}(\omega)\, E_j(\omega) \ . \tag{13.18}$$

(Here ij are spatial indices.) Here $\sigma_{ij}(\omega)$ is called *optical conductivity*.

In our language, ϕ_i is an external vector potential A_μ while $\mathcal{O}_i$ is the conserved current J_μ. We take a gauge where $A_t=0$, then $E_i=-\partial_t A_i$. Making a Fourier decomposition $A_i\sim e^{-i\omega t}$, we have

$$E_j = i\omega A_j \ .$$

[74] They follow from

$$G_R(z) = \oint_\Gamma \frac{d\zeta}{2\pi i}\, \frac{G_R(\zeta)}{\zeta - z} \qquad \text{for } \mathbb{I}\mathrm{m}\, z>0 \tag{13.14}$$

where Γ is within $\mathbb{I}\mathrm{m}\,\zeta>0$, right above the real axis and then closed in the upper half-plane. Then one takes the limit $z\to\omega+i0$.

We see that the optical conductivity is related to the current-current correlator:

$$\sigma_{ij}(\omega) = \frac{G_R^{ij}(\omega, 0)}{i\omega} \; . \tag{13.19}$$

By considering correlators of the current and the stress tensor, one can similarly construct a matrix of transport coefficients that include also the heat conductivity and the thermoelectric coefficients.

This is good new for AdS/CFT, because two-point functions is precisely one of the things that we can easily compute with AdS/CFT in strongly-coupled theories.

13.2 Real-time correlators from AdS/CFT

We have learned how to compute generic correlators in Euclidean signature. To compute transport coefficients we need two-point functions in Lorentz signature. In principle, they are related by analytic continuation, but if we have a numerical result we don't really know how to do the continuation. We should then understand how to directly compute correlators in real time.

At the boundary there is not much difference. The EOM for a scalar in AdS_{d+1} gives asymptotic behavior

$$\Phi(z,t,x) = e^{-i\omega t + ikx} \Big(z^{d-\Delta} \phi_0 \left(1 + O(z)\right) + z^\Delta \phi_1 \big(1 + O(z)\big) \Big) \; . \tag{13.20}$$

We impose Dirichelet boundary conditions for one of the two modes, typically ϕ_0. This makes ϕ_0 be identified with the source and ϕ_1 with the VEV.

Things are quite different in the interior, *i.e.* at the horizon. For instance, in AdS the two exact solutions for time-like k^μ (*i.e.* on-shell, since $\omega^2 > \vec{k}^2$) are

$$z^{d/2}\, K_{\pm\nu}(iqz) \;, \qquad\qquad q = \sqrt{\omega^2 - \vec{k}^2} > 0 \;, \qquad \nu = \Delta - \frac{d}{2} \; .$$

They behave as

$$e^{\pm iqz} \qquad\qquad \text{for } z \to \infty \; .$$

They are both regular at the horizon. This is because there are many different real-time Green's functions we can construct in QFT.

To construct *retarded* Green's functions we have to choose *infalling* boundary conditions, *i.e.* energy should move towards larger z as time passes, thus falling inside the horizon. It can be motivated in three ways:

- Such boundary conditions correspond to a causal behavior.
- Such boundary conditions reproduce the correct analytic structure of $G_R(\omega)$, *i.e.* no poles in the upper half-plane.

- They can be derived from a holographic version of the Schwinger-Keldysh prescription.

Infalling boundary conditions break time reversal. In the example of AdS, we should choose the solution that behaves like $e^{-i\omega t+iqz}$ (for $\omega > 0$), that is

$$e^{-i\omega t+ikx} z^{d/2} K_{+\nu}(iqz) \ .$$

More generally, we should

Impose regular boundary conditions at future horizons.

A future horizon is a null surface beyond which events cannot causally propagate back to the boundary. On a future horizon, regularity requires that modes are infalling.

For a horizon at non-zero temperature located at $z = z_+$, the standard black hole metric looks like[75]

$$ds^2 = -f(z)\, dt^2 + \frac{dz^2}{f(z)} \qquad\qquad f(z) \simeq 4\pi T(z_+ - z) \tag{13.21}$$

and g_{tt} has a simple zero at the horizon. To analyze the modes around the horizon, we can use Kruskal coordinates

$$\rho \pm \tau = e^{\frac{1}{2}\log(z_+ - z) \pm 2\pi T t} \ , \tag{13.22}$$

which bring the metric to the simple form $ds^2 = (\pi T)^{-1}(-d\tau^2 + d\rho^2)$. The future horizon is at $\rho = \tau > 0$, the past horizon is at $\rho = -\tau > 0$. The solutions to the massless Klein-Gordon equation are $f_+(\rho+\tau)$ and $f_-(\rho-\tau)$. Imposing the time dependence $\Phi \sim e^{-i\omega t}$ we find

$$f_+ = (\rho+\tau)^{-i\omega/2\pi T} \ , \qquad\qquad f_- = (\rho-\tau)^{i\omega/2\pi T} \ . \tag{13.23}$$

The solution f_- is singular at the future horizon, while f_+ is regular. This singles out f_+, and therefore the infalling mode behaves as

$$\phi(z) \sim e^{-\frac{i\omega}{4\pi T}\log(z_+ - z)} \ . \tag{13.24}$$

For a zero-temperature horizon located at $z = z_+$, the component g_{tt} has a double zero and the metric looks like AdS_2:

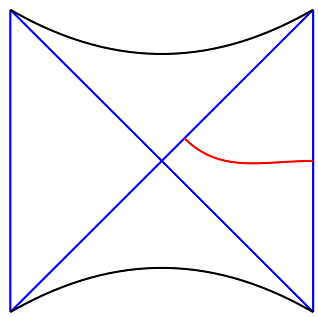
$$ds^2 \simeq -\frac{(z_+ - z)^2}{R^2}\, dt^2 + \frac{R^2}{(z_+ - z)^2}\, dz^2 \ . \tag{13.25}$$

This is mapped to our standard AdS metric by $z_+ - z = r = R^2/\zeta$. We already saw that infalling modes behaves as $e^{i\omega\zeta}$ (for zero momentum), thus

$$\phi(z) \sim e^{i\omega R^2/(z_+ - z)} \ . \tag{13.26}$$

[75]Go to Euclidean time and perform the coordinate change $\rho^2 = (z_+ - z)/\pi T$. The metric becomes approximately $ds^2 \simeq d\rho^2 + 4\pi^2 T^2 dt^2$, which is smooth $\mathbb{R}^2$ if $t \cong t + T^{-1}$.

We fix the boundary conditions on the future horizon because a space-like surface terminating on a future horizon provides a good Cauchy surface from which to evolve initial data. A space-like surface ending on a past horizon does not: one would need to specify what is coming out of the "white hole".



Furthermore, the presence of a future horizon allows energy to be lost behind the horizon, which is the holographic manifestation of dissipation.

At this point, the two-point functions are computed in the standard way:

$$G_R^{\phi\phi}(\omega, k) = \frac{\delta\phi_1}{\delta\phi_0}\Big|_{\omega,k} . \tag{13.27}$$

13.3 A setup for condensed matter problems

We have studied in some detail the solutions to Einstein gravity with negative cosmological constant Λ. The simplest solution is

$$\mathrm{AdS}_{d+1} ,$$

representing a CFT in its conformal vacuum. A more interesting solution is

$$\text{Schwarzschild BH in } \mathrm{AdS}_{d+1} ,$$

representing the same CFT but in a thermal ensemble.

To do more, we need to add other fields. It is common in condensed matter systems to have a $U(1)$ symmetry.

The most common situation is that it is the electromagnetic $U(1)$. Of course that is a gauge symmetry, while AdS/CFT describes *global* symmetries on the boundary. However in many condensed matter systems the photon can be thought of as non-dynamical (in the low-energy effective theory).

1. The electromagnetic coupling is small.

2. Electromagnetic interactions are usually screened in a charged medium, thus the effective theory usually does not contain photons.

Thus, we consider bulk theories with a $U(1)$ gauge field, dual to CFTs with a $U(1)$ global symmetry.

This allows us to obtain *background* electric and magnetic fields on the boundary, as well as to add a chemical potential μ that induces a charge (and particle) density ρ.

For instance, consider the following model in AdS$_4$:

$$S_{(4)} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\Big(\mathcal{R} + \frac{6}{L^2}\Big) - \frac{1}{4g^2}\int d^4x\,\sqrt{-g}\,F_{MN}F^{MN}\;. \tag{13.28}$$

One obvious solution is AdS$_4$ with radius L. This is dual to the conformally-invariant vacuum of some $2+1$ dimensional CFT.

More interesting solutions are the dyonic black membranes, *i.e.* black holes with temperature, electric and magnetic charges, that span $\mathbb{R}^{2,1}$. The metric takes the familiar form

$$ds^2 = L^2\bigg[\frac{dz^2}{z^2 f(z)} + \frac{1}{z^2}\Big(-f(z)\,dt^2 + dx^2 + dy^2\Big)\bigg] \tag{13.29}$$

with

$$f(z) = 1 + (h^2+q^2)\alpha\frac{z^4}{z_h^4} - \big(1+(h^2+q^2)\alpha\big)\frac{z^3}{z_h^3}\;, \qquad \alpha = \frac{\kappa^2 z_h^2}{2g^2L^2}\;, \tag{13.30}$$

and there is also a non-trivial electro-magnetic field

$$A = \frac{h}{z_h}\,x\,dy - q\Big(1-\frac{z}{z_h}\Big)dt\;. \tag{13.31}$$

These are closed cousins of the Reissner-Nordström black hole. The boundary is at $z=0$ and the metric is asymptotically AdS$_4$, while there is a horizon at $z=z_h$. An integration constant in A is chosen in such a way that $A_t(z_h)=0$, hence $A_MA_Ng^{MN}<\infty$ is regular.

As we learned, regularity of the Euclidean solution at the horizon gives the temperature (both of the BH and of the CFT):

$$T = \frac{3-(h^2+q^2)\alpha}{4\pi z_h}\;. \tag{13.32}$$

The boundary value of the spatial components of the field strength give a magnetic field F_{xy}:

$$F_{xy}(z=0) = \frac{h}{z_h}\;. \tag{13.33}$$

Therefore the magnetic charge of the BH corresponds to a constant magnetic field in the boundary theory. To extract the charge density, we analyze the asymptotic behavior of the gauge field:

$$A_\mu = a_\mu + b_\mu z + \ldots \tag{13.34}$$

The first constant is the source:

$$a_t = -q \equiv \mu \qquad\qquad \text{chemical potential } \mu\;. \tag{13.35}$$

The second constant is, up to a proportionality constant that follows from differentiating the action, the VEV:

$$b^t = g^2 \langle J^t \rangle \equiv g^2 \rho = -\frac{q}{z_h} = \frac{\mu}{z_h} \qquad\qquad \text{charge density } \rho \; . \tag{13.36}$$

Therefore the electric field of the BH corresponds to a chemical potential, which induces a charge density in the system.

13.4 The holographic superconductor

We have discussed scalar fields ϕ in the bulk. They can be dual to relevant operators, which can be used to trigger RG flows. If these fields are charged — meaning that the dual operators are charged under the global $U(1)$ symmetry on the boundary — they can become *order parameters* for broken symmetries.

This is very interesting because symmetry breaking is at the heart of CM physics (recall that the Higgs mechanism has been discovered by Anderson, a CM physicist, first).

We consider a simple model in which we add a charged scalar ϕ to the Einstein-Maxwell theory discussed before:

$$S_\phi = -\int d^4x \, \sqrt{-g} \left(|D\phi|^2 - 2\frac{|\phi|^2}{L^2} \right) , \qquad\qquad D = \partial - iA \; . \tag{13.37}$$

In general we could take a potential $V\big(|\phi|\big)$: we have taken the simplest choice, a mass term. The mass has been chosen arbitrarily (to do numerics we have to choose one), such that

$$m^2L^2 = -2 \; , \qquad\qquad \Delta_+ = 2 \; , \qquad \Delta_- = 1 \; . \tag{13.38}$$

We are in the range of double quantization, therefore this model can describe an order parameter of dimension 2 or 1. We set

$$h = 0 \tag{13.39}$$

for simplicity: no magnetic field. But we do turn on temperature T and chemical potential μ.

The central observation is that the charge density acts as an effective z-dependent negative contribution to the mass of the scalar:

$$m_{\text{eff}}(z) = m^2 + g^{tt} A_t^2 = m^2 - \frac{q^2}{L^2} \frac{z^2}{f(z)} \big(1 - z/z_h \big)^2 \; . \tag{13.40}$$

When q is large enough, the mass becomes too negative, there is an instability and the scalar develops a non-trivial profile. However the effective mass vanishes at $z = z_h$ and $z = 0$, therefore the runaway direction is "stabilized by the curvature".

We can make the simplifying assumption

$$\kappa^2 \ll g^2 L^2 \ . \tag{13.41}$$

This is the weak gravity or *probe limit*: the gauge and scalar field do not have enough energy to curve spacetime, and we can study their dynamics on a fixed background. The metric simplifies to

$$f(z) = 1 - \frac{z^3}{z_h^3} \ . \tag{13.42}$$

One can write down the EOMs for ϕ and A_t, choosing a gauge where $\phi \in \mathbb{R}$:

$$z^2 \Big(\frac{f\,\phi'}{z^2} \Big)' = \Big(\frac{m^2}{z^2} - \frac{A_t^2}{f} \Big) \phi \ , \qquad\qquad A_t'' = \frac{2g^2}{z^2 f} \phi^2 A_t \ . \tag{13.43}$$

The boundary conditions at the boundary are

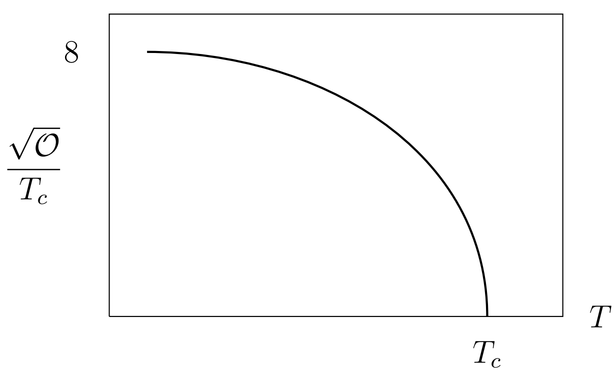
$$A_t = \mu - g^2 \rho\, z + \dots \ , \qquad\qquad \phi = a\, z + b\, z^2 + \dots \qquad\qquad \text{for } z \to 0 \ . \tag{13.44}$$

In the canonical ensemble we keep the density ρ fixed (and let μ be determined dynamically). If we choose $\mathcal{O}$ of dimension $\Delta = 2$, we insist that there is no source, $a = 0$. At the horizon we impose $\phi < \infty$ and $A_t = 0$ for regularity.

The equations have to be solve numerically. For T larger than a critical value T_c (proportional to $\rho^{1/2}$) the only solution is[76]

$$\phi = 0 \ , \qquad\qquad A_t = g^2 \rho (1 - z) \ . \tag{13.45}$$

At T_c, ϕ develops a normal mode: the linearized EOMs have a solution with no source. For $T < T_c$, there is a solution of the non-linear equations with non-trivial profile for ϕ (and it turns out that this solution has lower free energy):



The model has a second-order phase transition at T_c. The behavior of the condensate is

$$\langle \mathcal{O} \rangle \sim (T_c - T)^{1/2} \tag{13.46}$$

with the classical $\frac{1}{2}$ mean-field exponent of Landau-Ginzburg theory.

[76]Here we have set $L = 1$ and $z_h = 1$. All physical quantities depend on ρ/T^2.

13.5 Conductivity

To compute the conductivity, we need to study the linearized equations of motion for a fluctuation of the gauge field in the background of the condensate. The equation is

$$\left(f\,A_x'\right)' - \frac{\omega^2}{f}A_x = \frac{2g^2}{z^2}\phi^2 A_x\ . \tag{13.47}$$

The boundary conditions near the boundary are

$$A_x = \frac{E_x}{i\omega} + \frac{g^2}{\alpha}J^x z + \dots \tag{13.48}$$

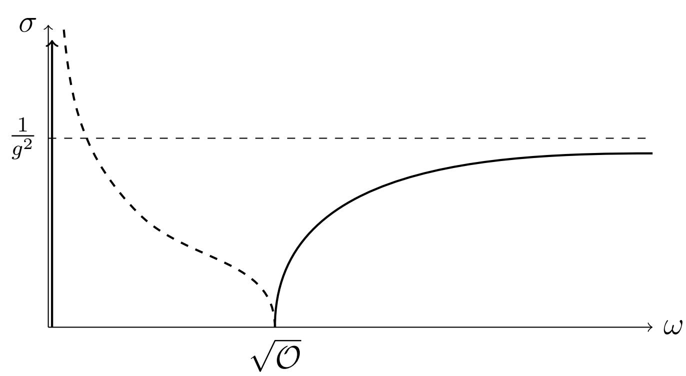
while at the horizon we impose infalling boundary conditions. Finally

$$\sigma_{xx}(\omega) = \left.\frac{J^x}{E_x}\right|_{\omega}\ . \tag{13.49}$$

Unfortunately the equations cannot be solved analytically, have to be solved numerically.

For $T > T_c$ the optical conductivity is constant and equal to $\sigma_{xx}(\omega) = 1/g^2$. The fact that it does not depend on ω is an artifact of the probe approximation.

For $T \ll T_c$ one finds, qualitatively:



For large frequencies, $\mathbb{R}\mathrm{e}\,\sigma_{xx}$ approaches the normal-phase value, but at low frequencies there is a "gap". This is interpreted as a gap in the Fermi surface, and the fact that to conduce one needs to break a Cooper pair. Thus the gap is associated to twice the gap in the Fermi surface. At $\omega = 0$ there is a delta function, characteristic of a superconductor, which is derived from the $1/\omega$ behavior of the imaginary part and the Kramers-Kronig relations. This shape is qualitatively similar to the textbook one of BCS superconductors, however the gap is much larger than in the BCS case.

14 Entanglement entropy

A good review is hep-th/0905.0932 by Nishioka, Ryu, Takayanagi [NRT09]. Consider a quantum mechanical system with many degrees of freedom, such as a spin chain, a lattice model or a QFT.

In general, we could consider pure states $|\psi\rangle$ or mixed states described by density matrices

$$\rho = \sum_i |\psi_i\rangle\langle\psi_i| \qquad \text{with} \qquad \operatorname{Tr}\rho = 1 \,. \tag{14.1}$$

Notice that ρ is Hermitian and positive definite, and can be diagonalized in an orthonormal basis $|\widetilde{\psi}_j\rangle$:

$$\rho = \sum_j \rho_j |\widetilde{\psi}_j\rangle\langle\widetilde{\psi}_j| \qquad \text{with} \qquad 0 \leq \rho_j \leq 1 \,, \qquad \textstyle\sum_j \rho_j = 1 \,. \tag{14.2}$$

Here ρ_j are probabilities. Density matrices can also describe pure states, if

$$\rho = |\psi\rangle\langle\psi| \tag{14.3}$$

for some (normalized) state $|\psi\rangle$. Expectation values are computed by

$$\langle\mathcal{O}\rangle_\rho = \operatorname{Tr}\rho\,\mathcal{O} \,. \tag{14.4}$$

For pure states this reduces to the standard $\langle\mathcal{O}\rangle = \langle\psi|\mathcal{O}|\psi\rangle$.

A density matrix is pure if and only if it is a projector,

$$\rho^2 = \rho \,, \tag{14.5}$$

and because of the normalization condition this automatically implies that it projects to a one-dimensional subspace. Alternatively, we can compute the von Neumann entropy:

$$S = -\operatorname{Tr}\rho\log\rho \,. \tag{14.6}$$

This is ≥ 0, and zero if and only if ρ is a pure state. If $S > 0$, then e^S is a rough measure of the number of states involved in ρ.[77]

Put the system at zero temperature. Assuming no ground-state degeneracy, the system is in its ground state $|\Psi\rangle$, which is a pure state. The density matrix is

$$\rho_{\text{tot}} = |\Psi\rangle\langle\Psi| \,. \tag{14.7}$$

The von Neumann entropy of this state, $S_{\text{tot}} = -\operatorname{Tr}\rho_{\text{tot}}\log\rho_{\text{tot}} = 0$, vanishes because ρ_{tot} is a pure state.

[77] If ρ is uniformly distributed over n orthogonal states, namely $\rho = \frac{1}{n}\sum_{j=1}^n |\widetilde{\psi}_j\rangle\langle\widetilde{\psi}_j|$, then $S = \log n$.

Next we divide the system into two subsystems, A and B, in such a way that the total Hilbert space factorizes into a tensor product:

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B . \tag{14.8}$$

For instance, we divide the sites of the spin chain or the lattice model into two groups (possibly connected), or we divide the space where a local QFT lives into two regions.[78] An observer that only has access to the subsystem A will feel as if the system is described by the *reduced density matrix*

$$\rho_A = \operatorname{Tr}_B \rho_{\text{tot}} \tag{14.9}$$

where the trace is over the subsystem B.

We define the entanglement entropy of the subsystem A as the von Neumann entropy of its reduced density matrix

$$S_A = -\operatorname{Tr}_A \rho_A \log \rho_A . \tag{14.10}$$

This quantity measures how much the systems A and B are entangled in the quantum state $|\Psi\rangle$ (or how much the state is "quantum").[79]

Example. Consider two particles of spin $\frac{1}{2}$: each has two states $|\uparrow\rangle$ and $|\downarrow\rangle$. Suppose they are in the state

$$|\Psi\rangle = |\uparrow\rangle_A \otimes |\downarrow\rangle_B .$$

Then

$$\rho_A = |\uparrow\rangle_A \langle\uparrow|_A , \qquad\qquad S_A = 0 . \tag{14.11}$$

In this state the two particles are not entangled: one is clearly $|\uparrow\rangle_A$ while the other one is clearly $|\downarrow\rangle_B$.

Suppose, instead, that they are in the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B \Big) .$$

This is the state if the two particles are created from the decay of a scalar particle. This time

$$\rho_A = \frac{1}{2}\Big(|\uparrow\rangle_A\langle\uparrow|_A + |\downarrow\rangle_A\langle\downarrow|_A \Big) , \qquad\qquad S_A = \log 2 . \tag{14.12}$$

In this state the two particles are entangled: we cannot determine the state of each one separately, rather, if one is $|\uparrow\rangle$ then the other one is $|\downarrow\rangle$ and vice versa. ■

We can define the entanglement entropy in arbitrary states, not just the ground state, and also in mixed states (described by general density matrices). For instance, we can define the

[78] The case of a QFT is tricky because the Hilbert space does not really factorize [Wit18]. For this discussion we will assume it does.

[79] As explained below, this is a good measure of entanglement only if the state is pure in $\mathcal{H}_{\text{tot}}$.

entanglement entropy at finite temperature $T = \beta^{-1}$ by using for ρ_{tot} the thermal density matrix

$$\rho_{\text{thermal}} = \frac{1}{Z}\, e^{-\beta H} \qquad\qquad Z = \operatorname{Tr} e^{-\beta H} \tag{14.13}$$

where H is the Hamiltonian. When A is the total system, $S_A(\beta)$ is simply the thermal entropy. Indeed

$$\log\rho = -\beta H - \log Z \tag{14.14}$$

and so

$$-\operatorname{Tr}\rho\log\rho = \beta\operatorname{Tr}\rho H + \log Z\operatorname{Tr}\rho = \beta\langle H\rangle + \log Z = \beta\big(\langle H\rangle - \mathcal{F}\big) = S\ , \tag{14.15}$$

where $Z = e^{-\beta\mathcal{F}}$ and $\mathcal{F} = E - TS$ is the Helmholtz free energy.[80]

Two important properties of the entanglement entropy are:

- When the density matrix ρ_{tot} is pure, then

$$S_A = S_B\ . \tag{14.16}$$

  Exercise. Prove it, for a finite-dimensional Hilbert space.

- *Strong subadditivity.* Given non-intersecting subsystems A, B and C (not necessarily covering the whole system):

$$\begin{aligned} S_{A+B+C} + S_B &\leq S_{A+B} + S_{B+C} \\ S_A + S_C &\leq S_{A+B} + S_{B+C}\ . \end{aligned} \tag{14.17}$$

  These relations are quite non-trivial and we will not prove them.

14.1 Entanglement entropy in QFT

Consider a local QFT on

$$\mathbb{R}_t \times N\ , \tag{14.18}$$

where N is a d-dimensional spatial manifold. We define the subregion A as a region $A \subset N$ at fixed time t_0. We call B its complement in N.

In a local QFT, let us suppose that the Hilbert space factorizes as a tensor product

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B\ . \tag{14.19}$$

This allows us to define S_A as before.

[80]The Helmholtz free energy corresponds to the thermodynamic ensemble at constant temperature and volume. This is the correct ensemble for a QFT, in which we are supposed to work at fixed finite volume and then take the infinite volume limit at the end.

Such entanglement entropy is usually divergent in a continuum theory, and its definition requires a UV cutoff $\Lambda = \frac{1}{a}$ (a is a lattice spacing). Then the coefficient in front of the divergence is proportional to the area of the boundary ∂A:

$$S_A = \gamma \cdot \frac{\text{Area}(\partial A)}{a^{d-1}} + \text{subleading} . \tag{14.20}$$

This behavior is intuitively understood as the fact that the entanglement between A and B occurs at the boundary ∂A most strongly. The coefficient γ depends on the theory, but usually also on the renormalization scheme. This is clear: if you redefine the cutoff $a \to \lambda a$ then $\gamma \to \lambda^{d-1}\gamma$.

The behavior in 2D is different (then $d = 1$, and the boundary has dimension zero). For instance in a CFT, the entanglement entropy of an interval of lenght ℓ in $\mathbb{R}$ is [CC04]

$$S_A = \frac{c}{3} \log \frac{\ell}{a} , \tag{14.21}$$

where c is the central charge. This time the coefficient in front of the divergent term is scheme-independent, because if we redefine $a \to \lambda a$ we add a constant, but do not change the coefficient in front of $\log a$.

The form of entanglement entropy in CFTs in generic dimension is

$$S_A = p_1 \Big(\frac{\ell}{a} \Big)^{d-1} + p_3 \Big(\frac{\ell}{a} \Big)^{d-3} + \ldots + \begin{cases} p_{d-2} \Big(\dfrac{\ell}{a} \Big)^2 + \tilde{c} \log \dfrac{\ell}{a} & (d+1) \text{ even} \\ p_{d-1} \Big(\dfrac{\ell}{a} \Big) + p_d & (d+1) \text{ odd,} \end{cases} \tag{14.22}$$

where ℓ is a typical length scale of A. This can be understood because the various terms are controlled by local counterterms constructed with the metric and the extrinsic curvature of the boundary. Most terms are scheme dependent, as one infers by considering redefinitions $a \to \lambda a$ of the cutoff. However, $\tilde{c}$ and p_d are *scheme independent* and carry physical information.

For instance, in $(3+1)$D $\tilde{c}$ is proportional to the central charge a — the one involved in the a-theorem which is monotonic along unitary RG flows. Indeed one can use strong subadditivity to provide an alternative proof of the a-theorem [CTT17].

In $(2+1)$D one can use

$$p_2 \equiv F \tag{14.23}$$

to define a "central charge", which also has been proven to be monotonic along RG flows [CH12, CHMY15].

Replica trick. To evaluate the entanglement entropy in QFT we use the "replica trick". First we evaluate

$$\text{Tr}_A \, \rho_A^n , \tag{14.24}$$

which is simpler because there is no log. This is called Rényi entropy. Then we analytically continue from $n \in \mathbb{N}$ to $n \in \mathbb{R}$, and then use

$$\frac{\partial}{\partial n} \operatorname{Tr} \rho^n = \frac{\partial}{\partial n} \operatorname{Tr} e^{n \log \rho} = \operatorname{Tr} \rho^n \log \rho \tag{14.25}$$

to write:

$$S_A = -\frac{\partial}{\partial n} \operatorname{Tr}_A \rho_A^n \Big|_{n=1} = -\frac{\partial}{\partial n} \log \operatorname{Tr}_A \rho_A^n \Big|_{n=1} . \tag{14.26}$$

The second equality is because $\operatorname{Tr}_A \rho_A = 1$.

To compute $\operatorname{Tr}_A \rho_A^n$ we use the path-integral formalism. Let us consider 2D for simplicity, and take A to be an interval at $t = 0$ in Euclidean signature. The ground-state wavefunctional Ψ is found by integrating from $t = -\infty$:

$$\Psi\big(\phi_0(x)\big) = \int_{t=-\infty}^{\phi(t=0,x)=\phi_0(x)} \mathcal{D}\phi \, e^{-S[\phi]} . \tag{14.27}$$

The density matrix $\rho = |\Psi\rangle\langle\Psi|$ has matrix elements

$$[\rho]_{\phi_-,\phi_+} = \Psi(\phi_-)\, \overline{\Psi}(\phi_+) . \tag{14.28}$$

$\overline{\Psi}$ is obtained by integrating from $t = 0$ to $t = +\infty$.

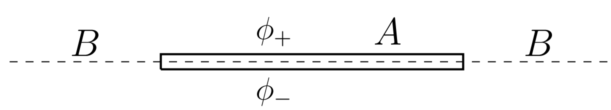
To obtain the reduced density matrix ρ_A we need to integrate over B, namely we set

$$\phi_-(x) = \phi_+(x) \equiv \phi_0 \qquad \text{for } x \in B \tag{14.29}$$

and integrate over ϕ_0 on B. We are left with discontinuous boundary conditions along A:

$$[\rho_A]_{\phi_-,\phi_+} = \frac{1}{Z_1} \int_{t=-\infty}^{t=+\infty} \mathcal{D}\phi \, e^{-S[\phi]} \prod_{x \in A} \delta\big(\phi(0_-, x) - \phi_-(x)\big)\, \delta\big(\phi(0_+, x) - \phi_+(x)\big) . \tag{14.30}$$

Here Z_1 is the vacuum partition function, necessary to guarantee that $\operatorname{Tr}_A \rho_A = 1$.



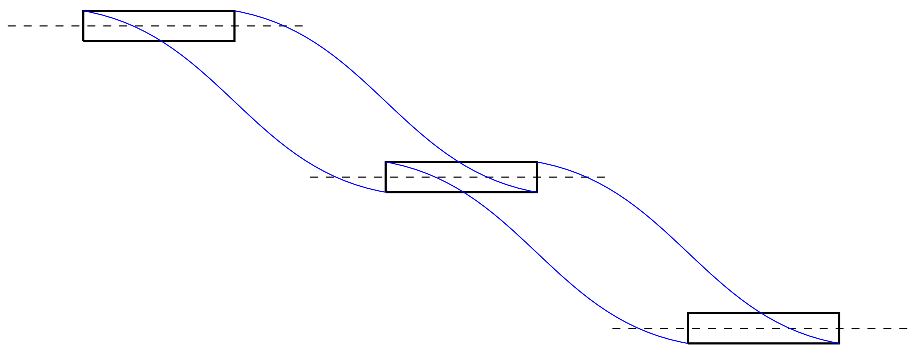
Now, ρ_A^n is given by n copies of ρ_A:

$$[\rho_A]_{\phi_{1,-}\phi_{1,+}} \cdots [\rho_A]_{\phi_{n,-}\phi_{n,+}} .$$

Multiplication is obtained by gluing the boundary conditions,

$$\phi_{j,+}(x) = \phi_{j+1,-}(x) \qquad \text{for } x \in A , \tag{14.31}$$

and integrating.



The trace is given by gluing and integrating the first and last boundary conditions.

Thus $\mathrm{Tr}_A\, \rho_A^n$ is the path-integral over an n-sheeted Riemann surface Σ_n, which is an n-fold covering of $\mathbb{R}^2$ branched over ∂A:

$$\mathrm{Tr}_A\, \rho_A^n = \frac{1}{(Z_1)^n} \int_{\Sigma_n} \mathcal{D}\phi\, e^{-S[\phi]} \equiv \frac{Z_n}{(Z_1)^n} \; . \tag{14.32}$$

The construction in higher dimensions is the same.

14.2 Holographic entanglement entropy

Given a CFT with holographic dual, how do we compute the entanglement entropy of a region A?

Let us consider the Poincarè patch of AdS_{d+2}:

$$ds^2 = R^2\, \frac{dz^2 + dx_{d,1}^2}{z^2} \; . \tag{14.33}$$

First we choose a cutoff $z \geq a$. The region A is at the boundary on $\mathbb{R}^d$, and ∂A is a $(d-1)$-dimensional surface in $\mathbb{R}^d$.

The prescription is that we should find a d-dimensional surface γ_A in AdS_{d+2}, at fixed time, that ends on ∂A, is homotopic to A, and has *minimal surface*. Then

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N^{(d+2)}} \; . \tag{14.34}$$

This is called the Ryu-Takayanagi formula. This prescription can be derived with a version of the replica trick in AdS_{d+2}.

Sketch of a derivation. We compute entanglement entropy as the analytically continued limit of the Renyi entropies. The latter is the partition function on an n-sheeted space $\mathcal{R}_n$, with deficit angle

$$\delta = 2\pi(1-n) \tag{14.35}$$

along the boundary ∂A. We should find a $(d+2)$-dimensional geometry $\mathcal{S}_n$ that solves Einstein equations and asymptotes to $\mathcal{R}_n$ at the boundary $z \to 0$. This is a technically difficult and unsolved problem, so we will use a trick.

We assume that $\mathcal{S}_n$ is an n-sheeted covering of AdS_{d+2}, with a deficit angle δ along a codimension-2 surface γ_A that asymptotes to the boundary ∂A. The Ricci scalar of the bulk spacetime has a δ-function along the surface:

$$R = 4\pi(1-n)\,\delta(\gamma_A) + R_0 \,, \tag{14.36}$$

where R_0 is the curvature of AdS_{d+2}. Then we plug this in the supergravity action

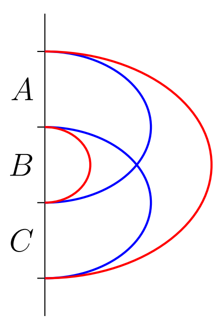
$$S_{\mathrm{AdS}} = -\frac{1}{16\pi G_N}\int d^{d+2}x\,\sqrt{g}\,(R+\Lambda) + \dots \tag{14.37}$$

The missing terms give a contribution that cancels out in the ratio as we send $n \to 1$. Applying AdS/CFT:

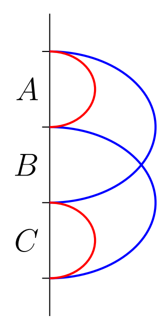
$$S_A = -\frac{\partial}{\partial n}\log \operatorname{Tr}\rho_A^n\Big|_{n=1} = -\frac{\partial}{\partial n}\left[\frac{(1-n)\,\mathrm{Area}(\gamma_A)}{4G_N}\right]_{n=1} = \frac{\mathrm{Area}(\gamma_A)}{4G_N}\,. \tag{14.38}$$

Moreover, the action principle in gravity becomes the variational principle for the Area and selects the minimal-area surface γ_A.

Holographic proof of strong subadditivity. It is easy to prove with pictures:



$S_{A\cup B} + S_{B\cup C} \geq S_{A\cup B\cup C} + S_B$



$S_{A\cup B} + S_{B\cup C} \geq S_A + S_C$

In both pictures, the sum of the lengths (in AdS) of the blue lines is greater than that of the red line, because in both cases one can decompose the blue lines in such a way that they connect the same pair of points as the red lines, but the red lines are minimal length for those pairs, while the blue lines are not (they are minimal for different pairs).

Entanglement entropy of interval in 2D CFT. We use the relation between the radius of AdS_3 and the central charge c of the CFT_2:

$$c = \frac{3R}{2G_N^{(3)}}\,. \tag{14.39}$$

Take AdS_3 (its Poincarè patch) and an interval of length ℓ. We need the geodesic between the two points

$$\Big(-\frac{\ell}{2}, a\Big) \qquad \text{and} \qquad \Big(\frac{\ell}{2}, a\Big) . \tag{14.40}$$

This is given by the half-circle

$$(x, z) = \frac{\ell}{2}\big(\cos u\,, \sin u\big) , \qquad \epsilon \leq u \leq \pi - \epsilon\ , \qquad \epsilon = \frac{2a}{\ell} \tag{14.41}$$

for $a \to 0$. Then the length is easily computed ($ds^2 = R^2 du^2/\sin^2 u$ along the curve):

$$\mathrm{Length}(\gamma_A) = 2R \int_\epsilon^{\pi/2} \frac{du}{\sin u} \simeq -2R \log \frac{\epsilon}{2} = 2R \log \frac{\ell}{a} . \tag{14.42}$$

The the entanglement entropy is

$$S_A = \frac{\mathrm{Length}(\gamma_A)}{4G_N^{(3)}} = \frac{c}{3} \log \frac{\ell}{a} , \tag{14.43}$$

reproducing the field theory result.

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