A NOTE ON ABELIAN CATEGORIES — TRANSLATING ELEMENT-CHASING PROOFS, AND EXACT EMBEDDING IN ABELIAN GROUPS

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Motivation. Given a theorem about abelian groups or R-modules which one proves by diagram-chasing arguments, one would like to get a nice proof of the same result in arbitrary abelian categories; in particular, one would like a reliable way of "translating" an elementary element-proof into an elementary arrow-proof.

A useful approach is the following. Given an object X of an abelian category, take an arbitrary object A, and think of a map $x: A \to X$ as representing a "parametrized family of elements of X". We can add such "families of elements", and apply maps $X \to Y$ to them, by adding or composing morphisms. The one kind of diagram-chasing argument that we unfortunately cannot apply to them is lifting: if $Z \twoheadrightarrow X$ is an epimorphism in our category, we cannot necessarily say that every map $x: A \to X$ arises from a map $A \to Z$.

One solution is to allow ourselves to modify the "parametrizing" object A. Let us define a "refinement" x' of a map $x: A \to X$ to mean the composite of this map with an epimorphism $B \twoheadrightarrow A$. Then given any epimorphism $f: Z \twoheadrightarrow X$, some refinement of the given map x can indeed be lifted to Z, namely the pullback

$$\begin{array}{ccc} B & \twoheadrightarrow & A \\ {\scriptstyle x'}\downarrow & & \downarrow{\scriptstyle x} \\ Z & \overset{f}{\twoheadrightarrow} & X \end{array} .$$

In the category-theoretic translation of an element-chasing proof, any initial choices of elements would be replaced by choices of maps from an arbitrary test-object A_0 into objects of the given diagram. In the course of the proof, a number of successive refinements would be made, giving maps $A_n \twoheadrightarrow \ldots \twoheadrightarrow A_1 \twoheadrightarrow A_0$, to make the liftings involved in the element-proof possible for the maps in question, and the proof would be completed like the element proof.

Empirically, I have found that this works quite well. I will sketch some further details in §2. But first I will show how the idea of this technique can be adapted to give a rather elementary proof of the exact embedding theorem for abelian categories (which is a kind of universal diagram-chasing theorem.)

Let us think of the idea described above loosely as follows: The functors Hom(A, -) are not satisfactory substitutes for an "underlying set" because, though left exact, they are not in general (right) exact. So to get around this, we look at inverse systems $\ldots \twoheadrightarrow A_n \twoheadrightarrow \ldots \twoheadrightarrow A_0$, and find that the direct limits of the system of functors $\mathrm{Hom}(A_0, -) \twoheadrightarrow \ldots \twoheadrightarrow \mathrm{Hom}(A_n, -) \twoheadrightarrow \ldots$ behave better.

But they do not behave well enough (in general) for what we now want — no <u>one</u> system of refinements need work for <u>all</u> liftings, though we <u>can</u> find such systems that will work for any given <u>finite</u> set of lifting problems.

There is a frequently effective trick for going from constructions that can be made to satisfy <u>any finite subset</u> of a set of conditions, to a construction satisfying them all: take a <u>reduced product</u>, with respect to a filter, G, on the set of all constructions, defined in terms of the conditions in question. It is this that we shall use below to get our exact embedding.

We shall also find that, because we will be considering simultaneously "all" systems of refinements rather than just a single one, the assumptions that the arrows are epimorphisms can be dropped; we will get the desired faithfulness without it. This will allow us to avoid having to assume the given abelian category is small; it will be enough for it to have a <u>set</u> of generators (Theorem 1).

Since our construction is based on the functors Hom(A,-); we shall reformulate it for proof in a form (Theorem 2) that assumes only that we are given an appropriate set of functors — not necessarily represented by objects A.

§1. The exact embedding Theorem.

We shall call a category "small" if it is a set, "legitimate" if for any two of its objects, A and B, Hom(A, B) is a set, and "large" if we do not assume either of these conditions. Ab will denote the (legitimate) category of abelian groups.

Theorem 1. Let A be a legitimate abelian category, having a generating set of objects, $S \subseteq Ob(A)$. Then there exists a faithful exact functor $e: A \to Ab$.

(Note: stronger exact embedding theorems are known.)

We shall prove Theorem 1 from Theorem 2. By a "small class", let us mean one which can be indexed by a set (even though it may have "large" elements).

Theorem 2. Let A be a large abelian category, P a small class of covariant left exact functors $A \to Ab$, and Q a small class of morphisms (natural transformations) among these functors. Suppose that given any $p \in P$ (and additional data as indicated below) one can find $p' \in P$ and $q: p \to p'$ in Q such that, simultaneously:

(i) (Lifting property) For any previously specified epimorphism $f: Z \twoheadrightarrow X$ in A, and element $x \in p(X)$, there exists $z \in p'(Z)$ such that $q_X(x) = f_{p'}(z)$:

$$Z \xrightarrow[f]{} \!\!\!\!\to X. \qquad \begin{array}{ccc} & & x \in p(X) \\ & & \downarrow q_X \\ z \in p'(Z) & \xrightarrow[f_{p'}]{} & p'(X); \end{array}$$

and

(ii) ("Faithfulness") For any previously specified object Y of A and nonzero element $y \in p(Y)$, one has $q_Y(y) \neq 0$ in $p'(Y)$.

Then the functor $\prod_P p : A \to Ab$ can be embedded in an exact functor $e: A \to Ab$.

Note: With quantifiers written in strict logical order, the condition on P and Q reads: $(\forall\, p \in P;\ X, Y, Z \in Ob(A);\ f \in Epi(Z,X);\ x \in p(X),\ y \in p(Y)-\{0\})(\exists\ p' \in P;\ (u: p \to p') \in Q;\ z \in p'(Z)):\ f_{p'}(z) = q_X(x),\ q_Y(y) \neq 0.$

<u>Proof of Theorem 1 from Theorem 2.</u> Given S as in Theorem 1, we set $P = \{Hom(A,-) \mid A \in S\}$, and take Q to be the class of all morphisms among these functors, which by the Yoneda Lemma are induced by morphisms among the objects of $\underline{A}$. Clearly, P and Q are small classes.

To establish the "simultaneous faithfulness and lifting" condition, suppose we are given $A \in S$, an epimorphism $f\colon Z \twoheadrightarrow X$ in $\underline{A}$, an element $x \in Hom(A,X)$, an object $Y \in \underline{A}$, and a nonzero element $y \in Hom(A,Y)$. We form the pullback:

$$\begin{array}{ccc} B & \overset{g}{\twoheadrightarrow} & A \\ w\downarrow & & \downarrow x \\ Z & \overset{f}{\twoheadrightarrow} & X \end{array} .$$

Now because f is an epimorphism, the induced map g is also, hence as $y \neq 0$, the composite $yg\colon B \to Y$ is also nonzero. Since S is a generating set for $\underline{A}$, there exists some $A' \in \underline{A}$ and some morphism $h\colon A' \to B$ such that the composite $ygh\colon A' \to Y$, is nonzero. Let us write $q = gh\colon A' \to A$. Then $yq \neq 0$, and xq factors through f, namely $xq = xgh = fwh$. This establishes the hypotheses of Theorem 2.

Hence by that Theorem, the functor $\prod_S Hom(A,-)$ can be embedded in an exact functor e. Since S is a generating set, this product functor is faithful, so e is faithful, establishing Theorem 1.

<u>Proof of Theorem 2.</u> For every $n \geq 0$, let P_n denote the set of all systems of functors and morphisms of the form:

(1) $$p_* = (p_0 \xrightarrow{q_1} p_1 \xrightarrow{q_2} \cdots \xrightarrow{q_n} p_n) \quad (p_i \in P,\ q_i \in Q).$$

In particular, P_0 may be identified with P.

For every $n > 0$, we have an obvious truncation map, $t_n\colon P_n \to P_{n-1}$.

Let us define $a_n\colon \underline{A} \to \underline{Ab}$ to be the functor $\prod_{p_* \in P_n} p_n$. Then for each $n > 0$ we have a morphism of functors $b_n\colon a_{n-1} \to a_n$, induced by the last morphisms, q_n, of the systems $p_* \in P_n$.

Now given any epimorphism $f\colon Z \twoheadrightarrow X$ in $\underline{A}$, any $n > 0$, and any $x \in a_{n-1}(X)$, let $U_{f,x} \subseteq P_n$ denote the set of those $p_* \in P_n$ such that $q_n(x_{p_*}) \in \operatorname{Im} p_n(Z)$. Note that the intersection of two such sets is again one: $U_{f,x}\ U_{f',x'} = U_{f'',x''}$, where $f'' = (f,f')\colon Z\oplus Z' \twoheadrightarrow X\oplus X'$, and $x'' = (x,x') \in a_{n-1}(X) \oplus a_{n-1}(X') \cong a_{n-1}(X \oplus X')$. Hence the sets $U_{f,x}$ form the basis of a filter F_n on P_n.

But we will want a slightly stronger filter, G_n, which we define inductively: G_0 is the trivial filter $\{P_0\}$ on P_0. For $n > 0$,

$$G_n = F_n \vee \bar{t}_n(G_{n-1})$$

(where $\bar{t}_n(G_{n-1})$ means the filter induced on P_n by the filter G_{n-1} on P_{n-1}, via the truncation map $t_n\colon P_n \to P_{n-1}$.) Thus, a typical member of G_n is a set V of systems $p_* \in P_n$ such that V includes all systems p_* that satisfy a certain lifting condition at the first "step", another at the second, and so forth. (Because of the occurrence of $x \in a_{n-1}(X) = \prod p_{n-1}(X)$ in the definition of our basis-sets, the lifting conditions in question will be functions of the system p_* <u>up to</u> the point at which the condition is imposed.)

We now define our "reduced products": For any $X \in \underline{A}$, let $c_n(X)$ be the quotient of the abelian group $a_n(X) = \prod_{P_n} p_n(X)$ by the subgroup consisting of all elements x such that $\{p_* \in P_n \mid x_{p_*} = 0\} \in G_n$. I.e., two elements of $a_n(X)$ fall together in $c_n(X)$ if they agree on all systems p_* satisfying some sufficiently strong set of lifting conditions.

By the inductive construction of the filters G_n, we see that the maps $b_n\colon a_{n-1} \to a_n$ induce maps $d_n\colon c_{n-1} \to c_n$. I claim these c_n are monomorphic. Indeed, suppose some nonzero element of $c_{n-1}(Y)$ $(Y \in \operatorname{Ob}(\underline{A}))$ had zero image in $c_n(Y)$. Let the given element be the coset of $y \in a_{n-1}(Y)$. Then $b_n(y)$ must be zero on some set $U_{f,x} \cap p_n^{-1}(V) \subseteq P_n$ $(V \in G_{n-1})$. Since we assumed y had <u>nonzero</u>

image in a_{n-1}, it is not everywhere zero on $V \subseteq P_{n-1}$; let $p_* \in V$ be such that $y_{p_*} \neq 0$.

We now use the "simultaneous lifting and faithfulness" hypothesis of our Theorem; it tells us that we can find a morphism $q_n: p_{n-1} \to p_n$ which does not annihilate y_{p_*}, and which has the lifting property determined by f and x. If we now extend $p_* \in P_{n-1}$ to $p'_* \in P_n$ by means of this morphism, we see that p'_* will be a point of $U_{f,x} \cap p_n^{-1}(V)$ at which $b_n(y)$ is nonzero. This contradicts our assumption, proving that the maps c_n are indeed monomorphisms. In particular, $c_0 = a_0 = \prod_P p$ embeds in every c_n.

Also, by the construction of our filters from "lifting properties", we see that given any epimorphism $f: Z \twoheadrightarrow X$ in $\underline{A}$, and any element $\xi \in c_{n-1}(X)$, (say the image of $x \in a_{n-1}(X)$), we will have $d_n(\xi) \in \operatorname{Im} f_{c_n}$. Loosely speaking, the functors c_n respect epimorphisms, "up to" the maps d_n. Hence, we see that if we let e denote the direct limit of the system of functors

$$c_0 \xrightarrow{d_1} c_1 \xrightarrow{d_2} \cdots ,$$

then e will respect epimorphisms. Also, e will be left exact, because the morphisms in P were, and this property is preserved by reduced products, and directed limits in $\underline{Ab}$. Hence e is exact. Since the morphisms $c_0 \to c_n$ are monic for each n, the induced map $c_0 \to e$ is monic, completing the proof of the Theorem.

§2. Further remarks on element-chasing. I will not go through an actual diagram-chasing proof here; the reader can work out his own favorite example. But I will give a few guidelines to bear in mind.

When the original element-proof calls on the definition of exactness, the translation will make use of the following easy Lemma: a sequence $Z \xrightarrow{g} Y \xrightarrow{f} X$ with $fg = 0$ is exact in $\underline{A}$ if and only if, for every $A \in \underline{A}$, and $y: A \to Y$ such that $fy = 0$, there is a refinement y' of y that factors through g.

Given a map $f: Y \to X$, if one forms the image object, $\operatorname{Im}(f) \rightarrowtail X$, then maps $A \to \operatorname{Im}(f)$ may be identified with maps $A \to X$ having refinements that factor through f. Suppose we have two maps, $f: Y \to X$ and $f': Y' \to X$. Then "Im(f) + Im(f')" may be translated as $\mathrm{Im}((f,f'): Y \oplus Y' \to X)$. Hence a statement in an element-chasing proof that some element $x \in X$ decomposes as the sum of an element in the image of f and an element in the image of f' translates to say that some refinement of the corresponding map $x: A \to X$ in the "element" proof factors through (f,f') — which is equivalent to saying that this refinement is the sum of an arrow which factors through f, and an arrows which factors through f'.

What about a proof where one <u>constructs</u> a map from some object X to some object Z by picking an element x, performing a series of liftings and mappings, and finally showing that the element $z \in Z$ one gets does not depend on the choices of lifting one has made, and that the resulting map $x \mapsto z$ is a homomorphism? The arrows proof may be done as follows: Start with any <u>epi</u>morphism $x: A \twoheadrightarrow X$ (e.g., the identity of X). Perform a series of refinements, liftings, and compositions, analogous to the liftings and mappings of the element proof, [illegible] ending with a diagram

$$\begin{array}{ccc} D & \twoheadrightarrow \cdots \twoheadrightarrow & A \\ z\downarrow & & \downarrow x \\ Z & & X. \end{array}$$

Then show, by the analogs of the "independent of choices" arguments, that z kills $\mathrm{Ker}(D \twoheadrightarrow X)$. Hence z induces a map $X \to Z$, as desired. Note that the verification that the resulting map "is a homomorphism" is avoided!

(S. Mac Lane, in "Categories for the working mathematician", [illegible] pp. 200-201, develops a similar approach to translating element proofs to arrow proofs. The main difference between his approach and mine is that he emphasizes the <u>equivalence</u> <u>relation</u> <u>generated</u> <u>by</u> the relation "is a refinement of". Since this equivalence relation does not respect the abelian group structure on Hom-sets, his approach seems rather awkward when applied to proofs that involve adding or [illegible] subtracting elements.)