Algebraic Theory of $D$-modules

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**Lecture 1. $D$-modules and functors.**

**§0. Introduction.**

1. In my lecture I will discuss the theory of modules over rings of differential operators (for short $D$-modules). This theory started about 15 years ago and now it is clear that it has very valuable applications in many fields of mathematics.

Names: Sato, Kashiwara, Kawai, Bernstein, Roos, Björk, Malgrange, Beilinson.

2. I will speak on an interpretation of the theory, given by Beilinson and myself. We restrict ourselves to purely algebraic theory of $D$-modules over any algebraically closed field $k$ of characteristic 0. Sato and Kashiwara worked for analytic varieties over $\mathbb{C}$, so as usual, our theories are interlapped like this . I should mention from the very beginning, that some of the most important technical notions and results are due to Kashiwara.

**§1. 1. $O$-modules.**

So we fix an algebraically closed field $k$ of char 0. One can assume $k = \mathbb{C}$.

Let $X$ be an algebraic variety (over $k$), $O_X$ the structure sheaf. Let $F$ be a sheaf of $O_X$-modules. I call $F$ quasi-coherent sheaf of $O_X$-modules (or $O_X$-module) if it satisfies the condition:

(*) If $U \subset X$ is open affine subset, $f \subset O(U), U_f = \{u \in U \big| f(u) \neq 0\}$, then $F(U_f) = F(U)_f \overset{\text{def}}{=} O(U_f) \bigotimes_{o(U)} F$.

By Serre's theorem this condition is local.

Let $\mu(O_X)$ be the category of $O$-modules. Locally, i.e., on an open affine subspace, $U \subset X$, I will identify $\mu(O_U)$ with the category of $C$-modules, where $C = O(U)$.

**2. Differential operators and $D$-modules.**

By definition, a differential operator of order $\leq k$ on $U$ is a $k$-linear morphism $d : C \to C$, such that $[\hat{f}_k \dots [\hat{f}_1[\hat{f}_o, d]]] = 0$ for any $f_o, \dots, f_k \in C$, where $\hat{f} : C \to C$ is an operator of multiplication by $f$.

The ring of different operators on $U$ I denote by $D(U)$, $O(U) \subset D(U)$.

**Proposition.** *$D(U_f) = O(U_f) \bigotimes_{o(U)} D(U) = D(U) \bigotimes_{o(U)} O(U_f)$. Hence $U \to D(U)$ is a quasicoherent sheaf of $O_X$-modules. I denote it by $D_X$ and call the sheaf of differential operators on $X$.*

$D$-module is by definition a sheaf $F$ of left $D_X$-modules which is quasi-coherent as $O_X$-module. Category of $D_X$-modules I will denote by $\mu(D_X)$. Locally, on affine open set $U$, $\mu(D_U) \cong D(U)$-mod.

If $X$ is singular, $D_X$ can be bad (for instance, it can be not locally noetherian). So from now on I assume $X$ *regular*, if I don't say otherwise.

**Lemma.** *1. For each $x \in X$ there exist an affine neighbourhood $U \supset x$ functions $x_1, \dots, x_n$ on $U$ and vector fields $\partial_1, \dots, \partial_n$ on $U$ such that $\partial_i(x_j) = \delta_{ij}, \partial_i$ generate tangent bundle of $X$.*

2. $D(U) = O(U) \bigotimes_k k[\partial_1, \dots, \partial_n]$.

The system $(x_i, \partial_i)$ I will call the coordinate system in $D_X$.

3. So I introduced main characters of my story and can begin the play. It is very useful, though formally not necessary, to have in mind some analytic picture, corresponding to $D$-modules. Let me describe it.

**Analytic picture.** Suppose we have a system $S$ of $p$ linear differential equations on $q$ functions $f_1 \dots f_q$, $S = \{\sum_{j=1}^{q} d_{ij} f_j = 0,\ i = 1, \dots, p\}$. Then we can assign to $S$ a $D$-module $M$ given by $q$ generators $e_1, \dots, e_q$ and $p$-relations $M = \oplus D \cdot e_j / (+D(\sum d_{ij} e_j))$. In this language, a solution $s$ of the system $S$ in some space of functions $F$ is nothing else than a morphism of $D$-modules $\alpha_S : M \to F$.

Having in mind this picture we can start investigation of $D$-modules.

**4. Left and right $D$-modules.**

Let us denote by $\mu^R(D_X)$ the category of right $D$-modules. How is it connected with $\mu(D_X)$?

*Motivation.* In analytic picture, the space of functions $F$ is a left $D$-module. But if we consider the space of distributions $F^*$, it has a natural structure of a right $D$-module. Hence systems of differential equations for distributions correspond to right $D$-modules.

But if we fix a differential form $w$ of highest degree, we can identify $F$ and $F^*$ by $\phi \in F \longmapsto \phi w \in F^*$.

**Proposition-Definition.** *Let $\Omega = \Omega_X$ be the $O_X$-module of differential forms of highest degree on $X$. For any $D_X$-module $F$ denote by $\Omega(F)$ the right $D_X$-module, given by $\Omega(F) = \Omega \bigotimes_{O_X} F$*

$$f(w \otimes u) = fw \otimes u, \quad \xi(w \otimes u) = -Lie_\xi(w) \otimes u - w \otimes u.$$

*Functor $\Omega : \mu(D_X) \to \mu^R(D_X)$ is an equivalence of categories.*

I prefer to use a slightly different description of $\Omega$. Consider the module ${}^\Omega D_X = \Omega(D_X) = \Omega \bigotimes_{O_X} D_X$. It has two different structures of a right $D_X$-module – one as $\Omega(\quad)$, and another from the endomorphism of left $D_X$-module $D_X$, which are given by right multiplications. It is easy to check that there exists a unique involution $v$ of ${}^\Omega D_X$, which interchanges these two structures of right $D_X$-module and is identical on $\Omega \subset {}^\Omega D_X$. By definition,

$$\Omega(F) = {}^\Omega D_X \bigotimes_{D_X} F.$$

The inverse function $\Omega^{-1} : \mu^R(D_X) \to \mu(D_X)$ is given by multiplication on the module $D_X^\Omega = D_X \bigotimes_{o_X} \Omega^{-1} = \mathrm{Hom}_{D_X}({}^\Omega d_X, D_X)$, which has two structures of left $D$-module.

We will work with left $D$-modules but remember that we can go freely to right $D$-modules and back.

**5. Inverse image of $D$-modules.**

Let $\pi : Y \to X$ be a morphism of algebraic varieties.

*Motivation.* We can lift a function from $X$ to $Y$. If they satisfy some system of equations $S$, then their images also would satisfy some system of equations $S'$. Is it possible to describe this system?

It turns out that we can do some algebraic version of this. Namely, I will describe a functor

$$\pi^\Delta : \mu(D_X) \longrightarrow \mu(D_Y).$$

First do it locally, i.e., suppose $X$ and $Y$ are affine, and $D_X$-module is given by a $D_X = D(X)$-module $M$. Then put

$$\pi^\Delta(M) = O_y \bigotimes_{o_X} M$$

and define the action of $D_Y$ on $\pi^\Delta(M)$ by

$$(*) \qquad f'(f \otimes m) = f'f \otimes m, \quad \xi(f \otimes m) = \xi f \otimes m + f(\sum_i \xi(x_i) \otimes \partial_i m),$$

where $(x_i, \partial_i)$ is a coordinate system in $D_X$. It is easy to check that this definition is correct. Intuitively, it is a version of the chain rule.

Now we can write the general definition

$$\pi^\Delta(F) = O_Y \bigotimes_{\pi^\cdot(o_X)} \pi^\cdot(F),$$

where $\pi^\cdot$ is an inverse image in the category of sheaves and the action of $D_Y$ is given by (*).

Again, it is convenient to rewrite this definition slightly. Put

$$D_{Y\to X} = \pi^\Delta(D_X).$$

$D_{Y\to X}$ is a sheaf on $Y$, which is $D_Y - \pi^\cdot(D_X)$-bimodule. By definition

$$\pi^\Delta(F) = D_{Y\to X} \bigotimes_{\pi^\cdot(D_X)} \pi^\cdot(F).$$

Note that as an $O_Y$-module $\pi^\Delta(F)$ coincides with an inverse image $\pi^*(F)$, but I would like to save notation $\pi^*$ for other case.

**Lemma.** *If $\tau : Y \to Z$ is a morphism we have $(\tau\pi)^\Delta = \pi^\Delta\tau^\Delta$.*

**6. Direct image of $D$-modules.**

*Motivation.* We can integrate functions on $Y$ (say with compact support) to get functions on $X$. How does this affect systems of equations they satisfy?

First of all, we should realize that there is no natural way of integrating functions, but there is a natural way of integrating distributions (namely $\langle \int E, \phi\rangle = \langle E, \pi^*(\phi)\rangle$). Hence we should try to construct a functor $\pi_+ : \mu^R(D_Y) \longrightarrow \mu^R(D_X)$.

First consider a local case. Then we can put $\pi_+(N) = N_{D_Y} \otimes D_{Y\to X}$, where $N$ is a right $D_Y$-module. Or, in terms of sheaves,

$$\pi_+(H) = \pi_\cdot(H_{D_Y} \otimes D_{Y\to X}),$$

where $\pi_\cdot$ is the direct image in the category of sheaves.

Since we can freely go from left $D$-modules to right $D$-modules and back, we can rewrite this functor for left $D$-modules. Since I prefer to work with left $D$-modules, let us do it.

Put $D_{X\to Y} = \Omega(\pi^\Delta(D_X^\Omega)) = \Omega_{Y\pi^\cdot(o_X)} \otimes \Omega_X^{-1})$. This is a sheaf on $Y$, which is $\pi^\cdot(D_X) - D_Y$-bimodule.

Now we define the functor $\pi_+ : \mu(D_Y) \longrightarrow \mu(D_X)$ by (*)

$$(*) \qquad \pi_+(H) = \pi_\cdot(D_{X\leftarrow Y} \bigotimes_{D_Y} H).$$

Now let us try to handle the general case ($X$ and $Y$ are not affine). Then we immediately run into trouble. The matter is, that formula (*) describes $\pi_+$ as a composition of left exact functor $\pi_\cdot$ and right exact functor $D_{X\leftarrow Y}\otimes$, and this composition apparently does not make much sense (formally, it affects the composition rule, $(\tau\pi)_+ \neq \tau_+\pi_+$).

Definition (*) makes some sense for affine morphism $\pi$ [when $\pi^{-1}$ (affine open subset) is affine], since in this case $\pi_\cdot$ is an exact functor. But in order to study the general case, we should work in derived categories.

**7. $D$-complexes and functors.**

Henceforth I assume all algebraic varieties to be quasiprojective.

**Proposition.** *Category $\mu(D_X)$ has enough injective and locally projective objects. It has a finite homological dimensin (we will see that it is $\leq 2dimX$).*

**Definition.** *$D_X$-complex is a bounded complex of $D_X$-modules. Corresponding derived category, which consists of $D_X$-complexes up to quasi-isomorphisms we denote by $D(D_X)$.*

*$D_X$-complexes I will often denote by $\dot F, \dot H, \dots$ . We define functors*

$\Omega : D(D_X) \xrightarrow{\sim} D^R(D_X)$

$L\pi^{\Delta} : D(D_X) \longrightarrow D(D_Y)$, *for* $\pi : Y \to X$, *by*

$$L\pi^{\Delta}(\dot{F}) = D_{Y\to X} \overset{L}{\underset{\pi^{\cdot}(D_X)}{\bigotimes}} \pi^{\cdot}(\dot{F}).$$

$\pi_* : D(D_Y) \to D(D_X)$ *by*

$$\pi_*(\dot{H}) = R\pi_{\cdot}(D_{X\leftarrow Y} \overset{L}{\underset{D_Y}{\bigotimes}} \dot{H}).$$

**Proposition.** $L(\tau \circ \pi)^{\Delta} = L\pi^{\Delta} \circ L\tau^{\Delta}$

$$(\tau\pi)_* = \tau_*\pi_*.$$

Usually we will decompose $\pi$ as a product of a locally closed imbedding and a smooth morphism. So let us consider these cases in more detail.

**8. Case of a closed imbedding** $i : Y \to X$**.**

Let us define functors

$i_+ : \mu(D_Y) \longrightarrow \mu(D_X)$ by $i_+ = i_{\cdot}(D_{X\to Y} \otimes_{D_Y} H)$

$i^+ : \mu(D_X) \longrightarrow \mu(D_Y)$ by $i^+(F) = \mathrm{Hom}_{i^{\cdot}D_X}(D_{X\leftarrow Y'} i^{\cdot}(F))$.

**Lemma.** *$i_+$ is left adjoint to $i^+$; $i_+$ is exact and $i^+$ is left exact.*

$$Ri_+ = i_{*'}$$

$$Ri^+ = Li^{\Delta}[dim\ Y - dim X].$$

It turns out that it is quite convenient to use shifted functor $L\pi^{\Delta}[\dim Y - \dim X]$ which in the case of imbedding coincides with $Ri^+$. So I put

$$\pi^! = L\pi^{\Delta}[\dim Y - \dim X] : D(D_X) \longrightarrow D[D_Y].$$

For any closed subset $Z \subset X$ I denote by $\mu_Z(X)$ the full subcategory of $\mu(X)$, consisting of $D_X$-modules $F$ such that supp $F \subset Z$.

**Theorem (Kashiwara).** *Let $i : Y \to X$ be a closed imbedding. Then functors $i_+ : \mu(D_Y) \to \mu_Y(D_X)$ and $i^+ : \mu_Y(D_X) \to \mu(D_Y)$ are inverse and define an equivalence of categories.*

This simple technical statement is very important and very useful.

**9. $D$-modules on singular varieites.**

Let $Z$ be a singular variety. Then the algebra $D_Z$ can be very bad, so it does not make sense to study modules over $D_Z$. But using Kashiwara's theorem we can define category of $D$-modules on $Z$ (which we denote by $\mu(D_Z)$ though it is not category of $D_Z$-modules) in the following way.

Let us realize $Z$ as a closed subvariety of a nonsingular variety $X$ and put by definition

$$\mu(D_Z) = \mu_Z(D_X).$$

Even if we cannot realize $Z$ as a subvariety, we can do it locally. Now, Kashiwara's theorem implies, that at least locally, $\mu(D_Z)$ is correctly defined. Glueing pieces together we can construct $\mu(D_Z)$ globally.

We define $D(D_Z)$ as derived category of $\mu(D_Z)$. If $Z$ is a closed subset of $X$, one can show that

$$D(D_Z) = D_Z(D_X) = \{\dot{F} \in D(D_X) \big| \text{supp } \dot{F} \subset Z, \quad i.e., \ \dot{F}\big|_{X\setminus Z} = 0\}.$$

Later I will discuss only nonsingular varieties, but all results can be transferred to the singular case.

**10. Proof of Kashiwara's theorem.**

We should prove that natural morphisms of functors

$$Id_{\mu(D_Y)} \longrightarrow i^+ i_+, \qquad\qquad i_+ i^+ \longrightarrow Id_{\mu_Y(D_X)}$$

are isomorphisms. It is sufficient to check locally, so I can assume that $X$ is affine, $Y$ is given by equations $x_1, \dots, x_\ell$. Using induction by $\ell$ I can assume that $Y$ is given by one equation $x$. Locally I can choose a vector field $\partial$ such that $\partial(x) = 1$, i.e., $[\partial, x] = 1$.

If $F \in \mu_Y(D_X)$, then supp $F \subset Y$ and since $F$ is quasicoherent, any section $\xi \in F$ is annihilated by large powers of $x$.

Consider the operator $I = x\partial$ and put $F^i = \{\xi | I\xi = i\xi\}$. Then it is clear that $x : F^i \to F^{i+1}$, $\partial : F^i \to F^{i-1}$, $x\partial : F^i \to F^i$ is an isomorphism for $i < 0$, $\partial x = x\partial + 1$ is an isomorphism for $i < -1$. Hence $x : F^i \to F^{i+1}$ and $\partial : F^{i+1} \to F^i$ are isomorphisms for $i < -1$. If $\xi \in F$ and $x\xi = 0$, then $x\partial\xi = \partial x \xi - \xi = -\xi$, i.e., $\xi \in F^{-1}$. By induction on $k$ it is easy to prove, that if $x^k\xi = 0$, then $\xi \in F^{-1} \oplus \ldots \oplus F^{-k}$. Hence $F = \bigoplus_{i=1}^{\infty} F^{-1} = k[\partial] \bigotimes_k F^{-1}$ and Ker $(x, F) = F^{-1}$. This is the statement of Kashiwara's theorem.

**Lecture 2.**

**1. Some applications of Kashiwara's theorem.**

**a) Structure of $O$-coherent $D_X$-modules.**

We say that $D_X$-module $F$ is $O$-coherent if locally it is a finitely generated $O_X$-module.

**Proposition.** *$O$-coherent $D_X$-module $F$ is locally free as $O_X$-module.*

*Proof.* Let $x \in X$, $m_X$ corresponding maximal ideal of $O_X$. The space $F_x = F/m_x F$ is called the fiber of $F$ at $x$. Since $F$ is coherent as $O_X$-module, it is sufficient to check that dim $F_x$ is a locally constant function on $X$. This we can check for restriction of $F$ on any nonsingular curve $C \subset X$. Hence we can replace $X$ by $C$ and $F$ by $i^{\Delta}_{C\to X}(F)$, and assume that $X$ is a curve.

If $F$ has a torsion at a point $x$, then $F$ contains a nonzero subsheaf $(i_x)_+ i_x^+(F)$, which is not $O$-coherent. Hence $F$ has no torsion, and, since $X$ is a curve, $F$ is locally free. Q.E.D.

Recall that locally free $O_X$-modules $F$ naturally correspond to the algebraic vector bundles $E$ on $X$ ($F$ is a sheaf of sections of $E$). Action of $D_X$ on $F$ defines a connection on $E$, by $\nabla_\xi(\phi) = \xi\phi$. Since $[\nabla_\xi, \nabla_\eta] = \nabla_{[\xi,\eta]}$ this connection is flat.

This gives an equivalence of categories

$$\{0 - \text{coherent } D_X - \text{modules}\} = \left\{ \begin{array}{l} \text{algebraic vector bundles on } X \\ \text{with flat connection} \end{array} \right. .$$

**b) $D$-modules on projective space.**

Let $V = k^n$ be an affine space over $k$, $V^* = V \setminus \{0\}$, $X = \mathbb{P}(V)$-corresponding projective space, $pr : V^* \to \mathbb{P}(V)$ the natural projection.

**Theorem.** *Functor of global sections $\Gamma : \mu(D_X) \to$ Vect, $F \to \Gamma(X, F)$ is exact, and each $D_X$-module $F$ is generated by its global sections (i.e., $D_X \otimes \Gamma(F) \to F$ is an epimorphism).*

*Remark.* Note that $\Gamma(F) = \mathrm{Hom}_{\mu(D_X)}(D_X, F)$. Hence theorem simply means that $D_X$ is a projective module and is a generator of category $\mu(D_X)$.

*Proof.* For any $D_X$-module $F$ put $F^\Delta = pr^\Delta(F) \in \mu(D_{V^*})$. This sheaf carries a natural action of the homotety group $k^*$ and hence the space of sections $\Gamma(F^\Delta)$ is a graded space $\bigoplus_{n=-\infty}^{\infty} \Gamma(F^\Delta)^n$. It is clear that $\Gamma(F) = \Gamma(F^\Delta)^0$ – zero component. If we denote by $I \in D_V$ the Euler operator $\sum x_i \partial_i$, which is an infinitesimal generator of the group $k^*$, then it defines a grading on $\Gamma(F^\Delta)$, i.e., its action on $\Gamma(F^\Delta)^n$ is multiplication by $n$.

Functor $F \to F^\Delta$ is exact, hence all nonexactness can come only from the functor $\Gamma_{V^*}$. Let us decompose it as $\Gamma_{V^*} = \Gamma_V \circ j_+ : \mu(D_{V^*}) \to \mu(D_V) \to$ Vect, where $j : V^* \hookrightarrow V$. Since $V$ is affine, functor $\Gamma_V$ is exact.

Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence of $D_X$-modules. Then the sequence $0 \to j_+(F_1^\Delta) \to j_+(F_2^\Delta) \to j_+(F_3^\Delta) \to 0$ is exact when restricted to $V^*$, hence its cohomologies are sheaves on $V$, concentrated at 0.

By Kashiwara's theorem each sheaf concentrated at 0 is a direct sum of many copies of a standard $D_V$-module $\Delta = P \circ 1[\partial_1, \dots, \partial_k]\delta$, where $x_i\delta = 0$. This implies that eigenvalues of $I$ on $\Gamma(\Delta)$ are equal $-n, -n-1, -n-2, \dots -$. Hence the sequence

$$0 \longrightarrow \Gamma(F_1^\Delta)^o \longrightarrow \Gamma(F_2^\Delta)^o \longrightarrow \Gamma(F_3^\Delta)^o \longrightarrow 0$$

is exact, since $\Gamma = \Gamma_V$ is an exact functor and sheaves, concentrated at 0, do not affect 0-graded part.

The statement, that any $D_X$-module is generated by its global sections can be reduced, using exactness of $\Gamma$, to the statement $F \neq 0 \Longrightarrow \Gamma(F) \neq 0$. This is proved in the same way as exactness of $\Gamma$.

**2. Case of an open imbedding.**

Let $j : V \to X$ be an open imbedding. Then $j^\Delta$ is an exact functor of restriction, i.e., $j^! = j^\Delta$, and $j_+$ is the usual functor of direct image in category of sheaves. Its derived functor $Rj_+$ equals $j_*$. In particular case when $j$ is an affine imbedding the functor $j_+$ is exact, i.e., $j_* = j_+$.

Functor $j^\Delta$ is left adjoint to $j_+$ and $j^\Delta j_+ = Id_U$. For arbitrary $D_X$-module $F$ the kernel and cokernel of the morphism $F \xrightarrow{\alpha} j_+ j^\Delta F$ are supported on the closed subset $Z = X \setminus U$.

Let us consider the functor $\Gamma_Z : \mu(D_X) \longrightarrow \mu(D_X)$ given by $\Gamma_Z(F) = \{\xi \in F | \text{supp } \xi \subset Z\}$. Then we have an exact sequence

$$0 \longrightarrow \Gamma_Z(F) \longrightarrow F \xrightarrow{\alpha} j_+ j^\Delta F.$$

If $F$ is an injective $D_X$-module, $\alpha$ is onto. Hence in derived category we always have an exact triangle

$$(*) \qquad R\Gamma_Z(\dot F) \longrightarrow \dot F \longrightarrow j_* j^! \dot F.$$

We will call this triangle a decomposition of $\dot F$ with respect to $(U, Z)$.

Denote by $D_Z(D_X)$ the full subcategory of $D(D_X)$, consisting of $D_X$-complexes $\dot F$ such that $\dot F|_U = 0$. Then (*) implies that the natural inclusion $D(\mu_Z(D_X)) \to D_Z(D_X)$ is an equivalence of categories.

**3. Base change.**

**Theorem.** *Consider Cartesian square*

$$\begin{array}{ccc} Z & \xrightarrow{\check\alpha} & Y \\ \big\downarrow{\scriptstyle \check\pi} & & \big\downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\alpha} & X \end{array}$$

*i.e.,* $Z = Y \times_X S$.

*Then functors* $\alpha^! \pi_*$ *and* $\check\pi_* \check\alpha^! : D(D_Y) \longrightarrow D(D_S)$ *are naturally isomorphic.*

**Corollary.** *If $Z = \emptyset$, i.e., $\alpha(S) \cap \pi(Y) = \emptyset$, then $\alpha^! \pi_* = 0$.*

*Sketch of the proof.* It is sufficient to consider 2 cases

i) $\alpha$ is a projection $T \times X \to X$

ii) $\alpha$ is a closed imbedding.

The case (i) is straightforward. In (ii) let $U$ be a complement of $S$,

$$V = \pi^{-1}(U) = Y \setminus Z, \qquad j : U \to X, \qquad \check{j} : V \to Y.$$

We have natural exact triangles

$$\check{\alpha}_* \check{\alpha}^! \dot{H} \longrightarrow \dot{H} \longrightarrow j_* \check{j}^{\,!} \dot{H}$$

$$\alpha_* \alpha^! \dot{F} \longrightarrow \dot{F} \longrightarrow j_* j^! \dot{F}.$$

Put $\dot{F} = \pi_* \dot{H}$. Then since we clearly have a base change for an open subset $U$, we have $\pi_*(j_* \check{j}^{\,!} H) \cong j_* j^! \dot{F}$. Hence, since $\pi_*$ is an exact functor in derived categories, we have

$$\pi_*(\check{\alpha}_* \check{\alpha}^! \dot{H}) \cong \alpha_* \alpha^! \dot{F}.$$

But $\pi_* \check{\alpha}_* \cong \alpha_* \check{\pi}_*$, i.e., $\alpha_*(\check{\pi}_* \check{\alpha}^! H) \cong \alpha_*(\alpha^! \pi_* H)$. By Kashiwara's theorem we can remove $\alpha_*$, which gives us the base change.

4. Let $S = \{X = \bigcup_{i=0}^{n} X_i\}$ be a smooth stratification of $X$, i.e., each $X_i$ is a locally closed nonsingular subvariety, and $X_o \cup X_i \cup \ldots \cup X_j$ is closed for each $j$. For each $i$ consider the functor $S_i : D(D_X) \to D(D_X)$, where $S_i = r_{I^*} r_i^!$, $r_i : X_i \to X$. Then each $D_X$-complex $\dot{F}$ is glued from $S_i(\dot{F})$, i.e., we have $D_X$-complexes $\dot{F}_i$ and exact triangles

$$\dot{F}_{i-1} \longrightarrow \dot{F}_i \longrightarrow S_i(\dot{F}) \text{ such that}$$
$$\dot{F}_{-1} = 0, \ \dot{F}_n = \dot{F}.$$

We will call $\{S_i(\dot{F})\}$ the stratification of $\dot{F}$ with accordance to $S$, and $D_{X_i}$ complexes $r_i^!(\dot{F})$ components of the stratification.

**5. Case of smooth (submersive) morphism $\pi : Y \to X$.**

For any smooth variety $Y$ let us denote by $DR_Y$ the de Rham complex $\Omega_Y^0 \to \Omega_Y^1 \to \ldots \to \Omega_Y^k$ of sheaves on $Y$. More generally, if $H$ is a $D_Y$-module, we can by the same formulae define de Rham complex $DR_Y(H)$ with components $DR_Y(H)^i = \Omega_Y^i \otimes_{o_Y} H$.

It is clear that $DR_Y(D_Y)$ is a complex of right $D_Y$-modules. Now, let $\pi : Y \to X$ be a smooth morphism. Denote by $\Omega_{Y/X}^i$ sheaves of relative $i$-forms on $Y$. In the same way as earlier we can define the relative de Rham complex $DR_{Y/X}(H)$ for any $D_Y$-module $H$.

**Lemma.** *$DR_{Y/X}(D_Y)[k] = D_{X\leftarrow Y}$ as a complex of right $D_Y$-modules.*

Hence we can calculate the direct image functor $\pi_*$ using this complex:

$$\pi_*(H) = R\pi.(D_{X\leftarrow Y}\overset{L}{\underset{D_Y}{\bigotimes}}H) = R\pi.(DR_{Y/X}(D_Y)\underset{D_Y}{\bigotimes}H)[k] = R\pi.(DR_{Y/X}(H)[k].$$

The only trouble here is that this formula defines $\pi_*(H)$ only as a complex of $O$-modules. Action of vector fields in general is described by quite unpleasant formulae. In the case when $\pi$ is a projection $\tau : Y = T \times X \to X$, action of vector fields is given by their action on $H$.

### 6. Coherent $D_X$-modules and $D_X$-complexes.

$D_X$-module $F$ is called coherent if locally it is finitely generated. We'll see that locally $D_X$ is a noetherian ring, hence any submodule of a coherent $D_X$-module $F$ is coherent.

Any $D_X$-module $F$ is a union of coherent $O_X$-submodules $L_\alpha$. If we put $F_\alpha = D_X L_\alpha$ we see that $F$ is a union of coherent $D_X$-submodules $F_\alpha$. It implies:

(i) Any coherent $D_X$-module $F$ is generated by a coherent $O_X$-submodule $F^o$.

(ii) Extension principle. If $H$ is a $D_X$-module, $U \subset X$ an open subset, $F \subset H|_U$ – a coherent $D_U$-submodule, then then exists a coherent $D_X$-submodule $H' \subset H$ such that $H'|_U = F$. Category of coherent $D_X$-modules I denote by $\mu_{\text{coh}}(D_X)$.

$D_X$-complex $\dot{F}$ is called coherent if all its homology sheaves $H^i(\dot{F})$ are coherent $D_X$-modules. The full subcategory of $D(D_X)$ consisting of coherent $D_X$-complexes I will denote by $D_{\text{coh}}(D_X)$.

Properties of coherent $D_X$-modules imply

**Lemma.** *The natural morphism $D(\mu_{\text{coh}}(D_X)) \longrightarrow D_{\text{coh}}(D_X)$ is an equivalence of categories.*

### 7. Direct image of proper morphism.

**Proposition.** *Let $\pi : Y \to X$ be a proper morphism. Then $\pi_* D_{\text{coh}}(D_Y) \subset D_{\text{coh}}(D_X)$.*

*Proof.* If $\pi$ is a closed imbedding, proposition follows from Kashiwara's theorem. So consider the case when $\pi : Y = \mathbb{P} \times X \longrightarrow X$ is a projection, where $\mathbb{P}$ is a projective space.

We can assume $X$ to be affine. Then by 1(b) $D_Y$ is a generator in $\mu_{\text{coh}}(D_Y)$ and hence it is sufficient to prove that $\pi_*(D_Y) \subset D_{\text{coh}}(D_X)$. But

$$\begin{aligned}\pi_*(D_Y) &= R\pi.(D_{X\leftarrow Y}\underset{D_Y}{\bigotimes}D_Y) = R\pi.(D_X\underset{k}{\bigotimes}\Omega_{\mathbb{P}}) \\ &= D_X\underset{k}{\bigotimes}R\pi.(\Omega_{\mathbb{P}}) = D_X[-\dim\,\mathbb{P}] \in D_{\text{coh}}(D_X).\end{aligned}$$

**8. Good filtration and singular support of a $D$-module.**

Consider the filtration $D_X^0 \subset D_X^1 \subset \dots$ of $D_X$ by order of an operator. Each $D_X^i$ is a coherent $O$-module, $D_X^0 = O_X$ and $D^i \cdot D^j \subset D^{i+j}$.

Let $\Sigma = \bigoplus_{i=0}^{\infty} \Sigma^i$, $\Sigma^i = D^i/D^{i-1}$ be the associated graded sheaf of algebras. Then $\Sigma$ is commutative and naturally isomorphic to the algebra of regular functions on the cotangent bundle $T^*(X)$.

Let $F$ be a $D_X$-module. A filtration on $F$ is a filtration $\Phi = \{F^o \subset \dots F^k \subset \dots\}$ of $F$ by $O$-submodules such that $F = \cup F^j$, $D^i F^j \subset F^{i+j}$. The associated graded module $F_\Sigma = \oplus F^i/F^{i-1}$ has a natural structure of $\Sigma$-module.

We say that filtration $\Phi$ is *good* if $F_\Sigma$ is a coherent $\Sigma$-module. An equivalent condition is

$$(*) \qquad \text{Each } F^j \text{ is a coherent } O_X - \text{module and } D^1 F^j = F^{j+1} \text{ for large } j.$$

It is clear that $D_X$-module $F$ with a good filtration is coherent. Conversely, if $F$ is a coherent $D_X$-module, then it is generated by a coherent $O_X$-module $F^o$ and we can define a good filtration $\Phi$ on $F$ by $F^j = D^j F^o$.

Let $F$ be a coherent $D_X$-module. Choose a good filtration $\Phi$ on $F$ and denote by $F_\Sigma$ the corresponding $\Sigma$-module. As a coherent $\Sigma$-module $F_\Sigma$ has a support $\mathrm{supp}(F_\Sigma) \subset T^*(X)$ (this support is a closed subvariety which is defined by the ideal $J_F \subset \Sigma$, equal to the annulator of $F_\sigma$ in $\Sigma$).

**Proposition.** *Supp*$(F_\Sigma)$ *depends only on* $F$ *and not on a particular choice of a filtration* $\Phi$.

We will denote this $\mathrm{supp}(F_\Sigma)$ as S.S.$(F) \subset T^*X$ and call it the *singular support* or the *characteristic variety* of $F$.

*Proof.* Let $\Phi, \Psi$ be two good filtrations of $F$. We say that $\Phi$ and $\Psi$ are neighbour if $F_\Phi^{i+1} \supset F_\Psi^i \supset F_\Phi^i$ for all $i$. For neighbour filtrations consider the natural morphism $F_{\Sigma\Phi} \longrightarrow F_{\Sigma\Psi}$ and include it in the exact sequence

$$0 \longrightarrow K \longrightarrow F_{\Sigma\Phi} \longrightarrow F_{\Sigma\Psi} \longrightarrow C \longrightarrow 0.$$

It is easy to check that $\Sigma$-modules $K$ and $C$ are isomorphic (only the grading is shifted by 1). This proves the proposition for neighbour filtrations.

If $\Phi$ and $\Psi$ are arbitrary good filtrations, we define the sequence of filtrations $\Phi_k$ by $F_{\Phi_k}^i = F_\Phi^i + F_\Psi^{i+k}$. It is clear that $\Phi_k$ and $\Phi_{k+1}$ are neighbour, $\Phi_k = \Phi$ for $k \ll 0$ and $\Phi_k = \Psi$ shifted on $k$ for $k \gg 0$. This proves the proposition.

*Remarks.* 1. Let $F$ be a $D_K$-module with a good filtration $\Phi$, $H \subset F$ a $D_X$-submodule. Consider induced filtrations on $H$ and $F/H$. Then we have an exact sequence $0 \to H_\Sigma \to F_\Sigma \to F/H_\Sigma \to 0$. In particular, filtration on $H$ is good, i.e., $H$ is $D_X$-coherent. Also we have

$$\text{S.S.}F = \text{ S.S.}H \ \cup \ \text{S.S.}(F/H).$$

Moreover, let $k = \dim$ S.S.$F$. Then we can assign to each $k$-dimensional component $W$ of S.S.$F$ some multiplicity (the multiplicity of supp $F_\Sigma$ at $W$; the proposition above really proves that this multiplicity is well defined). Put $m_k(F)$ = sum of multiplicities of all $k$-dimensional components of S.S.$F$. Then

$$m_k(F) = m_k(H) + m_k(F/H).$$

2. It is easy to see that $D_X$-module $F$ is $O$-coherent if and only if S.S.$F \subset X \subset T^*X$.

**9. Singular support and functors.**

Usually it is very difficult to describe the effect of functors $\pi_+, \pi^\Delta$ on singular support. (For instance, these functors usually do not preserve $D$-coherency.) But there are 2 cases when it can be done.

a) Let $i : Y \to X$ be a closed imbedding, $H \in \mu(D_Y)$. Then $i_+(H)$ is coherent if and only if $H$ is coherent and

$$\text{S.S.}(i_+H) = \{(x,\xi) | x \in Y, \quad (x, Pr_{T^*(X)\to T^*(Y)}\xi) \in \text{S.S.}H\}.$$

b) Let $\pi : Y \to X$ be a smooth (i.e., submersive) morphism, $F \in \mu(D_X)$. Then $\pi^\Delta(F)$ is coherent if and only if $F$ is coherent and

$$\text{S.S.}(\pi^\Delta F) = \{(y,\xi) | \xi = d^*\pi_{T^*(\pi y)\to T^*(y)}\eta, (\pi(y), \eta) \in \text{S.S.}F\}.$$

Let us note that in these two cases one important characteristic of S.S. is preserved. Namely, if we define the defect of $F$ as def$(F) = \dim$ S.S.$F - \dim X$, then the defect is preserved.

**10. Theorem on defect.**

**Theorem.** *Let $F \neq 0$ be a coherent $D_X$-module. Then def$(F) \geq 0$, i.e., dim S.S.$(F) \geq$ dim $X$.*

*Proof.* Suppose that dim S.S.$(F) < n = \dim X$. Then $F$ is supported on some proper closed subset $Z \subset X$. Restricting to an appropriate open subset we can assume that $Z$ is not empty and nonsingular. By Kashiwara's theorem $F = i_+(H)$, where $i : Z \to X$, $H$ be a coherent $D_Z$-module. Then $d(F) = d(H) < 0$ and we have a contradiction by induction on dim $X$.

**11. Holonomic $D$-modules.**

Coherent $D_X$-module $F$ is called *holonomic* if def$(F) \leq 0$, i.e., dim S.S.$(F) \leq$ dim $X$, i.e., $F$ has "minimal possible size". Holonomic modules will play a central role in our discussion.

*Example.* $O$-coherent $D$-modules are holonomic. The full subcategory of $\mu_{\text{coh}}(D_X)$, consisting of holonomic $D_X$-modules I will denote by Hol$(D_X)$.

**Proposition.** *a) Sub-category Hol is closed with respect to subquotients and extensions.*

*b) Each holonomic $D_X$-module has a finite length.*

*c) If $F$ is a holonomic $D_X$-module, then there exists an open dense subset $U \subset X$ such that $F|U$ is $O$-coherent $D_U$-module.*

*Proof.* a) and b) easily follow from Remark 1 in 8. Indeed if $n = \dim X$, then $m_n(H)$ is an additive characteristic on subquotients of $F$ which is strictly positive by the theorem on defect. Hence $F$ has a finite length. Another proof is based on the existence of a contravariant duality $D : \mathrm{Hol} \to \mathrm{Hol}$, such that $D^2 = \mathrm{id}_{\mathrm{Hol}}$, which will be proved next time. This duality implies that $F$ satisfies together ascending and descending chain conditions, i.e., $F$ has a finite length.

In the proof of c) put $S = \mathrm{S.S.}(F) \setminus X$. Since $F_\Sigma$ is a graded $\Sigma$-module, $S$ is invariant with respect to homotety in fibers of $T^*X$. It means that projection $p : T^*X \to X$ has at least 1-dimensional fibers on $S$. Hence $\dim p(S) < \dim S \leq \dim X$. After replacing $X$ by a suitable open subset $U \subset X \setminus p(S)$ we can assume that $S = \emptyset$, i.e., $\mathrm{S.S.}F \subset X$, i.e., $F$ is $O$-coherent.

**1. Main theorem A.**

We call a $D_X$-complex $\dot{F}$ *holonomic* if all its cohomology sheaves $H^i(\dot{F})$ are holonomic $D_X$-modules. The full subcategory of $D(D_X)$ consisting of holonomic $D_X$-complexes we denote by $D_{\mathrm{hol}}(D_X)$.

*Remark.* I do not know whether the natural inclusion $d(\mathrm{Hol}(D_X)) \to D_{\mathrm{hol}}(D_X)$ is an equivalence of categories. In a sense, I do not care.

**Main theorem A.** *Let $\pi : Y \to X$ be a morphism of algebraic varieties. Then*

$$\pi_* D_{\mathrm{hol}}(D_Y) \subset D_{\mathrm{hol}}(D_X), \quad \pi^! D_{\mathrm{hol}}(D_X) \subset D_{\mathrm{hol}}(D_Y).$$

The proof of the theorem is based on the following

**Key lemma.** *Let $i : Y \to X$ be a locally closed imbedding, $\dot{H} \subset D_{\mathrm{hol}}(D_Y)$. Then $i_*(\dot{H}) \subset D_{\mathrm{hol}}(D_X)$. We will prove the lemma in the subsection 8.*

**2. Proof of theorem A for $\pi^!$.**

It is sufficient to check 2 cases

a) $\pi$ is a smooth morphism (e.g., $\pi$ is a projection $\pi : Y = T \times X \to X$). In this case $\pi^\Delta$ is exact and $\pi^\Delta(\mathrm{Hol}) \subset \mathrm{Hol}$ by 2.9, i.e., $\pi^! T D_{\mathrm{hol}}(D_X) \subset D_{\mathrm{hol}}(D_Y)$.

b) $i : Y \to X$ is a closed imbedding. Let $j : U = X \setminus Y \to X$ be the imbedding of the complementary open set. For $\dot{F} \in D_{\mathrm{hol}}(D_X)$ consider the exact triangle

$$i_*(i^! \dot{F}) \longrightarrow \dot{F} \longrightarrow j_*(\dot{F}|U).$$

By the key lemma k$i_*(\dot{F}|U)$ is a holonomic $D_X$-complex. Hence $i_*(i^! \dot{F})$ is also holonomic. Now since the functor $i_*$ is exact and preserves the defect of a module, we can conclude that $i^! \dot{F}$ is a holonomic $D_Y$-complex.

**3. Criteria of holonomicity.**

**Criterion.** *Let $\dot{F}$ be a $D_X$-complex. Then $\dot{F}$ is holonomic iff $\dot{F}$ is coherent and for any point $x \in X$ the fiber $(i_x^! \dot{F})$ of $F$ at $x$ is finite dimensional.*

*Proof.* Direction "only iff" follows from 2. To prove "if" direction we need some general

**Lemma.** *Let $F$ be a coherent $D_X$-module. Then there exists an open dense subset $U \subset X$ such that $F|U$ is locally free as $O_U$-module.*

*Proof.* We assume $X$ to be affine and irreducible. Consider a good filtration $\Phi$ on $F$ and the associated $\Sigma$-module $F_\Sigma$. Since $F_\Sigma$ is a finitely generated $\Sigma$-module and $\Sigma$ is a finitely generated algebra over $O_X$, general results of commutative algebra imply that we can replace $X$ by an open dense affine subset $U \subset X$

such that $F_\Sigma|U$ is free as $O_U$-module (see EGA IV, 6.9.2). Since $F_\Sigma = \oplus F^n_\Sigma = \oplus(F^n/F^{n-1})$, all modules $F^n_\Sigma$ are projective as $O_U$-module. This proves the lemma.

Now let us prove that a coherent $D_X$-complex $\dot F$ with finite dimensional fibers is holonomic.

We use induction on dim $S$ = supp $\dot F$. Choose an open nonsingular subvariety $Y \subset S$ such that $\dim(S \setminus Y) <$ dim $S$ and put $\dot H = i^!_Y \dot F \subset D(D_Y)$. Then $\dot H$ is coherent and hence, replacing $Y$ by a suitable open dense subset, I can assume that all cohomology sheaves of $\dot H$ are locally free as $O_Y$-modules.

At each point $y \in Y$ the fiber $i^!_Y \dot H = i^!_Y \dot F$ is finite dimensional. Since $i^!_y$ up to a shift is equal to $Ri^\Delta_y$, and all cohomology sheaves of $\dot H$ are $i^\Delta_y$ acyclic (since they are $O$-free), it simply means that fibers of all these sheaves are finite dimensional, i.e., these sheaves are $O$-coherent. Hence $\dot H$ is holonomic and by the key lemma $i_*(\dot H)$ is also holonomic.

Replacing $\dot F$ by $\dot F' =$ cocone $(\dot F \to i_*(\dot H))$ we see that $\dot F'$ is coherent, since $\dot F$ and $i_*(\dot H)$ are, and all its fibres are finite dimensional (they are 0 outside of $S \setminus Y$ and coincide with fibers of $\dot F$ on $S \setminus Y$, since by base change $i^!_x i_*(\dot H) = 0$ for $X \notin Y$). Since dim supp $\dot F' <$ dim $S$, we see by induction that $\dot F'$ is holonomic and hence $\dot F$ is holonomic.

*Remark.* The proof above proves also the following

**Criterion.** *A $D_X$-complex $\dot F$ is holonomic if and only if there exists a smooth stratification $S = \{X = UX_i\}$ of $X$ such that all components (see 2.4) $H_i = r^!_i \dot F \subset D(D_{X_i})$ of the corresponding stratification of $\dot F$ are 0-coherent (i.e., all their cohomology sheaves are 0-coherent).*

**4. Proof of theorem A for $\pi_*$.**

Since the case of locally closed imbedding is contained in the key lemma, it is sufficient to consider morphism $\pi : Y = T \times X \to X$, where $T$ is a complete variety.

Let $\dot H \in D_{\mathrm{hol}}(D_Y)$, $\dot F = \pi_*(\dot H) \in D(D_X)$. In order to prove that $\dot F$ is holonomic we use criterion ¿from 3. Since $\pi$ is proper, $\dot F$ is coherent by 2.7. For any point $x \in X$ using base change we have

$$i^!_x \dot F = (\pi_x)_*(i^!_{T_x} \dot H), \quad \text{where}$$

$$T_x = \pi^{-1}(x) \simeq T, \quad i_{T_x} : T_x \longrightarrow Y \quad \text{and} \quad \pi_x : T_x \longrightarrow x$$

are natural inclusion and projection. By 2., $i^!_{T_x} \dot H$ is holonomic. Since $\pi_x$ is proper, it maps this complex into a coherent complex, i.e., $i^!_x \dot F$ is coherent, which means finite dimensional. QED

**5. Theoremof J.E. Roos.**

In order to prove the key lemma and introduce a duality on holonomic modules we need the following important result, due to J.E. Roos, which gives a connection between S.S.F. and homological properties of $F$.

Consider the $D_X$-module $D^\Omega_X$, described in 1.4, which has a second structure of a left $D_X$-module. For any coherent $D_X$-module $F$ this structure defines the structure of $D$-module on all sheaves $Ext^i_{D_X}(F, D^\Omega_X)$. Note that if $F$ is not coherent, these sheaves are not quasicoherent; we will not consider this case.

**Theorem.** *0. $F$ has a finite resolution by locally projective $D_X$-modules.*

*1. codim S.S.$(Ext^i_{D_X}(F, D^\Omega_X) \geq i$.*

*2. If codim S.S.$F = k$, then*

$$Ext^i_{D_X}(F, D^\Omega_X) = 0 \qquad for \quad i < k.$$

We postpone the proof of the theorem until 3.15.

**Duality functor.**

Let us define duality $D : D_{\text{coh}}(D_X)^o \longrightarrow D_{\text{coh}}(D_X)$ by

$$D(\dot{F}) = R\ \text{Hom}_{D_X}(\dot{F}, D^\Omega_X)[\dim\ X].$$

It means that we should replace $\dot{F}$ by a complex $\dot{P}$ of locally projective coherent $D$-modules $\dot{P} = \{\longrightarrow P_{-1} \longrightarrow P_0 \longrightarrow P_1 \longrightarrow \dots\}$ and put $D\dot{F} = D\dot{P}$, given by $D\dot{P}_i = *(P_{-\dim\ X-i})$, where $*P_j = \text{Hom}_{D_X}(P_j, D^\Omega_X)$.

Since $**P \simeq P$, we have $DD = \text{Id}$. Also by definition

$$H^i(DF) = Ext^{\dim\ X+i}_{D_X}(F, D^\Omega_X) \quad \text{for} \quad F \in \mu_{\text{coh}}(D_X).$$

**Corollary of J.E. Roos's theorem.** *Let $F$ be a coherent $D_X$-module. Then*

*a) complex $DF$ is concentrated in degrees between - dim $X$ and 0, i.e., $H^i(DF) = 0$ for $i \notin [-dim\ X, 0]$.*

*b) $F$ has a locally projective resolution of the length $\leq$ dim $X$.*

*c) $F$ is holonomic iff $DF$ is a module, i.e., $H^i(DF) = 0$ for $i \neq 0$.*

*d) $D$ gives an autoduality $D : Hol(D_X)^0 \longrightarrow Hol(D_X)$, i.e., $D$ is a contravariant functor, such that $DD = Id_{\text{Hol}}$. In particular, $D$ is exact.*

*Proof.*

a) Put $E^i = Ext^i_{D_X}(F, D^\Omega_X)$. By Roos's theorem def$(E^i) = \dim$ S.S.$E^i - \dim X = \dim X -$ codim S.S.$E^i$ is negative if $i > \dim X$. Hence by theorem of defect $E^i = 0$ for these $i$, and also for $i < 0$. This means that $H^i(DF) = 0$ for $i > 0$ and for $i < -\dim X$.

b) We should prove that locally $F$ has a projective dimension $\leq \dim X$. So we assume that $X$ is affine and $F$ has a finite projective resolution $\dot{P}$. Dual complex $D\dot{P}$ consists of projective modules and by a) is acyclic in degrees $i > 0$. This means that $D\dot{P} = \dot{P}' \oplus \dot{P}''$, where $\dot{P}'_i = 0$ for $i > 0$ and $i < -\dim X$ and $\dot{P}''$ is acyclic. Then $D(\dot{P}')$ gives a resolution of $P$ of the length $\leq \dim X$.

c) If $F$ is holonomic, then $H^i(DF) = Ext^{\dim\ \text{X+i}}_{D_X}(F, D^\Omega_X) = 0$ for $i < 0$ by Roos's theorem, i.e., $DF$ is a module. Conversely, if $F' = DF$ is a module, then $F = DF'$ again is a module, i.e., $F = Ext^{\dim\ X}_{D_X}(F', D^\Omega_X)$ and by Roos's theorem codim S.S.$F \geq \dim X$, i.e., dim S.S.$F \leq \dim X$.

d) follows from c) and $DD = \text{Id}$.

*Remark.* 1. Property c) was the reason for the normalization [dim $X$] in the definition of duality $D$.

2. It is clear from d) that $D_{\text{hol}}(D_X)) = D_{\text{hol}}(D_X)$.

**7. Extension lemma.** *Let $F \in D_X$, $U$ must be an open subset of $X$ and $H \subset F\big|_U$ a holonomic $D_U$-module. Then there exists a holonomic $D_X$-submodule $F' \subset F$, such that $F'\big|_U = H$.*

*Proof.* We can assume that $F$ is coherent and $F\big|_U = H$ (using extension principle for coherent $D$-modules). Consider $D_X$-complex $DF$. It has cohomologies in dimensions $\leq 0$. Put $G = H^0(DF)$, $F' = DG$. By Roos's theorem dim S.S.$G \leq$ dim $X$, i.e., $G$ is a holonomic $D_X$-module. Hence $F'$ is also harmonic.

Natural morphism $DF \to G$ defines a morphism $F' = DG$ into $F = DDF$ (one can check that this morphism is an imbedding). It is clear that $F\big|_U = H = F'\big|_U$. Hence $F'$ (or image of $F'$ in $F$) is the holonomic submodule we looked for.

*8. Proof of the key lemma.*

Step 1. For closed imbeddings the lemma follows from 2.9. Hence we can assume that $i : Y \to X$ is an open imbedding. Also we assume that $X$ is affine and $\dot{H} = H$ is a holonomic $D$-module, generated by one section $u$. Consider a covering of $Y$ by affine open subsets $Y_\alpha$ and replace $H$ by its Čech resolution, consisting of $(i_\alpha)_+(H\big|_{Y_\alpha})$. This trick reduces the proof to the case when $Y$ is affine, i.e, $Y$ has a form $Y = X_f = \{x | f(x) \neq 0\}$ for some regular function $f$ on $X$. In this case $i_* = i_+$ is an exact functor.

Step 2. Thus we have an affine variety $X$, a function $f \in O(X)$, an open subset $i : Y = X_f \hookrightarrow X$ and a holonomic $D_Y$-module $H$, generated by a section $u$, and we want to prove that $D_X$-module $F = i_+(H)$ is holonomic.

The difficult point is to prove that $F$ is coherent. What does it mean?

Since global sections $F(X)$ and $H(Y)$ coincide and $D(Y) = \bigcup_{n=-\infty}^{0} D(X) f^n$, we see that $D_X$-module $F$ is generated by sections $f^n u$ for all $n \in \mathbb{Z}$. Hence what we really want to prove is the statement:

$$\text{for all } n \ll 0 \quad f^n u \in D(X)(f^{n+1}u). \tag{*}$$

This follows immediately from the following

**Lemma on $b$-functions.** *There exists a polynomial in $n$ operator $d_0 \in D(X)[n]$ and a nonzero polynomial $b_0 \in k[n]$ such that*

$$d_0(f^{n+1}u) \equiv b_0(n) \cdot (f^n u). \tag{**}$$

**Step 3.** Proof of the lemma.

We extend our situation by extension of scalars $k \longrightarrow K = k(\lambda)$ – the field of rational functions. Denote by $\widehat{Y}$, $\widehat{X}$ extended varieties and define $D_{\widehat{Y}}$-module $\widehat{H}$ and $D_{\widehat{X}}$-module $\widehat{F}$ by

$\widehat{H} = f^\lambda \cdot (K \otimes_k H)\cdot$ and the structure of $D_{\widehat{Y}}$-module is given by

$\xi(f^\lambda h) = \frac{\lambda \xi(f)}{f} \cdot f^\lambda h h + f^\lambda \cdot \xi h$, $\xi$ a vector field on $Y$, $\widehat{F} = i_+(\widehat{H})$.

The $D_{\widehat{Y}}$-module $\widehat{H}$ is holonomic and by extension lemma 7, $\widehat{F}$ contains a holonomic $D_{\widehat{Y}}$-module $E$ such that $E\big|_{\widehat{Y}} = \widehat{H}$, i.e., the quotient $D_{\widehat{Y}}$-module $\widehat{F}/E$ is concentrated on $X \setminus Y$.

Consider the section $\widehat{u} = f^\lambda u \in F$. Since its image in $\widehat{F}/E$ is concentrated on $X \setminus Y$, it is annihilated by some power of $f$, i.e., $f^{n_0} \cdot \widehat{u} \in E$. Moreover, since $E$ is holonomic, it has a finite length, that implies that for some $n$ there exists $d \in D_{\widehat{X}}$ such that $d(f^{n+1}\widehat{u}) = f^n\widetilde{u}$. In other words, $d(f^{n+1+\lambda}u) = f^{n+\lambda}u$. Since we can everywhere replace $\lambda$ by $\lambda + n$ we have proved the existence of $d \in D_{\widetilde{X}}$ such that

$$d(f^{\lambda+1}u) = f^\lambda u.$$

Now we can write $d = d_o/b_o$, $d_o \in D(X)[\lambda],\ b_o \in k[\lambda]$. Then $d_o, b_o$ satisfy (**).

**Step 4.** Now, when we know that $F$ is coherent, let us prove that it is holonomic.

First of all, lemma on $b$-functions implies that $\widetilde{F} = E$ (notations from the step 3), i.e., $\widetilde{F}$ is holonomic and is generated by $\widetilde{u}$. It means that we can choose operators $d_1, \ldots, d_\ell \in D_{\widetilde{X}}$ such that the set $\Delta \subset T^*(\widetilde{X})$ of common zeroes of their symbols $\sigma_1, \ldots, \sigma_\ell \in \widetilde{\Sigma}$ has dimension dim $\Delta \leq$ dim $X$.

For almost any $n \in \mathbb{Z}$ we can substitute $n \longmapsto \lambda$, and we obtain operators $d_i^{(n)} \in D_X$, their symbols $\sigma_i^{(n)} \in \Sigma$ and the set $\Delta^{(n)} \in T^*(X)$ of their common zeroes, such that dim $\Delta \leq$ dim $X$ and $d_i^{(n)}(f^n u) = 0$. These formulae imply that $f^n u$ lies in a holonomic submodule of $F$. Since $F$ is generated by $f^n u$ for any $n$, which is $\ll 0$, it implies that $F$ is holonomic.

**9. Functors $\pi_!, \pi^*$ and their properties.**

For any morphism $\pi : Y \to X$ we define functors

$$\begin{aligned} &\pi_! : D_{\mathrm{hol}}(D_Y) \longrightarrow D_{\mathrm{hol}}(D_X) \\ &\pi^* : D_{\mathrm{hol}}(D_X) \longrightarrow D_{\mathrm{hol}}(D_Y) \qquad \text{by} \\ &\pi_! = D\pi_* D \\ &\pi^* = D\pi^! D. \end{aligned}$$

This definition makes sense as $\pi_*$ and $\pi^!$ maps holonomic complexes into holonomic.

Let us list some properties of $\pi_!$ and $\pi^*$.

1. There exists the canonical morphism of functors $\pi_! \to \pi_*$ which is an isomorphism for proper $\pi$.
2. The functor $\pi_!$ is left adjoint to $\pi^!$.
3. The functor $\pi^*$ is left adjoint to $\pi_*$.
4, If $\pi$ is smooth, $\pi^! = \pi^*[2(\dim Y - \dim X)]$.

Let us comment on these properties. By definition 3. follows from 2.

Consider in more detail the case when $\pi = j : Y \to X$ is an open imbedding. In this case $j^* = j^! =$ restriction on $Y$, i.e., $j^*$ is left adjoint to $j_*$ and hence $j_! = Dj_*D$ is left adjoint to $j^! = Dj^*D$. For any $\dot{H} \subset D_{\mathrm{hol}}(D_Y)$ the restriction of $j_!(\dot{H})$ on $Y$ coincides with $\dot{H}$, that gives a canonical morphism $j_!\dot{H} \to j_*\dot{H}$, identical on $Y$.

Thus it remains to prove properties 1 and 2 for proper $\pi$ and 4 for smooth $\pi$. But these properties have nothing to do with holonomicity, since $\pi_*$ for proper $\pi$ and $\pi^!$ for smooth $\pi$ map coherent $D$-modules into coherent. We will prove them in reasonable generality.

**10. The duality theorem for a proper morphism.**

**Theorem.** *Let $\pi : Y \to X$ be a proper morphism. Then on the category of coherent $D$-complexes*
*a) $D\pi_* = \pi_* D$ and*
*b) $\pi_*$ is left adjoint to $\pi^!$.*

*Proof of the statement a).*

**Case 1** $\pi$ is a closed imbedding. Let $P$ be a locally projective $D_Y$-module. I claim that $\pi_*(P)$ and $D\pi_*D(P)$ are $D_X$-modules and they are canonically isomorphic. It is sufficient to check this locally, so we can assume that $P = D_Y$. In this case it follows from the formula

$$R\ \mathrm{Hom}_{D_X}(D_{X\leftarrow Y}, D_X) = D_{Y\to X}[\dim\ Y - \dim\ X].$$

**Case 2** We call a $D_Y$-module $P$ elementary if it has the form $P = D_Y \otimes_{O_Y} \pi^{\Delta}(V)$ for some locally free $O_X$-module $V$. Considerations from 2.1b) show that each $D_Y$-module has a resolution, consisting from elementary modules. I claim that for elementary $D_Y$-module $P$ $D_X$-complexes $D\pi_*(P)[-\dim\ Y]$ and $\pi_*(DP)[-\dim\ Y]$ are sheaves and they are canonically isomorphic.

This fact is local, so I can assume $P = D_Y$.

The claim follows from the formulae

$R\ \mathrm{Hom}_{D_Y}(D_{Y\to X}) = D_{X\leftarrow Y}[\dim\ X - \dim\ Y]$

$D_{Y\to X} = D_X \otimes_k O_{\mathbb{P}}, \quad D_{X\leftarrow Y} = D_X \otimes_k \Omega_{\mathbb{P}},$

$R\Gamma(O_{\mathbb{P}}) = k, \quad R\Gamma(\Omega_{\mathbb{P}}) = k[-\dim\ \mathbb{P}].$

This proves a).

**11. $\mathbf{Hom}_{D_X}$ and internal Hom.**

Usually one can write homomorphisms of 2 sheaves as global sections of the sheaf of homomorphism. Let us look, how to set it for $D$-modules.

Of course, we have for $D_X$-modules $F, F'$ the following formula

$$\mathrm{Hom}_{D_X}(F, F') = \Gamma(\mathrm{Hom}_{D_X}(F, F'))$$

or, in derived category,

(*) $$R\mathrm{Hom}_{D_X}(\dot{F}, \dot{F}') = R\Gamma(R\mathrm{Hom}_{D_X}(F, F')).$$

This formula, by the way, implies that

(**) $$\begin{aligned}\text{Homol. dim Hom}_{D_X} &\leq \text{Homol. dim } \Gamma + \text{Homol. dim Hom}_{D_X}\\ &\leq \text{Homol. dim of } \mu(O_X) + \dim\ X \leq 2\dim\ X.\end{aligned}$$

But I want to write down $R\mathrm{Hom}$ in terms of functors, suitable for $D$-modules.

**Definition.** *Functors "!-tensor product"* $\boxtimes : D(D_X) \times D(D_X) \longrightarrow D(D_X)$ *and "internal Hom" Hom:* $D_{\mathrm{coh}}(D_X)^o \times D(D_X) \longrightarrow D(D_X)$ *are defined by*

$$\dot{F} \mathbin{\bigtriangleup} \dot{H} = \triangle^!(\dot{F} \boxtimes \dot{H}), \quad \mathit{Hom}(\dot{F}, \dot{H}) = D\dot{F} \mathbin{\bigtriangleup} \dot{H}$$

*where* $\triangle : X \longrightarrow X \times X$ *is the diagonal imbedding,* $\boxtimes$ *is the exterior tensor product over* $k$*.*

**Proposition.** $\mathit{RHom}_{D_X}(\dot{F}, \dot{H}) = \int_X \mathit{Hom}(\dot{F}, \dot{H})$ *where* $\int_X : D(D_X) \longrightarrow D(\mathit{Vect})$ *is the direct image of the projection of* $X$ *onto a point.*

*Proof.* If $F, H$ are $D_X$-modules, we define $D_X$-module structure on $F \otimes_{O_x} H$ by Leibniz rule. It is clear that $\bigtriangleup = L(\otimes_{O_X})[\dim X]$. (Left derived functor.)

Consider the case when $F$ is coherent and locally projective. Then

$$\mathrm{Hom}(F, H) = \mathrm{Hom}_{D_X}(F, D_X^\Omega) \otimes_{O_X} H = \mathrm{Hom}_{D_X}(F, D_X^\Omega \otimes_{O_X} H).$$

Let us compute $\Omega_X \otimes_{D_X} \mathrm{Hom}(F, H)$. We have

$$\Omega_X \otimes_{D_X} \mathrm{Hom}(F, H) = \Omega_X \otimes_{D_X} \mathrm{Hom}_{D_X}(F, D_X^\Omega \otimes_{O_X} H)$$

$$\mathrm{Hom}_{D_X}(F, \Omega_X \otimes_{D_X} (D_X^\Omega \otimes_{O_X} H)) = \mathrm{Hom}_{D_X}(F, H).$$

Applying this formula we see that

$$\int_X \mathrm{Hom}(F, H) = R\Gamma(\Omega_X \otimes_{D_X} \mathrm{Hom}(F, H) = R\Gamma(\mathrm{Hom}_{D_X}(F, H)) = R\mathrm{Hom}_{D_X}(F, H).$$

General case is proved using resolutions.

**12. Proof of the duality theorem, statement b).**

Using base change it is easy to check the projection formula

$$\pi_*(\dot{H} \mathbin{\bigtriangleup} \pi^! \dot{F}) = \pi_*(\dot{H}) \mathbin{\bigtriangleup} \dot{F}.$$

By duality theorem a) we can replace $\dot{H}$ and $\pi_* \dot{H}$ on dual complexes and obtain

$$\pi_*(\mathrm{Hom}(\dot{H}, \pi^! \dot{F})) = \mathrm{Hom}(\pi_*(\dot{H}), \dot{F}).$$

Now, applying integral $\int$, we have

$$R\mathrm{Hom}_{D_Y}(\dot{H}, \pi^! \dot{F}) = R\mathrm{Hom}_{D_X}(\pi_*(\dot{H}), \dot{F}),$$

QED.

**13. Functor $\pi^*$ for smooth morphisms.**

In order to finish the proof of property 4 in 9 we should check, that for a smooth morphism $\pi : Y \to X$ and a coherent $D_X$-complex $\dot{F}$ one has

$$D\pi^! \dot{F} = \pi^! D\dot{F}[-2k],$$

where $k = \dim\, Y - \dim\, X$.

As in 10. the proof can be reduced to the statement, that

$$R\mathrm{Hom}_{D_Y}(D_{Y\to X}, D_Y) = D_{X\leftarrow Y}[-k].$$

This statement is proved by studying the resolution $DR_{Y/X}(D_Y)$.

**14. Classification of irreducible holonomic modules.**

**Theorem.** *Let $i : Y \to X$ be an affine imbedding with $Y$-irreducible, $E$ an irreducible $O$-coherent $D_Y$-module. Put*

$$i_{!*}E = Im(i_!E \longrightarrow i_*E).$$

*a) $i_{!*}E$ is an irreducible holonomic module. It is a unique irreducible submodule of $i_*E$ (and unique irreducible quotient of $i_!E$). Also it can be characterized as the unique irreducible subquotient of $i_*E$ (or $i_!E$) which restriction to $Y$ is nonzero.*

*b) Any irreducible holonomic module $F$ has a form $i_{*!}E$ for some affine imbedding $i : Y \to X$ with irreducible $Y$ and irreducible $O$-coherent $D_Y$-module $E$.*

We will denote this irreducible holonomic module by $L(Y, E)$.

c) $L(Y, E) = L(Y', E')$ if and only if $\bar{Y} = \bar{Y}'$ and restrictions of $E$ and $E'$ to some subset $U \subset Y \cap Y'$, open in $Y$ and in $Y'$ are isomorphic.

*Remark.* We also will use notation $L(Y, E)$ for nonaffine imbeddings $i : Y \to X$. In this case we should replace $i_!E$ and $i_*E$ by their zero components $H^o(i_!E)$ and $H^o(i_*E) = i_+E$, and denote by $L(Y, E)$ the image of $i_!E \to i_*E$.

*Proof.* a) According to theorem A, $D_X$-modules $i_!E$ and $i_*E$ are holonomic, and hence have finite lengths.

Let $F$ be any irreducible submodule of $i_*E$. Then since $\mathrm{Hom}(F, i_*E) = \mathrm{Hom}(i^!F, E) \neq 0$ and $i^!F$ is irreducible, as well as $E$, we see that $E = i^!F$. Since $i^!i_*E = E$, there exists only one irreducible subquotient $F$ of $i_*E$ with the property that $i^!F \neq 0$ and in particular, only one irreducible submodule.

Applying the same arguments to $i_!E$ we see that it has a unique irreducible quotient.

Further, $\mathrm{Hom}(i_!E, i_*E) = \mathrm{Hom}(E, i^!i_*E) = k$, and the same is true for $\mathrm{Hom}(i_!E, F)$, where $F$ is a unique irreducible submodule of $i_*E$. This shows, that $F = \mathrm{Im}(i_!E \to i_*E)$.

b) Let $F$ be an irreducible holonomic $D$-module, $Y$ an open affine subset of an irreducible component of Supp $F$. Then sheaf $E = i^!(F)$ is irreducible holonomic $D_Y$-module and, decreasing $Y$, we can assume it is $O$-coherent. Since $\mathrm{Hom}(F, i_*E) \neq 0$, $F = L(Y, E)$.

c) The same proof that in a), b).

**15. Sketch of the proof of Roos's theorem.**

*Step 1.* Let $\dot{F} = \{\longrightarrow F^1 \longrightarrow F^2 \longrightarrow \ldots \longrightarrow F^k \longrightarrow\}$ be a complex of $D_X$-modules, $\{\phi_1\}$ good filtrations on $F_i$, which are compatible with $d$. Then it induces a complex of coherent $\Sigma$-modules

$$\dot{F}_\Sigma = \{0 \longrightarrow F^1_\Sigma \longrightarrow \ldots \longrightarrow F^k_\Sigma \longrightarrow 0\}.$$

**Lemma.** *$H^i(\dot{F})_\Sigma$ is a subquotient of $H^i(\dot{F}_\Sigma)$.*

In particular, if $\dot{F}_\Sigma$ is exact then $\dot{F}$ is exact. Also S.S.$H^i(\dot{F}) \subset$ Supp $H^i(\dot{F}_\Sigma)$.

*Step 2.* The statement of theorem is local, so I will assume $X$ to be small. Let $F$ be a $D_X$-module, $\phi$ a good filtration on $F, F_\Sigma$ the associated graded $\Sigma$-mdoule.

Since $T^*X$ is regular of dim $T^*X = 2n$, I can find a free resolution $\dot{C} = \{0 \to C_{-2n} \to \ldots C_o \to F_\Sigma \to o\}$ of $\Sigma$-mdoule $F_\Sigma$. Then it is easy to check that I can lift $\dot{C}$ to a complex of free $D_X$-modules with a good filtration $\phi$ $\dot{P} = \{0 \to P_{-2n} \to \ldots \to P_o \to F \to 0\}$ such that $\dot{P}_\Sigma = \dot{C}$. Then by step 1, $P$ is a free resolution of $F$.

*Step 3.* For any locally projective $D_X$-module $P$, I denote by $*P$ the $D_X$-module $\mathrm{Hom}_{D_X}(P, D_X^\Omega)$. By definition, $\mathrm{Ext}(F, D_X^\Omega)$ are calculated as homologies of the complex

$$*\dot{P} = \{0 \longrightarrow *P_0 \longrightarrow *P_{-1} \longrightarrow \ldots \longrightarrow *P_{-2n}\}.$$

If we consider the natural filtration on $D_X^\Omega$ and induced filtration on $*\dot{P}$, we will get precisely the complex $*\dot{C} = \{\mathrm{Hom}_\Sigma(C_{-i}, \Sigma)\}$. (Here I identify $\Omega_X$ with $O_X$.)

Now we should apply the fact, that the statement of the theorem is true for commutative regular ring $\Sigma$. Applying now step 1 we can deduce from this corresponding statement for $D_X$-modules.

## 4. Holonomic $D$-modules with regular singularities ($RS$-modules).

It turns out that class of holonomic $D$-modules contains a natural subclass, invariant with respect to all operations - - subclass of $RS$-modules.

### 1. $RS$-modules on a curve.

First of all, let us consider the classical case-modules with regular singularities on a curve.

Let $C$ be a curve. Choose a nonsingular curve $C^+$, which contains $C$ as an open dense subset and a point $c \in C^+ \setminus C$ (it plays a role of infinity for $C$). Let $t$ be a local parameter at $c$, $\partial = \partial/\partial_t$, $d = t\partial \in D_{C^+}$. We denote by $D_C^\nu$ the subsheaf of subalgebras of $D_{C^+}$, generated by $d$ and $O_{C^+}$. It is clear that $D_C^\nu$ and element $d$ in quotient algebra $D^\nu/tD^\nu$ do not depend on the choice of a local parameter $t$.

**Definition.** *a) Let $F$ be an $O$-coherent $D_C$-module. We say that $F$ has a $RS$ at the point $c$, if its direct image $F^+ = (i_{C\to C^+})_+F$ is a union of $O$-coherent $D_C^\nu$-submodules.*

*b) We say that an $O$-coherent $D_C$-module $F$ is $RS$, if it has $RS$ at all points on infinity (i.e., at all points $c \in \widehat{C} \setminus C$ of the nonsingular completion $\widehat{C}$ of the curve $C$).*

**Definition.** *Let $F$ be a holonomic $D_C$-module on a curve $C$. We say that $F$ is $RS$ if its restriction to an open dense subset $U \subset C$ is $O$-coherent $RS$ $D_C$-module.*

**Lemma.** *Let $C, C'$ be irreducible curves $\pi : C \to C'$ a dominant (nonconstant) morphism. Then $D_{C'}$-module $F$ is $RS$ iff $\pi'(F)$ is $RS$; also $D_C$-module $H$ is $RS$ iff $\pi_*(H)$ is $RS$.*

### 2. $RS$ $D$-modules.

**Definition.** *a) Let $F$ be an $O$-coherent $D_X$-module. Then $F$ is $RS$ if its restriction to any curve is $RS$.*
*b) Let ?$(Y, E)$ be an irreducible holonomic $D_X$-module. We say that $F$ is $RS$ if $E$ is $RS$ $O$-coherent $D_Y$-module.*
*c) A holonomic $D_X$-module $F$ is $RS$ if all its irreducible subquotients are $RS$.*
*d) A holonomic $D_X$-complex $\dot{F}$ is $RS$ if all its cohomology sheaves are $RS$.*

We denote by $RS\ (D_X)$ the full subcategory of $\mathrm{Hol}(D_X)$, consisting of $RS$-modules, and by $D_{RS}(D_X)$ the full subcategory of $D(D_X)$ consisting of $RS$ $D_X$-complexes.

**Proposition.** *The category $RS(D_X)$ is closed with respect to subquotients and extensions.*

*Proof.* By definition.

### 3. Main Theorem B.

**Main Theorem B.**
*a) Functors $D$, $\pi_*$, $\pi^!$, $\pi_!$,$\pi^*$ preserve subcategory*

$$D_{RS}(D) \subset D_{\mathrm{Hol}}(D).$$

*b) RS − criterion*

*An holonomic $D_X$-complex $\dot{F}$ is RS if and only if its restriction $i^!_C\dot{F}$ to any curve $C \subset X$ is RS.*

*Remark.* It would be more natural to take b) as a definition of $RS$ $D_X$-complexes. But then it would be difficult to prove "subquotient" properties, like lemma in 2. So we prefer the definition, which makes these properties trivial, and transfers all the difficulties into the "cohomological part", where we have an appropriate machinery to work with.

The proof of theorem B contains two technical results both due to P. Deligne. The first describes $RS$ property of $O$-coherent $D$-modules without referring to curves. The second proves that $\pi_*$ preserves $RS$ in a simplest case.

**4. $D$-modules with regular singularities along a divisor.**

Let $X$ be an algebraic variety. A regular extension of $X$ is a nonsingular variety $X^+$, containing $X$ as an open subset, such that $X^\nu = X^+ \setminus X$ is the divisor with normal crossings. We denote by $J \subset O_{X^+}$ the ideal of $X^\nu, T^\nu$ the subsheaf of vector fields preserving $J$ and $D^\nu_X$ the subalgebra of $D_{X^+}$, generated by $T^\nu$ and $O_{X^+}$.

Let $F$ be an $O$-coherent $D_X$-module $F^+ = (i_{X\to X^+})_+F$.

**Proposition (P. Deligne)..** *The following conditions are equivalent.*

*(i) $F^+$ is a union of $O$-coherent $D^\nu_X$ submodules*

*(ii) For any extended curve*

$\sigma : (C^+, C) \longrightarrow (X^+, X)$ *(i.e., $\sigma : C^+ \to X^+$, such that*

$\sigma(C) \subset X,\ \sigma(c) \in X^+ \setminus X)$ *$F|_C$ has $RS$ at $c$.*

*(iii) For each irreducible component $W$ of $X^\nu$ there is an extended curve $\sigma : (C^+, C) \longrightarrow (X^+, X)$ which intersects $WK$ transversally at $c$ such that $F|_C$ has $RS$ at $c$.*

**Corollary.** *Suppose $X^+$ is a complete regular extension of $X$, $F$ and $O$-coherent $D_X$-module. Then $F$ is RS iff $F^+$ is a union of $O$-coherent $D^\nu_X$-modules.*

**5. Proof of theorem B.**

**Key lemma.** *Let $\pi_* : Y \to X$ be a morphism, where $Y$ is a surface, $X$ is a curve, $X, Y$ are irreducible. Let $H$ be an $O$-coherent $kRS$ $D_Y$-module. Then for some open subset*

$$X_0 \subset X \qquad \pi_*(H)|_{X_0} \quad \text{is } RS.$$

*We will prove this lemma in 6.*

We also will use the following version of Hironaka's desingularisation theorem.

**Proposition.** *Let $\pi : Y \to X$ be a morphism. Then there exists a regular extension $i : Y \to Y^+$ and a morphism $\pi^+ : Y^+ \to X$ such that $\pi = \pi^+ oi$ and $\pi^+$ is a proper morphism.*

We will call the triple $(\pi^+, Y^+, i)$ the resolution of the morphism $\pi$.

Now let us start the proof of theorem B. By definition $RS$ is closed with respect to the duality $D$, and hence $D_{RS}$ is closed with respect to $D$.

*Proof of theorem B for $\pi_*$.* We have a morphism $\pi : Y \to X$ and an $RS$ $D_Y$-complex $\dot{H}$ and we want to prove that $\pi_*(\dot{H})$ is $RS$. The proof is by induction on the dimension of $S = Supp\ \dot{H}$. So we assume that the statement is true for dom $S < n$. Also we assume that $RS$-criterion of theorem B is true for dim $F < n$.

*Step 1.* Let $\pi = i : Y \to Y^+$ be an inclusion into a regular extension of $Y$, $H$ be an $RS$ $O$-coherent $D_Y$-module. Then $i_*(H)$ is $RS$ $D_{Y^+}$-module.

Since $i$ is an affine morphism $i_*(H) = i_+(H)$. Without loss of generality we can assume $Y^+$ to be complete. By Deligne's proposition $i_+(H)$ is a union of $O$-coherent $D_Y$-modules. Hence arbitrary irreducible subquotient $F$ of $i_+(H)$ has this property.

Let A$Z^+ =$ Supp $F$. Then it is easy to check that $Z^+$ is an irreducible component of an intersection of some components of the divisor $X^\nu$ and $F = L(Z, E)$, where $Z$ is an open subset of $Z^+$. It is clear that $E^+ = i_{Z \to Z^+}(E)$ is a union of $O$-coherent $D_Z^\nu$-modules, since $D_Z^\nu$ is a quotient of the algebra $D_Y^\nu$ and $E^+$ is a subquotient of $H^+$. Hence $E$ is $RS$, i.e., $F$ is $RS$.

## 6. Sketch of the proof of the key lemma.

We have a smooth morphism $\pi : Y \to X$ with dim $Y = 2$, dim $X = 1$. Then, after deleting several points from $X$, we can find a regular complete extension $Y^+$ of $Y$ and a morphism $\pi^+ : Y^+ \to X^+$, where $X^+$ is the regular completion of $X$, such that

(i) $\pi^{-1}(X^\nu) \subset Y^\nu$, where $X^\nu = X^+ \setminus X$, $Y^\nu = Y^+ \setminus Y$

(ii) $\pi^{-1}(X^\nu)$ contains all singularities of $Y^\nu$.

Denote by $T_Y^\nu$ and $T_X^\nu$ sheaves of vector fields on $Y^+$ and $X^+$, which preserve $Y^\nu$ and $X^\nu$. Conditions (i), (ii) imply that each local vector field $\xi \in T_X^\nu$ can be lifted locally to a vector field $\xi' \in T_Y^\nu$. This means that the natural morphism of sheaves on $Y^+$

$$\alpha : T_Y^\nu \longrightarrow (\pi^+)^* T_X^\nu = O_{Y^+} \bigotimes_{\pi^+ \cdot O_{X^+}} \pi^+ \cdot (T_X^\nu)$$

is epimorphic.

We denote by $T^{\nu}_{Y/X}$ the kernel of $\alpha$. Consider sheaves of algebras $D^{\nu}_{Y}$ and $D^{\nu}_{X}$ on $Y^{+}$ and $X^{+}$, generated by $T^{\nu}_{Y}$ and by $T^{\nu}_{X}$ and denote by $M^{R}(D^{\nu}_{Y})$, $M^{R}(D^{\nu}_{X})$ corresponding categories of right $D^{\nu}$-modules, and by $D^{R}(D^{\nu}_{Y}), D^{R}(D^{\nu}_{X})$ derived categories (here I prefer to work with right $D$-modules as all formulae are simple).

Let us put $D^{\nu}_{Y\to X} = O_{Y^{+}} \bigotimes_{\pi^{+}\cdot O_{X^{+}}} \pi^{+}\cdot(D^{\nu}_{X})$. This module is $D^{\nu}_{Y} - \pi^{+}\cdot(D^{\nu}_{X})$-bimodule. Using $D_{Y\to X}$ let us define the functor

$$\pi^{\nu}_{*} : D^{R}(D^{\nu}_{Y}) \longrightarrow D^{R}(D^{\nu}_{X}) \qquad \text{by}$$

$$\pi^{\nu}_{*}(E) = R(\pi^{+})\cdot(E \overset{L}{\bigotimes_{D^{\nu}_{Y}}} D^{\nu}_{Y\to X}).$$

*Statement.* (i) Let $H$ be a right $D_{Y}$-module, $H^{+} = (i_{Y})_{+}H \in M^{R}(D_{Y^{+}})$. Then, if we consider $H^{+}$ as $D^{\nu}_{Y}$-module, we have

$$\pi^{\nu}_{*}(H^{+}) = \pi_{*}(H^{+}) \qquad \text{as} \quad D^{\nu}_{X}\text{-module.}$$

(ii) if $E$ is an $O$-coherent $D^{\nu}_{Y}$-module, then

$$\pi^{\nu}_{*}(E) \quad \text{is} \quad O\text{-coherent} \quad D^{\nu}_{X}\text{-module.}$$

This statement implies the key lemma. Indeed, if $H$ is an $RS$ $O$-coherent (right) $D_{Y}$-module, then $H^{+}$ is an inductive limit of $O_{Y^{+}}$-coherent $D^{\nu}_{Y}$-modules and hence $\pi_{*}(H^{+}) = \pi^{\nu}_{*}(H^{+})$ is an inductive limit of $O_{X^{+}}$-coherent $D^{\nu}_{Y}$-modules, i.e., it is $RS$.

*Proof of statement.* (i) is an immediate consequence of the projection formula and the fact that $D^{\nu}_{Y}|_{Y} = D_{Y}$, $D^{\nu}_{Y\to X}|_{Y} = D_{Y\to X}$.

(ii) Consider "De Rham" resolution of $D_{Y\to X}$

$$0 \longrightarrow D^{\nu}_{Y} \bigotimes_{O_{Y}} T^{\nu}_{Y/X} \longrightarrow D^{\nu}_{Y} \longrightarrow D^{\nu}_{Y\to X} \longrightarrow 0.$$

Using it we see that as $O_{X^{+}}$-module

$$\pi^{\nu}_{*}(E) = R(\pi^{+})\cdot(E\otimes T^{\nu}_{Y/X} \longrightarrow E).$$

Since $\pi^{+}$ is a proper morphism, $R\pi^{+}$ maps coherent $O_{Y^{+}}$-modules into coherent $O_{X^{+}}$-modules, i.e., $\pi^{\nu}_{*}(E)$ is $O$-coherent for $O$-coherent $E$.

**2.** The following statement, due to P. Deligne, is a very useful criterion of $RS$.

**Criterion.** *Let $X^{+}$ be an irreducible complete normal (maybe singular) variety, $X \subset X^{+}$ an open nonsingular subset, $E$ an $O$-coherent $D_{X}$-module. Assume that for any component $W$ of $X^{\nu} = X^{+}\setminus X$ of* *codimension 1 in $X^+$, $S$ is RS along $W$ (i.e., $E$ satisfies conditions (i), (ii), (iii) in 4 along $W$). Then $E$ is RS.*

Unfortunately, the only proof of this criterion I know is analytic. I would like to have an algebraic proof.

**8. $RS$-modules with given exponents.**

Let us fix some $\mathbb{Q}$-linear subspace $\Lambda \subset kK$, containing 1. Let $C$ be a curve, $C^+$ its regular extension $c \in C^+ \setminus C$, $F$ an $RS$ $O$-coherent $D_C$-module, $F^+ = (i_C)_+F$. For any $O$-coherent $D^\nu$ submodule $E \subset F^+$ we denote by $\Lambda_c(E)$ the set of eigenvalues of the operator $d = t\partial$ in the finite-dimensional space $E/tE$ ($t$ is a local parameter at $c$, see 1). Now we define

$$\Lambda(F) = \bigcup_{c,E} \Lambda(E) \quad \text{for all } O\text{-coherent}$$

$D^\nu$-submodules of $F^+$ and all points $c \in C^+ \setminus C$.

The set $\Lambda(F)$ is called the set of exponents of $F$. We say that $F$ is $RS\Lambda$ if $\Lambda(F) \subset A$. We say that $D_X$-complex $\dot{F}$ is $RS\Lambda$ if for any curve $C \subset X$ all cohomology sheaves of $i_C^!(\dot{F})$ are $RS\Lambda$.

It is not difficult to prove that all functors $D, \pi_*, \pi^!, \pi_!, \pi^*$ preserve $D_{RS\Lambda(D_X)}$ – one should repeat proofs in 1-5 with minor modifications. Apparently criterion 6 is also true for $RS\Lambda$ (for $\Lambda = \mathbb{Q}$ it is proved by Kashiwara). I would like to have an algebraic proof of it.

**5. Riemann-Hilbert correspondence.**

In this lecture I will work over the field $k = \mathbb{C}$ of complex numbers.

**1. Constructible sheaves and complexes.**

Let $X$ be a complex algebraic variety. We denote by $X^{an}$ the correspondent analytic variety, considered in classical topology.

Let $C_X$ be the constant sheaf of complex numbers on $X^{an}$. We denote by $Sh(X^{an})$ the category of sheaves of $C_X$-modules, i.e., the category of sheaves of $\mathbb{C}$-vector spaces. Derived category of bounded complexes of sheaves we denote by $D(X^{an})$. I will call sheaves $F \in Sh(X^{an})$ $C_X$*-modules* and complexes $\dot{F} \in D(X^{an})$ $C_X$*-complexes.*

I call $C_X$-module $F$ *constructible* if there exists a stratification $X = UX_i$ of $X$ by locally closed *algebraic* subvarieties $X_i$, such that $F|X_I^{an}$ is finite dimensional and locally constant (in classical topology). I call $C_X$-complex $\dot{F}$ constructible if all its cohomology sheaves are constructible $C_X$-modules. The full subcategory of $D(X^{an})$ consisting of constructible complexes I denote by $D_{con}(X^{an})$.

Any morphism $\pi : Y \to X$ of algebraic varieties induces the continuous map $\pi^{an} : Y^{an} \to X^{an}$ and we can consider functors

$$\pi_!, \pi_* : D(Y^{an}) \longrightarrow D(X^{an})$$

$$\pi^*, \pi^! : D(X^{an}) \longrightarrow D(Y^{an})$$

also we will consider the Verdier duality functor

$$D; D(X^{an}) \longrightarrow D(X^{an}).$$

**Theorem.** *Functors $\pi_*, \pi_!, \pi^*, \pi^!$ and $Dk$ preserve subcategories $D_{con}(\quad)$. On this categories $DD = Id$ and*

$$D\pi^* D = \pi^!, \qquad D\pi_* D = \pi_!.$$

**2. De Rham functor.**

Denote by $O_X^{an}$ the structure sheaf of the analytic variety $X^{an}$. We will assign to each $O_X$-module $F$ corresponding "analytic" sheaf of $O_X^{an}$-modules $F^{an}$, which locally is given by

$$F^{an} = O_X^{an} \bigoplus_{O_X} F.$$

This defines an exact functor

$$an : M(O_X) \longrightarrow M(O_X^{an}).$$

In particular, sheaf $D_X^{an}$ is the sheaf of analytic differential operators on $X^{an}$, and we have an exact functor

$$an : M(D_X) \longrightarrow M(D_X^{ab}).$$

Since this functor is exact it induces a functor

$$an : D(D_X) \longrightarrow D(D_X^{an}).$$

**Definition.** *I define the De Rham functor*

$$DR : D(D_X) \longrightarrow D(X^{an}) = D(Sh(X^{an})) \qquad by$$

$$DR(\dot{F}) = \Omega_X^{an} \bigoplus_{D_X^{an}} \dot{F}^{an}.$$

*Remarks.* 1. We know that the complex $DR(D_X)$ is a locally projective resolution of the right $D_X$-module $\Omega_X$. Hence

$$DR(\dot{F}) = DR_X(D_X^{an}) \bigoplus_{D_X^{an}} \dot{F}^{an}|n| = DR_X(\dot{F}^{an})|n|,$$

where $n = \dim\ X$.

In particular, if $F$ is an $O$-coherent $D_X$-module, corresponding to bundle with a flat connection and $L = F^{\text{flat}}$ the local system of flat sections of $F$ (in classical topology), then by Poincaré lemma

$$DR(F) = L|n|.$$

2. Kashiwara usually uses slightly different functor $Sol : D_{coh}(D_X)^o \to D(X^{an})$,

$$Sol(\dot{F}) = R\ \mathrm{Hom}_{D_X^{an}}(F^{an}, O_X^{an}).$$

I claim that $Sol(\dot{F}) = DR(D\dot{F})| - \dim\ X|$. This follows ¿from the formula

$$\mathrm{Hom}_{D_X}(P, O_X) = \Omega_X \bigoplus_{D_X} (*P),$$

which is true for any locally projective coherent $D_X$-module $P$, where $*P = \mathrm{Hom}_{D_X}(P, D_X^{\Omega})$.

**3. Main Theorem C.**

*a)* $DR(D_{hol}(D_X)) \subset D_{con}(X^{an})$ *and on the subcategory*

$D_{hol}$ $D \circ DR = DR \circ D$ .

*If* $\dot{F} \in D_{hol}(D_X)$, $\dot{H} \in D(D_Y)$, *then*

$Dr(\dot{F} \boxtimes \dot{H}) \approx DR(\dot{F}) \boxtimes DR(\dot{H})$.

*b)* *On the subcategory* $D_{RS}$ *functor* $DR$ *commutes with* $D, \pi_*, \pi^!, \pi_!, \pi^*$ *and* $\boxtimes$

*c)* $DR : D_{RS}(D_X) \to D_{con}(X^{an})$ *is an equivalence of categories.*

4. First let us consider some basic properties of the functor $DR$.

(i) $DR$ commutes with restriction to an open subset. For an étale covering $\pi : Y \to X$ $DR$ commutes with $\pi_*$ and $\pi^!$.

(ii) There exists a natural morphism of functors $\alpha : DR\pi_* \to \pi_* \circ DR$ which is an isomorphism for proper $\pi$.

In order to prove this let us consider the functor

$$\pi_*^{an} : D(D_Y^{an}) \to D(D_X^{an}) \quad \text{on the categories of } D^{an}\text{-complexes,}$$

which is given by

$$\pi_*^{an}(\dot{F}) = R\pi_{\cdot}^{an}(D_{X\leftarrow Y}^{an} \bigotimes_{D_Y^{an}} \dot{F}).$$

I claim that $DR\pi_*^{an} = \pi_* \circ DR$. Indeed,

$$DR(\pi_*^{an}(\dot{F}) = \Omega_X^{an} \overset{L}{\bigoplus_{D_X^{an}}} R\pi_{\cdot}^{an}(D_{X\leftarrow Y}^{an} \overset{L}{\bigoplus_{D_Y^{an}}} \dot{F}) =$$

$$R\pi_{\cdot}^{an}(\pi^{\cdot}(\Omega_X^{an}) \overset{L}{\bigotimes_{\pi^{\cdot} D_X^{an}}} D_{X\leftarrow Y}^{an} \overset{L}{\bigoplus_{D_Y^{an}}} \dot{F}) = R\pi_{\cdot}^{an}(\Omega_Y^{an} \overset{L}{\bigotimes_{D_Y^{an}}} \dot{F}),$$

since $\pi^{\cdot}\Omega_X \bigotimes_{\pi^{\cdot} D_X} D_{X\leftarrow Y} \approx \Omega_Y$ as $D_Y$-module.

Now there exists in general the natural isomorphism of functors

$$an \circ R\pi_{\cdot}(\dot{F}) \longrightarrow R\pi_{\cdot}^{an}(an\dot{F}).$$

This functor is not an isomorphism in general, since direct image on the left and on the right are taken in different topologies. But according to Serre's "GAGA" theorem it is an isomorphism for proper $\pi$. Combining these 2 observations we obtain (ii).

(iii) On the category of coherent $D_X$-complexes there exists a natural morphism of functors

$$\beta : DR \circ D(\dot{F}) \longrightarrow D \circ DR(\dot{F})$$

which is an isomorphism for $O$-coherent $\dot{F}$ and which is compatible with the isomorphism $\pi_* DR = DR\pi_*$ for proper $\pi$, described in (ii).

By definition of the duality functor $D$ in the category $D(X^{an})$

$$D(\dot{S}) = RHom_{C_X}(\dot{S}, C_X|ddimX|).$$

(Here $C_X|2dimX|$ is the dualizing sheaf of $X^{an}$). Hence in order to construct $\beta$ it is sufficient to construct a morphism

$$\beta' : DR \circ D(\dot{F}) \otimes_{\mathbb{C}_X} DR(\dot{F}) \longrightarrow cln$$

where $cln$ is an injective resolution of $C_X|2 \dim X|$.

As we saw, $DR \circ D(\dot{F})$ is naturally isomorphic to $Sol(\dot{F})|\dim X| = RHom_{D_X^{an}}(\dot{F}^{an}, O_X^{an})|\dim X|$.

Let us realize $DR(\dot{F})$ as $DR_X(\dot{F}^{an})$ and $DR \circ D(\dot{F})$ as $Hom_{D_X^{an}}(\dot{F}^{an}, cl^{an})$ where $cl$ is an injective resolution of k$O_X|\dim X|$. Then we have the natural morphism

$$\beta'' : DR \circ D(\dot{F}) \otimes_{C_X} DR(\dot{F}) \longrightarrow DR_X(cl^{an}).$$

Since $DR_X(cl^{an}) \approx DR_X(O_X^{an})|\dim X| = C_X|2\dim X|$, we have a morphism $D_X(cl^{an}) \longrightarrow cln$, which composition with $\beta''$ gives us $\beta'$. It is easy to check that $\beta$ is an isomorphism for $O$-coherent $\dot{F}$. Compatibility condition with $\pi_*$ it is sufficient to check for imbeddings and projections $\mathbb{P} \times X \to X$, where it is straightforward.

(iv) There is a natural morphism of functors

$$\gamma : DR(F \boxtimes H) \longrightarrow DR(F) \boxtimes DR(H)$$

which is an isomorphism for $O$-coherent $F$.

Morphism $\gamma$ is defined by the natural imbedding $\Omega_X^{an} \boxtimes_C \Omega_Y^{an} \longrightarrow \Omega_{X\times Y}^{an}$. If $F$ is $O$-coherent and $H$ is locally projectively is an isomorphism by partial Poincaré lemma. This implies the general statement.

(v) There is a natural morphism of functors $\delta : DR \circ \pi^!(\dot{F}) \to \pi^! DR(\dot{F})$ which is an isomorphism for smooth $\pi$.

Indeed, for smooth $\pi$ the isomorphism of these functors can be constructed on generators – locally projective modules (for instance if $\pi : Y = T \times X \to X$ is the projection, then $\pi^!(\dot{F}) = O_T \boxtimes \dot{F}|\dim T|$, $\pi^! DR(\dot{F}) = C_T \boxtimes DR(\dot{F})|2\dim T| = Dr(O_T) \boxtimes DR(\dot{F})|\dim T|)$. Consider the case of a closed imbedding $i : Y \to X$. Using $i_*$, which commutes with $DR$, we will identify sheaves on $Y$ with sheaves on $X$, supported on $Y$. Then $i_* i^! \dot{F} = R\Gamma_{|Y|}\dot{F}$ in both categories, which gives the natural morphism

$$\delta : DR \circ i_* i^!(\dot{F}) = DR(R\Gamma_{|Y|}\dot{F}) \longrightarrow R\Gamma_{|Y|} DR(\dot{F}) = i_* i^! DR(\dot{F}).$$

**5. Proof of Theorem C a) (case of holonomic complexes).**

Let $\dot{F}$ be a holonomic $D_X$-complex. Consider the maximal Zariski open subset $U \subset X$ such that $DR(\dot{F})|_U$ is constructible. Since $F$ is $O$-coherent almost everywhere $U$ is dense in $X$.

Let $W$ be an irreducible component of $X \setminus U$. I want to show that $DR(\dot{F})$ is locally constant on some dense Zariski open subset $W_0 \subset W$.

*Claim.* I can assume that

$X = \mathbb{P} \times W$, $\quad W = p \times W$, $\quad$ where $p \in \mathbb{P}$,

$U$ and $V = U \cup W$ $\quad$ are open in $X$.

Indeed, consider an étale morphism of some open subset of $W$ onto an open subset of an affine space $\mathbb{A}^k$ and extend it to an étale morphism of a neighbourhood of $W$ onto an open subset of $\mathbb{A}^n \supset \mathbb{A}^k$. By changing base from $\mathbb{A}^k$ to $W$, I can assume that $V = U \cup W$ is an open subset of $X' = \mathbb{P}^{n-k} \times W$. Then I can extend $F$ to some sheaf of $X'$.

Now consider the projection $pr : X = \mathbb{P} \times W \to W$. Since it is a proper morphism $DR(pr_*(\dot{F})) = pr_* DR(\dot{F})$. Since $pr_*(\dot{F})$ is a holonomic $D_W$-complex, it is 0-coherent almost everywhere, i.e., $DR(pr_*(\dot{F}))$ is locally constant almost everywhere.

Put $\dot{S} = DR(\dot{F}) \subset D(X^{an})$. Replacing $W$ on an open subset, we can assume that $pr_*(\dot{S}) = DR(pr_*(\dot{F}))$ is locally constant. We have an exact triangle.

$$\dot{S}_V \to \dot{S} \to \dot{S}_{X\setminus V}, \quad \text{where} \quad \dot{S}_V = (i_V): (\dot{S}/V) \quad \text{is extension by zero.}$$

By the choice of $U$, $\dot{S}/_V$ is constructible, i.e., $\dot{S}_V$ is constructible. Hence $pr_*(\dot{S}_{X\setminus V})$ is constructible and going to an open subset we can assume it is locally constant.

Now $\dot{S}_{X\setminus V}$ is a direct sum of 2 sheaves $(i_W)_!\dot{S}/W$ and something concentrated on $X \setminus V \setminus W$. This implies that $\dot{S}/W$ is a direct summand of the locally constant sheaf $pr_*(\dot{S}_{X\setminus V})$ and hence itself is locally constant. QED

Now let $\dot{F}$ be a holonomic complex. Put

$$Err(\dot{F}) = \mathrm{Cone}(DR \circ D(\dot{F}) \to D \circ DR(\dot{F})).$$

This sheaf vanishes on a dense open subset, where $\dot{F}$ is 0-coherent. Also function $Err$ commutes with direct image for proper morphisms. Repeating the arguments above we see that $Err = 0$, i.e., $DR$ commutes with $D$ on $D_{\mathrm{hol}}(D_X)$.

The same arguments show that $DR(\dot{F} \boxtimes \dot{H}) = DR(\dot{F}) \boxtimes DR(\dot{H})$ for holonomic $\dot{F}$.

*Remark.* Of course this proof is simply a variation of Deligne's proof of "Théorèmes de finitude" in SGA 4 1/2.

**6. Proof of theorem C b) for direct image..**

Let us prove that the morphism

$$DR \circ \pi_*(\dot{H}) \to \pi_* \circ DR(\dot{H})$$

is an isomorphism for $H \in D_{RS}(D_Y)$.

*Case 1.* $\pi = i : Y \to X$ is a regular extension and $H$ is an $RS$ 0-coherent $D_Y$-module.

In this case the proof is straightforward, using the definition of $RS$ (it was done by P. Deligne). Namely, locally in the neighbourhood of a point $x \in X \setminus Y$ we can choose coordinates $x_1, \dots, x_n$ such that $X \setminus Y$ is given by $x_1, \dots, x_k$. Now we place $x$ by an *analytic* neighbourhood of $x$. Then $H$ and $H^+ = i_+(H)$ are determined by monodromy representation of the fundamental group $\pi, (X \setminus Y)$. Since this group is commutative, we can decompose $H$ into 1-dimensional subquotients. Using commutativity with $\boxtimes$ we can reduce to the case dim $Y = 1$. Hence as $O_Y$-module $H^+$ is generated by one element $e$, which satisfies the equation $x\partial(e) = \lambda e$. Now direct calculations show that

$$DR(H^+) = (i)_* DR(H).$$

*Case 2.* $H$ is an $RS$ 0-coherent $D_Y$-module.

In this case we decompose $\pi = \pi^+ \circ i$, where $i : Y \to Y^+$ is a regular extension and $\pi^+ : Y^+ \to X$ is a proper morphism. $DR$ commutes with $i$ by case 1 and with $\pi^+$ by 4 (ii).

*General Case.* It is sufficient to check the statement on generators. Hence we can assume that $\dot{H} = i_*(\xi)$, where $i : Z \to Y$ is a locally closed imbedding and $\xi$ an $RS$ 0-coherent $D_Z$-module. Then

$$DR\pi_*(\dot{H}) = DR(\pi i)_*(\xi) \overset{\text{case 2}}{=\!=\!=} (\pi i)_* \ DR(\xi) =$$
$$\pi_*(i_* DR(\xi)) \overset{\text{case 2}}{=\!=\!=} \pi_* DR(i_*(\xi)) = \pi_* DR(\dot{H}).$$

**7. Proof of theorem C b).**

Functors $D, \pi_*$ and $\boxtimes$ were considered in 5 and 6.

Functor $\pi^!$. In 4(v) I have constructed the morphism $\delta : DR\pi^! \to \pi^! DR$ which is an isomorphism for smooth $\pi$. Hence it is sufficient to check that $RS$ $D_Y$-complexes $\delta$ is an isomorphism for the case of a closed imbedding $\pi = i : Y \to X$. Denote by $j : V = X \setminus Y \to X$ the imbedding of the complementary open set. Then we have the morphism of exact triangles

$$\begin{array}{ccccc} DR(i_* i^! \dot{F}) & \longrightarrow & DR(\dot{F}) & \longrightarrow & DR(j_*(\dot{F}|_V)) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \alpha} \\ i_* i^! DR(\dot{F}) & \longrightarrow & DR(\dot{F}) & \longrightarrow & j_*(DR(\dot{F})|_V). \end{array}$$

Since $\alpha$ is an isomorphism by 6, $\delta$ is an isomorphism.

Functors $\pi_!$ and $\pi^*$. They commute with $DR$ since $\pi_! = D\pi_* D$ and $\pi^* = D\pi^! D$.

**8. Proof of theorem C c).**

First of all, let us prove that $DR$ gives an equivalence of $D_{RS}(D_X)$ with a full subcategory of $D_{\text{coh}}(X^{an})$. We should prove that for $\dot{F}, \dot{R} \in D_{RS}(D_X)$

$$DR : \mathrm{Hom}_{D_{RS}}(\dot{F}, \dot{H}) \longrightarrow \mathrm{Hom}_{D_{coh}}(DR(\dot{F}), DR(\dot{H}))$$

is an isomorphism.

It turns out that it is simpler to prove the isomorphism of $R\mathrm{Hom}$ ( ). We have shown in lecture 3 that

$$R\mathrm{Hom}(\dot{F}, \dot{H}) = \int_X \mathrm{Hom}(\dot{F}, \dot{H}) = \int_X \mathrm{Hom}(\dot{F}, \dot{H}) = \int_X D\dot{F} \triangle \dot{H}.$$

Let us prove that in the category $D_{\text{coh}}(X^{an})$ $R\mathrm{Hom}$ is given by the same formula we have

$$R\ \mathrm{Hom}(\dot{R}, DS^{\cdot}) = R\mathrm{Hom}(\dot{R}, R\ \mathrm{Hom}(S^{\cdot}, \mathrm{Dual})) =$$

$$R\ \mathrm{Hom}(\dot{R} \otimes S^{\cdot}, \mathrm{Dual}) = D(R^{\cdot} \otimes S^{\cdot}) = DR^{\cdot} \triangle DS^{\cdot}.$$

Hence

$$R\ \mathrm{Hom}(\dot{R}, S^{\cdot}) = \int R\ \mathrm{Hom}(\dot{P}, S^{\cdot}) = \int D(\dot{R}) \triangle S^{\cdot}.$$

This proves that $DR$ gives an equivalence of the category $D_{RS}(D_X)$ with a full subcategory of $D_{\mathrm{coh}}(X^{an})$.

Now let us prove that this subcategory contains all isomorphism classes of $D_{\mathrm{coh}}(X^{ab})$. Since it is a triangulated full subcategory, it is sufficient to check that it contains generators. As generators we can choose $C_X$-complexes of the form $i_*(L)$ where $i : Y \to X$ is an imbedding and $L$ is a local system on $Y$. Since $DR$ commutes with direct images it is sufficient to check that there exists an $RS$ 0-coherent $D_Y$-module $\xi$ such that $DR(\xi) \approx L|\mathrm{dim}\ Y|$, i.e., such that the sheaf of flat sections kof $\xi^{an}$ is isomorphic to $L$. This is a result by P. Deligne.

**9. Perverse sheaves, intersection cohomology and such.**

Main theorem C gives us a dictionary which allows to translate problems, statements and notions from $D$-modules to constructible sheaves and back.

Consider one particular example. The category $D_{RS}(D_X)$ of $RS$-complexes contains the natural full abelian subcategory $RS$-category of $RS$-modules.

How to translate it in the language of constructible sheaves.

¿From the description of the functor $i^!$ for locally closed imbedding one can immediately get the following

**Criterion.** Let $\dot{F}$ be a holonomic $D_X$-complex. Then $\dot{F}$ is concentrated in nonnegative degrees (i.e., $H^i(\dot{F}) = 0$ for $i < 0$) if and only if it satisfies the following condition.

$(*)_{RS}$ For any locally closed imbedding $i : Y \to X$ there exists an open dense subset $Y_0 \subset Y$ such that $i^!(\dot{F})\big|_{Y_0}$ is an 0-coherent $D_{Y_0}$-complex, concentrated in degrees $\geq 0$.

In terms of constructible complexes this condition can be written as

$(*)_{con}$ For any locally closed imbedding $i : Y \to X$ there exists an open dense subset $Y_0 \subset Y$ such that $i^!(\dot{S})\big|_{Y}$ is locally constant and concentrated in degrees $\geq$ - dim $Y$.

Thus we have proved the following.

**Criterion.** A constructible complex $S^{\cdot}$ lies in the abelian subcategory

$$DR(RS(D_X)) \quad \text{iff} \quad \dot{S} \quad \text{and} \quad DS^{\cdot} \quad \text{satisfy} \quad (*)_{con}.$$

Now it is easy to recognize this as a definition of a perverse sheaf on $X^{an}$.

**Exercise.** Let $L(Y, \xi)$ be an irreducible $RS$ $D_X$-module. Then $DR(L(Y, \xi))| - \mathrm{dim}\ Y|$ is the intersection cohomology sheaf, associated to $(Y, \mathrm{Loc.\ syst.}\ \xi)$.

Thus intersection cohomology sheaves just correspond to irreducible $RS$ $d$-modules.

**10. Analytic criterion of regularity.**

For any point $x \in X$ I denote by $O_x^{an}$ and $O_x^{\text{form}}$ algebras of convergent and formal power series on $X$ at the point $x$. For any $D_X$-complex $\dot{F}$ the natural inclusion $O_x^{an} \to O_x^{\text{form}}$ induces a morphism

$$\nu_x : R\ \text{Hom}_{D_X}(\dot{F}, O_x^{an}) \longrightarrow R\ \text{Hom}_{D_X}(\dot{F}, O_X^{\text{form}}).$$

We say that $\dot{F}$ is good at $x$ if $\nu_x$ is an isomorphism.

**Proposition.** *Let $\dot{F}$ be an RS $O_X$-complex. Then $\dot{F}$ is good at all points.*

*Remark.* One can show that conversely, if $X$ is a complete variety and $\dot{F}$ a holonomic $D_X$-complex good at all points $x \in X$, then $\dot{F}$ is $RS$.

*proof.* For locally projective $D_X$-module $P$ we have

$$\text{Hom}_{D_X}(P, O_x^{\text{form}}) = \text{Hom}_k(P/M_xP, k) = i_x^{\Delta}(P)^*.$$

Hence $R\ \text{Hom}_{D_X}(\dot{F}, O_x^{\text{form}}) = i_x'^{\cdot}(\dot{F})^*|\dim X|$. If we put $\dot{G} = D\dot{F}$ and remember that $i_x^* = Di_x^!D$ we see that

$$R\ \text{Hom}_{D_X}(\dot{F}, O_x^{\text{form}}) = i_x^*(\dot{G})|\dim X|.$$

¿From the other side

$$R\ \text{Hom}_{D_X}(\dot{F}, O_x^{an}) = \text{fiber at } x \text{ of } Sol(\dot{F}) = i_x^* DR(\dot{G})|\dim X|.$$

Thus we can reformulate our problem, using the $DR$ functor.

(*) Holonomic $D_X$-complex $\dot{F}$ is good at $x$ iff for $\dot{G} = D\dot{F}$ the canonical morphism

$$\nu_x : i_x^* DR(\dot{G}) \longrightarrow DR\ i_x^*(\dot{G})$$

is an isomorphism.

Hence the proposition is simply a particular case of theorem C.

The proof of the converse statement is based on the criterion of $RS$ which is discussed in 4.

**Lecture 6. $D$-modules and the proof of the Kazhdan-Lusztig conjecture.**

I would like to outline main steps of the proof of the Kazhdan-Lusztig conjecture. Only part of it is connected with $D$-modules, but somehow it has the same spirit as the theory of $D$-modules, as I presented it.

The amazing feature of the proof is that it does not try to solve the problem but just keeps translating it in languages of different areas of mathematics (further and further away from the original problem) until it runs into Deligne's method of weight filtrations which is capable to solve it.

So, have a seat; it is going to be a long journey.

**Stop 1.** $g$-modules, Verma modules and such.

Let $g$ be a semisimple Lie algebra over $\mathbb{C}$, $f \subset g$ a Carton subalgebra, $\Sigma$ $f^*$ root system, $\Sigma^+$ the system of positive roots and $n \subset g$ corresponding nilpotent subalgebra. To each weight $\chi \in f^*$ we assign $g$-module $M_\chi$ (it is called Verma module) which is a universal $g$-module, generated by 1 element $f_\chi$ such that $nf_X = 0$ and $f_\chi$ is an eigenvector of $f$ with the eigencharacter $\chi - \rho$ (here $\rho$ is the halfsum of positive roots). Each Verma module $M_X$ has unique irreducible quotient $L_X$, has finite length and all its irreducible subquotients are of the form $L_\psi$ for $\psi \in f^*$. Hence we can write in the Grothendieck group

$$M_\chi = b_{\chi\psi} L_\psi.$$

**Problem.** Calculate multiplicities $b_{\chi\psi}$.

It is usually more convenient to work with the inverse matrix $a_{\chi\psi}$, such that $L_\chi = \Sigma a_{\chi\psi} M_\psi$.

Also, using elements of the center $z(g) \subset U(g)$ it is easy to show that $a_{\chi\psi} \neq 0$ only if $\chi$ and $\psi$ lie on one orbit of the Weyl group. The most interesting case is the $W$-orbit of $(-\rho)$. So let us put for $w \in W$, $M_w = M_{w(-\rho)}$, $L_{w(-\rho)}$ and formulate the

**Problem A.** Calculate matrix $a_{ww'}$, given by

$$L_w = \Sigma\, a_{ww'} M_{w'}.$$

**Stop 2.** $D$-modules, Schubert cells ... .

Now we are going to translate Problem A into the language of $D$-modules.

Let $G$ be an algebraic group corresponding to $g$, $X$ the flag variety of $G$, i.e., $X = G/B$ where $B$ is a Borel subgroup of $G$. The natural action of $G$, i.e., $X = G/B$ where $B$ is a Borel subgroup of $G$. The natural action of $G$ on $X$ defines the morphism $U(g) \to D_X$. Hence for each $D_X$-module $F$ the space $\Gamma(F) = \Gamma(X, F)$ of global sections of $F$ has the natural structure of $g$-module. Our translation is based on the following

**Theorem (Beilinson, Bernstein).**

*The functor $\Gamma : \mu(D_X) \to \mu(g)$, $\quad F \to \Gamma(F)$ gives an equivalence of the category $\mu(D_X)$ with the category $\mu_\theta(g)$ of $g$-modules with trivial infinitesimal character $\theta$. Here $\theta$ is the character of the center $Z(g) \subset U(g)$, i.e., the homomorphism $\theta : Z(g) \longrightarrow \mathbb{C}$, corresponding to the trivial representation of $g$. We say that $g$-module $M$ has infinitesimal character $\theta$ if $Ker\, \theta \cdot M = 0$.*

The proof of the theorem consists of two parts:

1. We show that the functor $\Gamma$ is exact and each $D_X$-module $F$ is generated by its global sections. This implies that $\mu(D_X)$ is equivalent to the category of $D(X)$-modules, where $D(X) = \Gamma(X, D_X)$ is the algebra of global differential operators. We already saw that this fact is true for projective spaces (see lecture 2); though the proof is different, the effect has the same nature.

2. We show that $D(X) = U(g)/\text{Ker}\ \theta \cdot U(g)$.

This is pure luck. The proof is just a direct calculation, which uses Kostant's theorem on functions on nilpotent cone.

This theorem allows us to translate all the problems of the representation theory, involving modules in $\mu_\theta(g)$ into the language of $D$-modules. Since $M_w, L_w \in M(g)$ we can translate our problem. Let us indicate how to do it.

It is easy to prove that on any module $M = M_w$ or $L_w$ the nilpotent algebra $n$ acts locally nilpotent. It means that we can exponentiate this action and define some algebraic action of the corresponding nilpotent subgroup $N \subset G$. Hence on $M$ we have two actions: action $\kappa$ of the Lie algebra $g$, and the representation $\rho$ of the Lie group $N$. It is clear that $M$ is a $(g, N)$-module, i.e., it satisfies the following conditions:

(i) Representation $\rho$ is algebraic, i.e., $M$ is a union of finite dimensional algebraic representations of the algebraic group $N$.

(ii) Morphism $\kappa : g \otimes M \to M$ is $N$-invariant with respect to the adjoint action of $N$ on $g$ and action $\rho$ of $N$ on $M$.

(iii) On Lie algebra $n$, $g$ actions $\kappa$ and $d\rho$ coincide.

Translating in $D_X$-modules we see that the $D_X$-module $F$, corresponding to $M$ is really a $(D_X, N)$-module, i.e., it is endowed with an action $\rho$ of the group $N$ such that

(i) $\rho$ is algebraic, i.e., $F$ is a union of coherent $O$-modules with algebraic action of $N$ (compatible with the natural action of $N$ on $X$).

(ii) Action $\kappa : D_X \otimes F \to F$ is $N$-invariant.

(iii) On Lie algebra $n$ of the group $N$ action $\kappa$, given by the natural morphism $n \to$ Vect. fields on $X$, $D_X$ coincides with $d\rho$.

In particular, it means that Supp $F$ is $N$-invariant. Using Bruhat decomposition we see that $N$ haws a finite number of orbits on $X$. Namely,

$$X = \bigcup_{w \in W} Y_w, \quad \text{where } Y_w = N(wx_N),$$

and $x_N \in X$ is the point, corresponding to $N$. If $Y$ is an open orbit of $N$ in the Supp $F$, then $i_Y^!(F)$ is an $(D_Y, N)$-module. Now, since $N$ acts transitively on $Y$ it is not difficult to describe all $(D_Y, N)$-modules. They all are direct sums of many copies of the standard $(D_X, N)$-module $O_Y$.

Let us put $\mu_Y = (i_Y)_!(O_Y)$, $I_Y = (i_Y)_*(O_Y)$, $L_Y = \text{Im}(\mu_Y \to I_Y)$. Fortunately in this case $Y$ is affine (it is isomorphic to an affine space), so $\mu_Y, I_Y$ are $(D_X, N)$-modules, not complexes.

**Lemma.** *$u_w = u_{Y_w}$ corresponds to $M_w$*

*$L_w = L_{Y_w}$ corresponds to $L_w$*

It is not quite trivial to establish. But if we are interested only in the images of $M, L$ in the Grothendieck group, then it is easy to prove. Indeed, since each $L_w$ is selfdual (since $DO_Y = O_Y$), in Grothendieck group $\mu_w \simeq I_w$. Now it is very easy to directly compute $\Gamma(X, I_w)$ as $h$-module and to show that it coincides with $M_w/h$. Since an element in the Grothendieck group is determined by its restriction to $h$, this proves that $\mu_w \sim M_w$ (in Grothendieck group).

Now we an reformulate the problem.

**Problem B.** Calculate $a_{ww'}$ given by

$$L_w = \Sigma a_{ww'} \mu_{w'}.$$

**Stop 3.** Constructible sheaves.

Now we can use Hilbert-Riemann correspondence, I have described in lecture 5, and translate the whole problem into the language of constructible sheaves.

First of all, let us define the Grothendieck group $K(D_{RS})$ of the category $D_{RS}(D_X)$ as a group, generated by $RS$-complexes and relations $[\dot{F}] + [\dot{H}] = \dot{G}$ for any exact triangle $\dot{F} \to \dot{G} \to \dot{H}$. It is easy to prove that $K_{RS}$ coincides with the Grothendieck group $K(RS)$ of the category $RS(D_X)$; isomorphism $x : K(D_{RS}) \to K(RS)$ is given by Euler characteristic $x([\dot{F}]) = \Sigma(-1)^i[H^i(\dot{F})]$. In the same way $K(D_{con}) = K(con)$. For simplicity we restrict ourselves to the subcategories in $K(RS)$ and $K(con)$ generated by sheaves, which are $N$-invariant. Functor $DR$ gives us an isomorphism $DR : D(D_{RS}) = K(RS) \to K(D_{con}) = K(con)$. Let us look how to translate $\mu_w$ and $L_w$.

By definition $\mu_w = (i_{Y_w})_!(O_y)$. Hence $DR(\mu_w) = i_{Y_w})_!(1_{Y_w})[\mathrm{dim} Y_w]$, where $1_Y$ is the trivial sheaf on $Y$. If we denote by $T_w$ the element $(i_{Y_w})_!(1_{Y_w}) \in K(con)$, (extension by zero), we see that $DR(\mu_w) = (-1)^{\ell(w)} T_w$, where by definition $\ell(w) = dim Y_w$ (it is the usual length function on the Weyl group). As we discussed in lecture 5, $DR(L_w) = IC(Y_w)[dim Y_w]$, where $IC(Y)$ is the intersection cohomology sheaf of $Y$. Let us denote by $IC_w$ the element of $K(con)$, corresponding to $IC(Y_w)$. Then we can reformulate our problem.

**Problem C.** Find $a_{ww'}$ given by

$$IC_w = \Sigma a_{ww'} (-1)^{\ell(w) - \ell(w')} T_{w'}.$$

**Fast train.** Etale cohomologies, changing of the field, ... .

What we have done so far is the translation of the very difficult problem A to the not less difficult topological problem C. This problem is essentially the problem of calculating intersection cohomologies of the highly singular varieties $\overline{Y}_w$. The only general method of solving such problems known so far is based on algebraic geometry over finite fields. So we should go this way.

Let us fix the stratification $\Phi = (X = UY_w)$ and denote by $D_\Phi(X^{an})$ the subcategory of $D(X^{an})$, consisting of $C_X$-complexes, such that their cohomology sheaves are locally constant along each stratum $Y_w$ (since $Y_w$ is contractible, they in fact are constant along $Y_w$). Corresponding Grothendieck group we denote $K_\Phi$. It is clear that $K_\Phi = \bigoplus_{w\in W} \mathbb{Z}T_w$, and we just want to find the expression of elements of $IC_w \in K$ in this basis.

It turns out that we can replace everywhere classical topology by etale topology and all properties of constructible complexes, constructible sheaves, which can be expressed in terms of functors $D, \pi_*, \pi^!, \pi_!, \pi^*$ will not change.

Since etale topology is defined purely algebraically, we now can translate the whole situation to arbitrary field.

So, we now consider an algebraically closed field $k$ of arbitrary characteristic $p$, a flag variety $X$ of a reductive group $G$ over $k$, and $\Phi = (X = UY_w)$ the Bruhat stratification. We consider derived category $D_\Phi$ of complexes with cohomologies, constant along each stratum $Y_w$. In the Grothendieck group $K_\Phi$ of this category we have a basis $T_w$ and elements $IC_w$, corresponding to $IC$-sheaves, and we want to find an expression of $IC_w$ via $\{T_{w'}\}$.

There are theorems, which claim that the situation in etale topology over any field will be exactly the same as in classical topology over $\mathbb{C}$.

*Remark.* In etale topology we are working with $\ell$-adic sheaves whose stalks are vector spaces over the algebraic closure $\overline{Q}_\ell$ of the field of $\ell$-adic numbers, where $\ell \neq char k$. For simplicity we will identify $\overline{Q}_\ell$ with $\mathbb{C}$.

In fact, $\ell$-adic sheaves are not quite sheaves and elements of $D_\Phi$ are not quite complexes. But it does not matter since we can work with our functors $D, \pi_*, \dots$ in the usual way.

**Stop 4.** Weil sheaves, Tate twist, Lefschetz formula.

Now suppose we are working over the field $k$ which is the algebraic closure of a finite field $\mathbb{F}_q$. Also we assume that our stratification $\Phi$ is defined over $\mathbb{F}_q$, i.e., each stratum $Y_w$ is given by equations and inequalities with coefficients in $\mathbb{F}_q$. Denote by $Fr_q$ the automorphism of the field $k$, given by $c \longmapsto c^q$. For any variety $Y$, defined over $F_q$, $Fr_q$ induces a bijection $Fr_q : Y(k) \to Y(k)$, which turns out to be a homeomorphism in etale topology.

Let us call *Weil sheaf* an $\ell$-adic sheaf $F$ together with the action of $Fr_q$ on $F$. In a similar way we can consider Weil complexes of sheaves. Derived category of Weil complexes, whose cohomologies are constant along strata of stratification $\Phi$ we denote $D_\Phi^W$, and corresponding Grothendieck group $K_\Phi^W$. These definitions make sense since each stratum $Y_w$ is invariant under $Fr_q$.

**Important example.** Let us describe Weil sheaves on the variety *pt*, consisting of one point. Then any sheaf $F$ is given by a vector space $V$. Hence Weil sheaf on *pt* is just a $\overline{\mathbb{Q}}_\ell$-vector space $V$ together with a linear transformation $Fr_q : V \to V$.

**Definition.** *Tate sheaf $L$ over a point $p$ is defined by one-dimensional vector space $\overline{\mathbb{Q}}_\ell$ together with the morphism $Fr_q : \overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell$, which is the multiplication by $q$, i.e., $Fr_q(\alpha) = q\alpha$.*

If $\pi : Y \to X$ is a morphism of algebraic varieties, which is defined over $\mathbb{F}_q$, it induces functors $\pi_*, \pi_! : D^W(Y) \to D^W(X)$, $\pi^*, \pi^! : D^W(X) \to D^W(Y)$. Also there is a functor of Verdier duality $D : D^W(X) \to D^W(X)$. All these functors have the same properties, as we have discussed earlier. But there is one important improvement:

(*) If $X$ is a nonsingular variety, then $D(1_X) = L^{-dimX} \cdot 1_X[2dimX]$.

Here $1_X$ is the trivial sheaf on $X$, $L$ we consider as a sheaf on $X$ – this is the Tate sheaf lifted from the point, and $L^{-k}$ means $(L^{-1})^{\otimes k}$.

If we forget the action of $Fr$ we have an old formula for dualizing sheaf. So (*) simply means that though dualizing sheaf is essentially isomorphic to the constant sheaf, this isomorphism is not canonical; in particular, $Fr_q$ changes it in $q^{dimX}$ times.

**Exercise.** Over a point $D(L^k) = L^{-k}$.

**Digression.** Weil sheaves and functions.

For each variety $X$, defined over $F_q$ denote by $X(q)$ the finite set, consisting of points of $X$, which are defined over $\mathbb{F}_q$ (i.e., which are fixed points of $Fr_q$). To each Weil complex $\dot{F}$ I will assign the function $f_F$ on the finite set $X(q)$ given by

$$f_F(x) = \Sigma(-1)^i trFrq(stalkH^i(\dot{F})_x)$$

(it makes sense since $x$ is $Fr_q$ invariant). It is clear that $f_F$ depends only on the class of $\dot{F}$ in the Grothendieck group $k$.

**Theorem.** *Let $\pi : Y \to X$ be a morphism, defined over $\mathbb{F}_q$, and $\pi : Y(q) \to X(q)$ the corresponding map of finite sets. Then*

$$f_{\pi^*(\dot{F})} = \pi^*(_{\dot{F}}), \quad f_{\pi_!(\dot{H})} = \int_\pi f_{\dot{H}},$$

*where $\dot{F} \in D^W(X)$, $\dot{H} \in D^W(Y)$ and operations $\pi^*$ and $\int_\pi$ on functions are defined by*

$$\pi^*(f)(y) = f(\pi(y)) \ (\int_\pi f)(x) = \sum_{\pi(y)=x} f(y).$$

Here the first statement is triviality and the second is a deep generalization of Lefschetz fixed points theorem.

This theorem claims that all usual operations with functions on finite sets we can rewrite on the level of Weil sheaves (or at least, their Grothendieck group). The importance of this observation can be understood if you realize that starting from one Weil complex $\dot{F}$ we can construct the sequence of functions: to any $q'$ which is a power of $q$ we will assign the function $f_F^{q'}$ on the set $X(q')$; and any natural operation with all these functions can simultaneously be described by one operation with the complex $\dot{F}$. This notion gives the formal definition of the "natural sequence of functions" on sets $X(q')$, $q' = q^i$.

*Example.* Consider the projection $pr : \mathbb{A}^k \to pt$ of the affine space into a point. Then the theorem implies that $pr_!(a_{\mathbb{A}^k}) = L^k[-2k]$, (i.e., in $K^W$ $pr_!(1) = L^k$).

Indeed, comparison with the classical case shows that $dim H^i(pr_!(a_{\mathbb{A}^k})) = \delta_{i,2k'}$, and the theorem describes the action of $Fr_q$ on one-dimensional space $H^{2k}(pr_!(a_{\mathbb{A}^k}))$.

**Stop 5.** Weights and purity.

Let $\dot{F}$ be a Weil complex over a point $p$, which is defined over some field $\mathbb{F}_{q'}$. We say that $w(\dot{F})$ (weight of $\dot{F}$) is less or equal to $\ell$ (notation $w(\dot{F}) \leq \ell$) if for any $i$ all eigenvalues of $Fr_{q'}$ in the space $H^o(\dot{F})$ have absolute value $\leq (q')^{\frac{\ell+i}{2}}$.

(Hey, what do you mean? They are supposed to be $\ell$-adic numbers.)

Well, if you remember, we have identified $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$, so we consider them as complex numbers, and absolute value is the absolute value. Also Deligne proved that in all interesting cases they are algebraic numbers, so it is all not so bad. And in any case, in what we are going to consider they will always be powers of $q$. So do not worry).

It is clear that this notion does not depend on the choice of $q'$, i.e., if we change $q'$ by $q" = (q')^\ell$, it does not affect the condition.

Let now $\dot{F}$ be a Weil complex on $X$. Any point $x \in S$ is defined over $k = \mathbb{F}_{q'}$, i.e., it is defined over some field $\mathbb{F}_{q'}$. We say that $w(\dot{F}) \leq \ell$ if for any point $x \in X$ the stalk $\dot{F}_x = i_x^*(\dot{F})$ has weight $\leq \ell$.

We say that $W(\dot{F}) \geq \ell$ if $W(D\dot{F}) \leq -\ell$. We say that $\dot{F}$ is pure of the weight $\ell$ if

$$W(\dot{F}) \leq \ell \qquad \text{and} \quad W(\dot{F}) \geq \ell.$$

**Deligne's purity theorem.** Let $\pi : Y \to X$ be a morphism, defined over $\mathbb{F}_q$. Then $\pi^*$ and $\pi_!$ decrease weight, $\pi_*$ and $\pi^!$ increase weight, i.e.,

if $W(\dot{F}) \leq \ell$, then $W(\pi^*\dot{F}) \leq \ell$

if $W(\dot{H}) \leq \ell$, then $W(\pi_!\dot{H}) \leq \ell$

if $W(\dot{H}) \leq \ell$, then $W(\pi_*\dot{H}) \geq \ell$

if $W(\dot{F}) \leq \ell$, then $W(\pi^!\dot{H}) \geq \ell$ .

In particular, proper morphism preserves purity.

**Gabber's purity theorem.** Let $Y$ be an irreducible algebraic variety, $IC(Y)$ the intersection cohomology Weil sheaf of $Y$ (which coincides with $1_Y$ on the nonsingular part of $Y$). Then $IC(Y)$ is pure of the weight 0

**Stop 6.** Hecke algebra.

Get back to flag variety $X = UY_w$. Let us consider only complexes, for which all eigenvalues of all morphisms $Fr_q$ are powers of $q$. In general, this category is not invariant with respect to functors, but in our particular case it is.

Let $A = K(D^W(pt))$ be the Grothendieck group of the Weil sheaves over a point. Then $A = \mathbb{Z}[L^{\pm 1}]$, the algebra of Laurent polynomials.

Denote by $H$ the Grothendieck group $K(D^W_\Phi(X))$ of Weil sheaves constant along strata of $\Phi$. Then it is clear that $H$ is a free $A$-module with the basis $\{T_w\}$.

For any $w \in W$ the intersection cohomology sheaf $IC_w \in H$ satisfies the following relations

(i) $D(IC_w) = L^{-dim Y_w} \cdot IC_w$

(ii) $IC_w = T_w + \Sigma P_{w,w'} T_{w'}$,

where $P_{w,w'} \in A$ satisfy the condition

$$(*) \qquad P_{w,w'} = 0 \quad \text{if} Y_{w'} \not\subset \overline{Y}_w \qquad \text{and } \deg P_{w,w'} < 1/2(\ell(w) - \ell(w')).$$

Indeed, as a sheaf $IC_w$ is selfdual, and since in a neighborhood of $Y_w$ it coincides with $T_w$ and in this neighbourhood $DT_w = L^{-dim Y_w} \cdot T_{w'}$ we have (i).

In order to prove (ii) let us fix some point $x \in Y_{w'}$. Then by definition of $IC_w$ stalks of all cohomology sheaves $H^i(IC_w)_x$ equal 0 when $i \geq dim Y_w - dim Y_{w'} = \ell(w) - \ell(w')$. By Gabber's theorem $w(IC_w) \leq 0$, i.e., the action of $Fr_q$ on $H^i(IC_w)_x$ has eigenvalues $\leq q^{i/2}$. But it is clear that $\Sigma(-1)^i Tr Fr_q(H^i(IC_w)_x) = P_{w,w'}(L = q)$. This proves (ii).

Relations (i) and (ii) gives a hope that if we are able to describe the action of the duality operator $D$ on $H$, then we would be able to find Kazhdan-Lusztig polynomials $P_{w,w'}$. After this we can forget about Weil structure (i.e., specialize $L \to 1$) and obtain the formulae for $a_{ww'}$.

In order to describe the action of $D$ I will introduce on $H$ the structure of an algebra.

The motivation for this came from comparison with functions. Informally $H$ is a space of functions on $X(q)$ constant on $N(q)$ orbits. There is the natural identification of $N(q)$ orbits on $X(q)$ with $G(q)$ orbits on $X \times X(q)$, so we can consider elements of $H$ as $Q(q)$-invariant functions in 2 variables $f(x,y)$, $x, y \in X(q)$. But space of functions in 2 variables has the natural operation-convolution, given by

$$f * h(x,y) = \int F(x,z) h(z,y) dz$$

or, with more details

$$(f * h)(x,y) = \int f(x,u) h(v,y)_{\text{substitute } u=v=z} dz.$$

The discussion on the stop 4 allows us immediately to translate this operation in the derived category, or in the Grothendieck group.

First of all, consider the stratification $\Psi$ of $X \times X$ by $G$-orbits and consider category $D_\Psi^W(X \times X)$ and the corresponding group $K_\Psi^W$. This group is naturally isomorphic to $H = K_\Psi^W$; isomorphism is given by restriction of the sheaves $F$ on $X \times X$ to the fiber $x_0 \times X \simeq X$. I will identify $H$ and $K_\Psi^W$ using this isomorphism.

Now, let $\dot{F}, \dot{H} \in D_\Psi^W(X \times X)$. I will define their convolution $*$ by

$\dot{F} * H = pr_! \Delta^*(\dot{F} \boxtimes \dot{H})$, where

$\Delta : X \times X \times X \to X \times X \times X \times X, \quad \Delta(x, z, y) = (x, z, z, y)$

$pr : X \times X \times X \to X \times X, \quad pr(x, z, y) = (x, y)$.

**Proposition.** *$H$ is an associative $A$-algebra with respect to convolution $*$ with identity $1 = T_e$. If $\ell(ww') = \ell(w) + \ell(w')$, then $T_w * T_{w'} = T_{ww'}$.*

The last statement can be checked straightforwardly. Also it follows from the fact that it is true for usual Hecke algebras, which consist of $G(q)$ invariant functions on $X(q) \times X(q)$.

These formulae imply that $H$ as an $A$-algebra is generated by elements $T_{\sigma'}$ for simple reflections $\sigma$.

In order to describe the action of $D$ on $H$ we use the following trick due to Lusztig.

**Proposition.** *Let $\sigma \in W$ be a simple reflection. Then for any $h \in H$ we have*

$$D((T_\sigma + 1) * h) = L^{-1}(T_\sigma + 1) * Dh$$

*also* $(T_\sigma + 1)^2 = (L + 1)(T_\sigma + 1)$.

**Corollary.** *$D$ is the automorphism of the algebra $H$. On generators $T_\sigma$ $D$ is given by $DT_\sigma = L^{-1}T_\sigma + (L^{-1} - 1)$.*

Indeed, the proposition shows that $D((T_\sigma + 1) * h) = D(T_\sigma + 1) * Dh$ for all $h$. Since elements $T_\sigma + 1$ generate $H$, we have $D(f * h) = Df * Dh$. The formula $D(T_\sigma + 1) = L^{-1}(T_\sigma + 1)$ gives the action of $D$ on $T_\sigma$.

The proof of the proposition is based on the following observation. Denote by $p_\sigma$ the parabolic subgroup of $G$, obtained by adding to the Borel subgroup the simple root, corresponding to $\sigma$, and consider the algebraic variety $X_\sigma = G/P_\sigma$. The natural $G$-equivariant projection $p_\sigma : X \to X_\sigma$ has fibers, isomorphic to the projective line $\mathbb{P}'$. For instance, if we put $x_\sigma = p_\sigma(x_0)$, then $p_\sigma^{-1} = Y_e \cup Y_\sigma$ is the projective line with the natural stratification. It means that $T_\sigma + 1$ correspond to the sheaf $R_\sigma$ which is the trivial sheaf on $p_\sigma^{-1}(x_\sigma)$, extended by zero. After this it is not difficult to prove that for any $F \in D^W$ we have

$$(*) \qquad R_\sigma * F = p_\sigma^*(p_\sigma)_! F.$$

Now, since $p_\sigma$ is proper, direct image $(p_\sigma)_! = (p_\sigma)_*$ commutes with $D$. Since $p_\sigma$ is smooth, $p_\sigma^! = L^{-1}p_\sigma^*$, i.e., $Dp_\sigma^* = Lp_\sigma^* D$ (locally $X \simeq X_\sigma \times \mathbb{P}'$, so $p_\sigma^*(F) = F \boxtimes 1_{\mathbb{P}'}$, i.e., $Dp_\sigma^*(F) = DF \boxtimes D(1_{\mathbb{P}}) = LDF \boxtimes 1_{\mathbb{P}} = Lp_\sigma^*(DF)$.

Also, it is clear that $(p_\sigma)_!(R_\sigma) \simeq (L+1)T_{x_\sigma}$ (in Grothendieck group) and $p^*_\sigma(T_{x_\sigma}) = R_\sigma$. This gives the second formula of the proposition.

**Last stop.** Combinatorial problem.

**Proposition.** *(simple combinatorics).*

*(i) There exists an $A$-algebra $H$ which is free with basis $T_w$, such that*
$T_w \cdot T_{w'} = T_{ww'}$ *if* $\ell(w) - \ell(w') = \ell(ww')$.
$(T_\sigma + 1)^2 = (L+1)(T_\sigma + 1)$ *for simple reflections* $\sigma \in W$.

*(ii) There exists a unique automorphism $D$ of the algebra $H$, such that*
$D(L) = L^{-1}$
$D(T_\sigma + 1) = L^{-1}(T_\sigma + 1)$ *for simple reflections* $\sigma \in W$.

*(iii) For each $w \in W$ there exists a unique element $C_w \in H$ such that*
$C_w = T_w + \sum_{w' \lneqq w} p_{w,w'} T_{w'}$, *where $P_{ww'} \in A$ has degree*
$< \frac{1}{2}(\ell(w) - \ell(w'))$ *and* $DC_w = L^{-\ell(w)} C_w$.
*In this case $P_{ww'} \in \mathbb{Z}[L]$.*

**Example.** $C_\sigma = T_\sigma + 1$.

Polynomials $P_{w,w'}$ are called Kazhdan-Lusztig polynomials. Now, if we summarize our discussion, we will obtain the combinatorial formula for multiplicity matrix $a_{w,w'}$.

**Answer.** $a_{ww'} = (-1)^{\ell(w)-\ell(w')} P_{ww'}(1)$.

**Some questions.**

**Question 1.** Where is the solution? How can I find these polynomials?

In a sense there was no solution. We have just translated our original problem, adding a new parameter $L$ for rigidity, to a combinatorial problem and proved that this problem has a unique solution. Of course, now we can obtain some recursive formulae for calculation of Kazhdan-Lusztig polynomials, but they are quite complicated.

Whether there exist explicit formulae for $p_{ww'}$, I think not, i.e., I think that some type of combinatorial complexity is built into the problem.

In some cases one can ge explicit formulae for $P$. For instance, one can calculate intersection cohomology sheaves for Schubert varieties on usual Grassmannians (see Lascoux and Schutzenberger). But Zelevinsky showed that in this case it is possible to construct small resolutions of singularities. I would say that if you can compute a polynomial $P$ for intersection cohomologies in some case without a computer, then probably there is a small resolution, which gives it.

**Question 2.** What is the geometrical meaning of other coefficients of $p_{ww'}$?

Kazhdan and Lusztig showed that all stalks of the sheaves $IC_w$ are pure. Hence, if we choose a point $x \in Y_{w'}$, then

$$\dim H^i(IC_w)_x = 0 \quad \text{for odd } i$$

$$= i/2 \quad \text{coefficient of } p_{ww'} \text{ for even } i.$$

In the proof they used an observation, that transversal section to $Y_{w'}$ of the variety $\overline{Y}_w$ is conical, i.e., it has an action of $k^*$ which contracts everything into a point $x \in Y_{w'}$.

In general, stalks of $IC$ sheaves are not pure. But there is one more case, calculated by Vogan and Lusztig, namely the stratification of the flag variety byorbits of complexified maximal compact subgroup, in which stalks always are pure. I do not know why.

Untwisting the situation back we can connect $H^i(IC_w)$ with

$$Ext^i_{\mu_\phi(g)}(M_{w'}, L_w) \quad \text{or, if you want, with } H^i(n, L_w).$$

**Question 3.** It is all very nice but is it really necessary to go into all this business with varieties over finite fields? How are finite fields connected with $g$-modules?

In fact, it is not necessary. You can obtain the same results using Hodge theory for constructible sheaves or, even better, directly Hodge theory for $D$-modules.

One small detail – these theories do not exist yet (there is a Hodge theory for locally constant sheaves – this is Deligne's theory of variations of Hodge structures – and it is quite powerful, but it is clearly not enough). But at least we know what to think about.