Stable Maps and Gromov-Witten Invariants

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Abstract

The classical complex Grassmann variety of lines in projective space generalizes to moduli of higher degree "stable maps" of rational curves with marked points, but only as a Deligne-Mumford stack, not as a variety (or even as a scheme). Schubert calculus on the Grassmannian generalizes to Gromov-Witten invariants on the stack of stable maps, but unlike Schubert calculus, these invariants only comprise a small portion of the intersection numbers on the moduli stack of stable maps. Equivariant cohomology and reconstruction theorems produce efficient methods for calculating Gromov-Witten invariants without the full knowledge of the intersection ring.

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1 Introduction

I want to begin by reminding you of the Grassmannian. Of course this is a homogeneous space, but as a model for the moduli spaces and intersection theory questions you will see in my talks (and others), it is better to take the "Grothendieck" point of view. Fix positive integers $m < n$.

The Functor of Points: The Grassmann *functor* $\mathcal{G}(m,n)$ is the functor from schemes (of finite type over $\mathbf{C}$) to sets, given by:

$$\mathcal{G}(m,n)(S) = \{\text{vector subbundles } E \subset \mathcal{O}_S^{\oplus n} \text{ of rank } m\}$$

(a vector subbundle (as opposed to subsheaf) has a vector bundle quotient)

$$\mathcal{G}(m,n)(\phi : S \to T) = (\phi^* : \{\text{subbundles of } \mathcal{O}_T^{\oplus n}\} \to \{\text{subbundles of } \mathcal{O}_S^{\oplus n}\})$$

To say that this functor is *represented* by a Grassmann scheme $G(m,n)$ is to say that there is an isomorphism of functors:

$$\mathcal{G}(m,n) \cong h_{G(m,n)}$$

where $h_{G(m,n)}$ is the *functor of points* of $G(m,n)$:

$$h_{G(m,n)}(S) = \mathrm{Hom}(S, G(m,n)) := \{f : S \to G(m,n)\}$$

$$h_{G(m,n)}(\phi : S \to T) \mapsto (\phi^* : \mathrm{Hom}(T, G(m,n)) \to \mathrm{Hom}(S, G(m,n))).$$

This not only says that the (closed) *points* of $G(m,n)$ are in a bijection with the rank m subspaces of $\mathbf{C}^n$ via:

$$\mathcal{G}(m,n)(\mathrm{Spec}(\mathbf{C})) = \{\text{rank } m \text{ subspaces } W \subset \mathbf{C}^n\}$$

$$\leftrightarrow h_{G(m,n)}(\mathrm{Spec}(\mathbf{C})) = \{\text{closed points of } G(m,n)\}$$

but even more that there is a *universal vector subbundle:*

$$(U \subset \mathcal{O}_{G(m,n)}^{\oplus n}) \leftrightarrow (\mathrm{id}_{G(m,n)} : G(m,n) \to G(m,n)) \in h_{G(m,n)}(G(m,n))$$

such that each vector subbundle is pulled back from the universal one:

$$(f^*U \subset \mathcal{O}_S^{\oplus n}) \leftrightarrow (f : S \to G(m,n)) = (f^*\mathrm{id}_{G(m,n)}) \in h_{G(m,n)}(S)$$

The Construction of $G(m,n)$: The open subset:

$$V \subset \mathbf{P}(\mathrm{Hom}(\mathbf{C}^m, \mathbf{C}^n))$$

parametrizing *injective* linear maps $l : \mathbf{C}^m \to \mathbf{C}^n$ is a principal $\mathrm{PGL}(m, \mathbf{C})$ bundle over the subvariety $G(m,n) \subset \mathbf{P}(\wedge^m \mathbf{C}^n)$ via $l \mapsto l(e_1) \wedge ... \wedge l(e_m)$ (the e_i are the basis of $\mathbf{C}^m$) and the "tautological" injective homomorphism:

$$\mathbf{C}^m \otimes \mathcal{O}_V \hookrightarrow \mathbf{C}^n \otimes \mathcal{O}_V$$

descends to the desired universal subbundle $U \subset \mathbf{C}^n \otimes \mathcal{O}_{G(m,n)}$.

Some of the local properties of the Grassmannian can be seen with:

Simple Deformation Theory: The Zariski tangent space at each point of the Grassmannian $(= (f : \mathrm{Spec}(\mathbf{C}) \to G(m,n)) = (W \subset \mathbf{C}^n))$ is:

$$\{\text{morphisms } f_\epsilon : \mathrm{Spec}(\mathbf{C}[\epsilon]) \to G(m,n) \text{ extending } f : \mathrm{Spec}(\mathbf{C}) \to G(m,n)\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{ccc} E_\epsilon \subset \mathcal{O}^{\oplus n}_{\mathrm{Spec}(\mathbf{C}[\epsilon])} & & W \subset \mathbf{C}^n \\ \downarrow & \text{extending} & \downarrow \\ \mathrm{Spec}(\mathbf{C}[\epsilon]) & & \mathrm{Spec}(\mathbf{C}) \end{array} \right\}$$

($\mathbf{C}[\epsilon] \cong \mathbf{C}[x]/x^2$ are the dual numbers) and this is naturally identified with:

$$\mathrm{Hom}(W, \mathbf{C}^n/W)$$

The smoothness of $G(m,n)$ is checked by the infinitessimal lifting property. A point $f : \mathrm{Spec}(\mathbf{C}) \to G(m,n)$ is smooth if and only if, for every surjective map of local Artinian $\mathbf{C}$-algebras $B \to A$, every extension of f to a morphism $f_A : \mathrm{Spec}(A) \to G(m,n)$ extends further to some $f_B : \mathrm{Spec}(B) \to G(m,n)$. This follows from the fact that every extension of

$$\begin{array}{ccccc} W \subset \mathbf{C}^n & & E_A \subset \mathcal{O}^{\oplus n}_{\mathrm{Spec}(A)} & & E_B \subset \mathcal{O}^{\oplus n}_{\mathrm{Spec}(B)} \\ \downarrow & \text{to} & \downarrow & \text{extends further to} & \downarrow \\ \mathrm{Spec}(\mathbf{C}) & & \mathrm{Spec}(A) & & \mathrm{Spec}(B) \end{array}$$

because each E_A is a *free* submodule!

Key Point: The tangent space and smoothness (hence the dimension) were detected using only the functor of points applied to local Artinian $\mathbf{C}$-algebras! Similarly, the valuative criteria for properness and separatedness only require knowledge of the functor of points for $\mathrm{Spec}(D)$ where D is a valuation ring.

Classical Schubert calculus, on the other hand, is a global property of the Grassmannian. In some sense, the complete story is captured with:

The Intersection Ring: Let $1, c_1(U^\vee), c_2(U^\vee), ..., c_m(U^\vee) \in A^*(G(m,n))$ be the Chern classes of the dual of the universal subbundle. Then:

$$\mathbf{Z}[x_1, ..., x_n] \to A^*(G(m,n)); \quad x_i \mapsto c_i(U^\vee)$$

is surjective, and the Segre polynomials $s_{n-m+1}(x_1, ..., x_m), ..., s_n(x_1, ..., x_m)$ generate the kernel ideal. Recall that $s_i(x_1, ..., x_m)$ is the coefficient of t^i in:

$$\frac{1}{1 + x_1 t + x_2 t^2 + ... + x_m t^m}$$

and the ith Segre class of $U^\vee$ is $s_i(U^\vee) = s_i(c_1(U^\vee), ..., c_m(U^\vee))$. Since:

$$\int_{G(m,n)} c_m(U^\vee)^{n-m} = 1$$

any intersection number:

$$\int_{G(m,n)} p(c_1(U^\vee), c_2(U^\vee), ..., c_m(U^\vee)) = N_p$$

can be computed (in principle) by calculating:

$$p(x_1, ..., x_m) \equiv N_p x_m^{n-m} \pmod{\langle s_{n-m+1}, ..., s_n \rangle}$$

This can be done using a classical theorem of Pieri and the Schubert calculus of the Grassmannian, but we will investigate other methods in these notes. For example, here is a direct computation involving "residues":

Vafa-Intriligator Formula: Let $q_1, ..., q_m$ be the "Chern roots" of $U^\vee$, i.e.

$$c_1(U^\vee) = q_1 + ... + q_m, ..., c_i(U^\vee) = \sigma_i(q), ..., c_m(U^\vee) = q_1 \cdots q_m$$

where $\sigma_i(q)$ is the ith elementary symmetric polynomial in $q_1, ..., q_m$. Then:

$$N_p = \frac{(-1)^{\binom{m}{2}}}{m! n^m} \sum_{\{(\zeta_1, ..., \zeta_m) | \zeta_i^n = 1\}} p(\sigma_1(\zeta), ..., \sigma_m(\zeta)) \cdot \sigma_m(\zeta) \cdot v(\zeta)$$

where $v(\zeta)$ is the Vandermonde determinant $v(\zeta) = \prod_{i \neq j} (\zeta_i - \zeta_j)$.

Note: Frequently the polynomial $p(c_1, ..., c_m)$ is given more naturally as a symmetric polynomial in the Chern roots than as a polynomial in $c_i(U^\vee)$, and in those cases formulas like these (with the assistance of a computer) are particularly attractive.

Example: The "universal projective line" in $\mathbf{P}^{n-1}$ is:

$$\begin{array}{ccc} \mathbf{P}(U) & \stackrel{e}{\rightarrow} & \mathbf{P}^{n-1} \\ \pi \downarrow & & \\ G(2,n) & & \end{array}$$

and the basic enumerative classes:

$$\sigma_{H^m} := \pi_* e^* H^m \text{ (lines meeting a subspace of codimension } m\text{)}$$

$$\tau_{\mathcal{O}_{\mathbf{P}^n}(k)} := c_{k+1}(\pi_* \mathcal{O}_{\mathbf{P}^n}(k)) \text{ (lines in a hypersurface of degree } k\text{)}$$

are expressed very naturally in terms of the Chern roots as:

$$\sigma_{m-1} = \sum_{m_1+m_2=m-1} q_1^{m_1} q_2^{m_2} \text{ and } \tau_{k+1} = \prod_{m_1+m_2=m-1} (m_1 q_1 + m_2 q_2)$$

Thus any intersection number of the form:

$$\int_{G(m,n)} \sigma_{H^{m_1}} \cdots \sigma_{H^{m_a}} \cdot \tau_{\mathcal{O}_{\mathbf{P}^n}(k_1)} \cdots \tau_{\mathcal{O}_{\mathbf{P}^n}(k_b)}$$

lends itself to the Vafa-Intriligator formula. Such intersection numbers are now known as the basic "Gromov-Witten invariants" of lines in $\mathbf{P}^n$.

Basic genus-zero Gromov-Witten invariants are analogues of the σ and τ classes above when the "universal line":

$$\mathbf{P}(U) \stackrel{e}{\rightarrow} \mathbf{P}^{n-1}$$

is replaced by a "universal map to a rational curve":

$$\mathcal{C} \stackrel{e}{\rightarrow} X$$

of arbitrary degree in an arbitrary projective variety. In these talks, I want to consider two separate problems. First, how to carry out the Grothendieck program above to find the generalized Grassmannian of rational curves of arbitrary degree in an arbitrary projective manifold (we will start with $\mathbf{P}^n$), and second, how to compute Gromov-Witten invariants (in good cases) with only a partial knowledge of the intersection ring of this "Grassmannian."

2 Maps from $\mathbf{P}^1$ to $\mathbf{P}^n$ and Deligne-Mumford Stacks

We start by looking at the PGL$(2, \mathbf{C})$-quotient of the open subset:

$$V_d := \mathrm{Map}_d(\mathbf{P}^1, \mathbf{P}^n) \subset \mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^d(\mathbf{C}^2), \mathbf{C}^{n+1})) =: \mathbf{P}^n_d$$

parametrizing the *regular* (not rational) maps $f : \mathbf{P}^1 \to \mathbf{P}^n$ of degree d

$$(\text{i.e. } f = (P_0(x,y) : ... : P_n(x,y)) \in V_d \Leftrightarrow \gcd(P_0, ..., P_n) = 1)$$

But there are two problems with this:

(i) The action on V_d isn't free, and

(ii) The (GIT) quotient $V_d/\mathrm{PGL}(2, \mathbf{C})$ isn't proper.

Already (i) will put us in the realm of stacks, but let's consider (ii) first. We could take the quotient of the larger open subset of GIT (semi)-stable points for the SL$(2, \mathbf{C})$ action on $\mathbf{P}^n_d$ to get a proper quotient, but this will parametrize *rational* maps to $\mathbf{P}^n$. Instead, we enlarge V_d to a bigger open set of a "better" compactification. To see what this compactification should be, suppose e^* is a family of regular maps of degree d over a punctured t-disk:

$$\begin{array}{ccc} \mathbf{P}^1 \times \Delta^* & \stackrel{e^*}{\to} & \mathbf{P}^n \\ \downarrow & & \\ \Delta^* & & \end{array}$$

Then e^* extends generically across the central fiber to a *rational* map $\mathbf{P}^1 \times \Delta - - > \mathbf{P}^n$ defined outside of finitely many points in $\mathbf{P}^1 \times \{0\}$. We can resolve the map by blowing up points of $\mathbf{P}^1 \times \{0\}$, but possibly at the cost of introducing non-reduced fibers in the map to Δ. We can extend across a family with reduced fibers, but this may require a base change, giving us:

$$\begin{array}{ccccccc} C_0 & \subset & \mathcal{C} & \to & \mathbf{P}^1 \times \Delta & -- > & \mathbf{P}^n \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \in & \Delta & \stackrel{t=s^k}{\to} & \Delta & & \end{array}$$

(for some k) that *uniquely* extends e^* across the s-disk subject to:

Restrictions on the family:

(o) $\mathcal{C}$ has only the mildest (canonical) surface singularities

(i) C_0 is nodal (i.e. a divisor with normal crossings in $\mathcal{C}$).

(ii) C_0 is minimal with respect to the map to $\mathbf{P}^n$, i.e. if any component introduced by blowing up maps to a point in $\mathbf{P}^n$, then it must meet the rest of C in at least 3 points. Equivalently (by a happy coincidence) the group of automorphisms of the map $e_0 : C_0 \to \mathbf{P}^n$ (fixing the map to $\mathbf{P}^1$) is finite.

Now that we have a characterization of what limits of regular maps should be, the next step is to actually construct a compactification of $\mathrm{Map}_d(\mathbf{P}^1, \mathbf{P}^n)$ with such limits as the "points of the boundary."

Compactification of $\mathbf{Map}_d(\mathbf{P}^1, \mathbf{P}^n)$

We start by giving this the Grothendieck treatment:

Functor of Points: The functor $\overline{\mathcal{M}ap_d}(\mathbf{P}^1, \mathbf{P}^n)$ is defined by:

$$\overline{\mathcal{M}ap_d}(\mathbf{P}^1, \mathbf{P}^n)(S) = \left\{ \text{families} \quad \begin{array}{ccccc} \mathbf{P}^1 \times S & \overset{f_S}{\leftarrow} & \mathcal{C} & \overset{e_S}{\rightarrow} & \mathbf{P}^n \\ & \searrow & \downarrow & & \\ & & S & & \end{array} \right\}$$

such that $\mathcal{C}$ is flat over S, each C_s is nodal, each $f_s : C_s \to \mathbf{P}^1$ has degree 1, and with respect to f_s, each $e_s : C_s \to \mathbf{P}^n$ satisfies (ii) above.

$$\overline{\mathcal{M}ap_d}(\mathbf{P}^1, \mathbf{P}^n)(\phi : S \to T) = (\text{base change by } \phi)$$

Theorem (Kontsevich-Manin [11]): This functor is a smooth, proper Deligne-Mumford stack.

Before we get into D-M stacks, consider what the theorem implies:

When d > 1, this is NOT the functor of points of a proper scheme: For example, consider the family of maps of degree two from $\mathbf{P}^1$ to $\mathbf{P}^1$:

$$e_t = (x^2 : ty^2 + x^2) : \mathbf{P}^1 \times \Delta^* \to \mathbf{P}^1$$

This family cannot be extended across 0 without the base change $t = s^2$. (Every $e_t^{-1}(1 : a) = (1 : \pm\sqrt{\frac{a-1}{t}})$ has nontrivial monodromy around $t = 0$.) But if there were a proper scheme $\overline{\mathrm{Map}_d}(\mathbf{P}^1, \mathbf{P}^n)$ then the associated map $e_t : \Delta^* \to \overline{\mathrm{Map}_d}(\mathbf{P}^1, \mathbf{P}^n)$ would extend across Δ by the valuative criterion for properness and the family of maps would then extend *without base change*!

The problem with this functor is a subtle question of automorphisms. Since (unlike the Grassmann functor) there exist non-trivial automorphisms:

$$\begin{array}{ccccc} \mathbf{P}^1 & \stackrel{f}{\leftarrow} & C & \stackrel{e}{\rightarrow} & \mathbf{P}^n \\ \| & & \wr\| & & \| \\ \mathbf{P}^1 & \stackrel{f}{\leftarrow} & C & \stackrel{e}{\rightarrow} & \mathbf{P}^n \end{array}$$

of some limits of regular maps (for example, the limit of the e_s above), $\overline{\mathcal{M}ap_d}(\mathbf{P}^1, \mathbf{P}^n)$ is not actually a functor from schemes to sets "at" such maps. For example, the monodromy problem for the family e_t is reflected in the fact that for the following commuting diagram of morphisms:

$$\begin{array}{ccc} 0 & \in & \Delta \\ \| & & n \downarrow \\ 0 & \in & \Delta \end{array} \qquad n(s) = -s$$

the induced isomorphism of the pull-backs $e_s|_0$ and $e_{-s}|_0$ is not the identity map (otherwise the family would descend to an extension of e_t). Thus the "functor" of points isn't functorial! But it *is* (2-)functorial, as a functor from schemes to *groupoids.* And a Deligne-Mumford stack is the (2-)functor of points of a "groupoid-like scheme."

Recall: A groupoid is a category where all the morphisms are isomorphisms.

Basic Observation: A groupoid $\mathcal{C}$ consists of the following data:

(a) Two sets $R = \text{Mor}(\mathcal{C})$ and $U = \text{Ob}(\mathcal{C})$ together with

(b) Five maps:

- $s : R \to U$ the "source" map $s(f : X \to Y) = X$
- $t : R \to U$ the "target" map $t(f : X \to Y) = Y$
- $e : U \to R$ the "identity" map $e(X) = (\text{id}_X : X \to X)$
- $i : R \to R$ the "inverse" map $i(f) = f^{-1}$
- $m : R \times_U R \to R$ the "composition map" $m(f, g) = g \circ f$

(where q_1, q_2 are the first and second projections in:

$$\begin{array}{ccc} R \times_U R & \stackrel{q_2}{\rightarrow} & R \\ q_1 \downarrow & & s \downarrow \\ R & \stackrel{t}{\rightarrow} & U \end{array}$$

so that composition is well-defined, i.e. $s(g) = t(f)$)

(c) Six (sets of) conditions on the maps

(i) $s \circ e = \mathrm{id}_U = t \circ e$ (the source and target of id_X are X)

(ii) $s \circ m = s \circ q_1$ and $t \circ m = t \circ q_2$ (source and target of compositions)

(iii) $m \circ (\mathrm{id}_R, m) = m \circ (m, \mathrm{id}_R) : R \times_U R \times_U R \to R$ (associativity)

(iv) $m \circ (e \circ s, \mathrm{id}_R) = \mathrm{id}_R = m \circ (\mathrm{id}_R, e \circ t) : R \to R \times_U R \to R$
(the identity morphism is a left and right identity)

(v) $i^2 = \mathrm{id}_R, s \circ i = t$, and $t \circ i = s$ (source and target of the inverse)

(vi) $m \circ (\mathrm{id}_R, i) = e \circ s$ and $m \circ (i, \mathrm{id}_R) = e \circ t : R \to R$
(the inverse is a left and right inverse).

Definition: A pair of étale surjective morphisms of *schemes* $s, t : R \rightrightarrows U$ together with three additional morphisms:

$$e : U \to R, \quad i : U \to U, \quad m : R \times_U R \to R$$

constitute an étale equivalence relation if they satisfy (i)-(vi) above.

Fundamental examples of étale equivalence relations:

(I) A finite group acting on a scheme. The action and projection:

$$s = \sigma : G \times U \to U \text{ and } t = \pi : G \times U \to U$$

together with the additional morphisms:

- $e : U \to G \times U;\ u \mapsto (1_G, u)$
- $m : G \times G \times U \cong (G \times U) \times_U (G \times U) \to G \times U;$
  $(g, g', u) \leftrightarrow ((g, g'u), (g', u)) \mapsto (gg', u)$
- $i : G \times U \to G \times U; (g, u) \mapsto (g^{-1}, gu)$

constitute an étale equivalence relation.

(II) An étale surjective map $U \to M$ (e.g. a Zariski open cover!). Then

$$s = p_1 : U \times_M U \to U \text{ and } \ t = p_2 : U \times_M U \to U$$

(the two projections) together with:

- $e = \delta : U \to U \times_M U$, • $i = (p_2, p_1) : U \times_M U \to U \times_M U$ and

- $m = \pi_{13} : U \times_M U \times_M U \cong (U \times_M U) \times_U (U \times_M U) \to U \times_M U$

constitute and étale equivalence relation.

The raison d'étre for étale equivalence relations (which I'll denote U/R) is that for many purposes, they can be treated as though they were schemes. For example, the following theorem is abstracted into the *definitions* of the category of quasi-coherent sheaves on U/R and the functor of points of U/R:

Descent Theorem (Grothendieck): For the particular étale equivalence relation in Example (II) above:

(a) A quasi-coherent sheaf $\mathcal{F}$ on U together with "descent data":

$$g : s^*\mathcal{F} \xrightarrow{\sim} t^*\mathcal{F} \quad \text{satisfying} \quad e^*g = \mathrm{id}_{\mathcal{F}}, i^*g = g^{-1}, \text{and} \quad m^*g = q_2^*g \circ q_1^*g$$

descends to a uniquely determined quasi-coherent sheaf on M. Moreover, sheaf homomorphisms $\phi : \mathcal{F} \to \mathcal{G}$ that commute with the descent data also descend uniquely, and there is an equivalence of categories between quasi-coherent sheaves with descent data on U and quasi-coherent sheaves on M

(b) A morphism $\phi : U \to S$ descends to $\overline{\phi} : M \to S$ if and only if

$$s^*\phi = t^*\phi : U \times_M U \to S$$

Remark: If $U = \coprod U_i \to M$ is a (Zariski) open cover and $\mathcal{F} = \mathbf{C}^n \otimes \mathcal{O}_U$, then descent data is the same thing as a collection of ordinary transition functions $g_{ij} : \mathbf{C}^n \otimes \mathcal{O}_{U_i \cap U_j} \to \mathbf{C}^n \otimes \mathcal{O}_{U_i \cap U_j}$ satisfying the ordinary cocycle condition $g_{ik} = g_{jk} \circ g_{ij} : \mathbf{C}^n \otimes \mathcal{O}_{U_i \cap U_j \cap U_k} \to \mathbf{C}^n \otimes \mathcal{O}_{U_i \cap U_j \cap U_k}$. Thus descent data can be thought of as the generalization of transition functions with cocycle conditions from trivial bundles to arbitrary quasi-coherent sheaves, and from Zariski open covers to arbitrary étale surjective morphisms.

Definition: For an arbitrary étale equivalence relation U/R:

(a) The category of quasi-coherent sheaves on U/R is *by definition* the category of quasi-coherent sheaves on U with descent data.

(b) The functor of points of U/R is the "sheafification" $\overline{h}_{U/R}$ of:

$$h_{U/R}(S) = \ (s_*, t_* : \mathrm{Hom}(S, R) \rightrightarrows \mathrm{Hom}(S, U) \text{ with } e_*, m_*, i_*)$$

$$h_{U/R}(\phi : S \to T) = \phi^*$$

This allows one to define, not just $\overline{h}_{U/R}(S)$, but also (in a consistent way) $\overline{h}_{U/R}(U'/R')$ for any étale equivalence relation. These are, of course, the scheme analogues of *functors* of groupoids $F : \mathcal{C}' \to \mathcal{C}$, which naturally form a 2-category, with natural transformations as the 2-morphisms.

Remark: Sheafifying the functor $h_{U/R}(S)$ is a very reasonable thing to do! For example, if U is a Zariski open cover of M and $R = U \times_M U$, then:

$$h_{U/R}(S) = \{\text{morphisms } f : S \to M \text{ that factor through } U\}!$$

and it is only after sheafifying that we get the desired:

$$\overline{h}_{U/R}(S) = \text{Hom}(S, M)$$

Definition: A **Deligne-Mumford stack** is the (sheafified) functor of points of some étale equivalence relation U/R.

Example: In Example (I) above,

(a) The category of quasi-coherent sheaves on U/R is same thing as the category of G-sheaves on U.

(b) The "presheaf" functor of points is:

$$h_{U/R}(S) = \{\text{Hom}(S, U) + \text{trivial } G\text{-bundle on } S\}$$

and the sheafified functor of points is:

$$\overline{h}_{U/R}(S) = \{\text{Hom}(S, U) + \text{principal } G\text{-bundles on } S\}$$

which is therefore a Deligne-Mumford stack.

Getting back to $\overline{\mathcal{M}ap_d}(\mathbf{P}^1, \mathbf{P}^n)$, we need to construct an étale equivalence relation U/R such that:

$$\overline{\mathcal{M}ap_d}(\mathbf{P}^1, \mathbf{P}^n) = \overline{h}_{U/R}$$

We will do this in the next lecture. The equivalence relation will be a mixture of Examples (I) and (II), i.e. patched out of equivalence relations $U_i/G_i \times U_i$.

3 Pointed Curves and Stacks of Maps

We start with the two basic moduli spaces of pointed genus-zero curves:

Unparametrized: For each $n \geq 3$, $\overline{M}_{0,n}$ is the smooth projective $n-3$-fold that represents the functor:

$$\overline{\mathcal{M}}_{0,n}(S) = \left\{ \text{flat families} \quad \begin{array}{c} \mathcal{C} \\ \pi \downarrow\Uparrow \rho_i \\ S \end{array} \right\}, \quad \overline{\mathcal{M}}_{0,n}(\phi) = (\text{base-change})$$

where the C_s are nodal, projective genus 0 curves, the $\rho_1, ..., \rho_n$ are sections that avoid the nodes of the fibers and avoid each other, and finally, each pointed curve fiber $(C_s; p_{1_s}, ..., p_{n_s})$ is automorphism-free.

Remark: The same definition works for genus g curves, except that we need to allow finite automorphism groups. In genus 0, there is a minor miracle: any pointed curve with a finite automorphism group is automorphism-free. For this reason, $\overline{M}_{0,n}$ is a scheme, but $\overline{\mathcal{M}}_{g,n}$ is only a Deligne-Mumford stack.

Examples: $\overline{M}_{0,3} = \text{pt}$, $\overline{M}_{0,4} = \mathbf{P}^1$ (via the cross-ratio).

Kapranov has a very beautiful description of $\overline{M}_{0,n}$ in general. A family $(\mathcal{C}; \rho_1, ..., \rho_n) \in \overline{\mathcal{M}}_{0,n}(S)$, uniquely determines a morphism $f_S : \mathcal{C} \to \mathbf{P}^{n-3}$ of degree $n-3$ on each C_s with the condition that the $f_S \circ \rho_i$ are constant maps for $i = 1, ..., n-1$:

$$f_S \circ \rho_1(S) = (1 : 0 : ... : 0), ..., f_S \circ \rho_{n-2}(S) = (0 : ... : 0 : 1) \text{ and}$$

$$f_S \circ \rho_{n-1}(S) = (1 : 1 : ... : 1)$$

This gives a natural transformation of functors $\overline{\mathcal{M}}_{0,n} \to h_{\mathbf{P}^{n-3}}; \mathcal{C}_S \mapsto f_S \circ \rho_n$, hence a morphism of their moduli spaces:

$$\Phi : \overline{M}_{0,n} \to \mathbf{P}^{n-3}$$

which is an isomorphism off the codimension 2 (and higher) linear subspaces spanned by the images of the $f_S \circ \rho_i$, and which is a blow-up of these subspaces in a particular order, starting with the points themselves. Thus, for example,

$$\overline{M}_{0,5} \to \mathbf{P}^2 \text{ is the blow-up of 4 independent points}$$

Parametrized: $\mathbf{P}^1[n]$ is the smooth projective n-fold that represents:

$$\mathcal{P}^1[n](S) = \left\{ \begin{array}{ccc} \mathbf{P}^1 \times S & \stackrel{f}{\leftarrow} & \mathcal{C} \\ & \searrow & \pi \downarrow\Uparrow \rho_i \\ & & S \end{array} \right\}, \quad \mathcal{P}^1[n](\phi) = (\text{base-change})$$

where, in addition to the previous conditions on the fibers and sections, each $f_s : C_s \to \mathbf{P}^1$ is a degree 1 "parametrization" of C_s, and each n-pointed parametrized curve $(C_s; p_{1s}, ..., p_{ns}, f_s)$ is automorphism-free.

Remark: This is the Fulton-Macpherson configuration space for $\mathbf{P}^1$.

Examples: $\mathbf{P}^1[0] = \text{pt}, \mathbf{P}^1[1] = \mathbf{P}^1, \mathbf{P}^1[2] = \mathbf{P}^1 \times \mathbf{P}^1$, $\mathbf{P}^1[3] = \text{bl}_\Delta((\mathbf{P}^1)^3)$ (the blow-up along the small diagonal) and in general,

$$\Psi : \mathbf{P}^1[n] \to (\mathbf{P}^1)^n$$

is the blow-up along certain diagonals (in a very particular order).

Relationships between the two moduli spaces: The action of $\mathrm{PGL}(2, \mathbf{C})$ is free on the open subset

$$U_{\text{aut}-\text{free}} := \{(C; p_1, ..., p_n, f) | (C; p_1, ..., p_n) \text{ is automorphism-free}\} \subset \mathbf{P}^1[n]$$

(i.e. the parametrized component should have ≥ 3 nodes + marked points) and then:

$$\overline{M}_{0,n} = U_{\text{aut}-\text{free}}/PGL(2, \mathbf{C}).$$

But there are also interesting maps going the other way! For example:

$$\overline{M}_{0,n+1} \times \mathbf{P}^1 \to \mathbf{P}^1[n]$$

attaches $p_{n+1} \in C$ to $p \in \mathbf{P}^1$ to make an n-pointed parametrized curve, where all the marked points are on the unparametrized component(s). This is $\Psi^{-1}(\Delta) \subset \mathbf{P}^1[n]$ (again, the small diagonal) and it lives in the complement of $U_{\text{aut}-\text{free}}$. More generally, $\mathbf{P}^1[n]$ is "stratified" by quasi-finite maps:

$$\sigma_{n_0, n_1, \ldots, n_\nu} : \mathbf{P}^1[n_0 + \nu] \times \prod_{i=1}^{\nu} \overline{M}_{0,n_i+1} \to \mathbf{P}^1[n]; \quad \sum n_i = n$$

attaching ν extra marked points on a parametrized curve to single marked points on ν unparametrized curves. The resulting curve is a marked "comb."

Construction of the étale equivalence relation for $\overline{\mathcal{M}ap}_d(\mathbf{P}^1, \mathbf{P}^n)$.

Choose a basis $\mathbf{H} = \{H_0, ..., H_n\}$ of hyperplanes for $\mathbf{P}^n$, and with respect to this basis, consider only the families (denoted $\overline{\mathcal{M}ap}_d(\mathbf{P}^1, \mathbf{P}^n)_{\mathbf{H}}(S)$):

$$\begin{array}{ccccc} \mathbf{P}^1 \times S & \stackrel{f}{\leftarrow} & \mathcal{C} & \stackrel{e}{\rightarrow} & \mathbf{P}^n \\ & \searrow & \downarrow & & \\ & & S & & \end{array}$$

that meet the hyperplanes transversely, in the sense that each $e^{-1}(H_i) \subset \mathcal{C}$ is a union of d sections $\rho_{ij} : S \to \mathcal{C}$ that avoid the nodes of the fibers. This looks like it determines an element of $\mathcal{P}^1[d(n+1)](S)$, but actually only gives:

$$\overline{\mathcal{M}ap}_d(\mathbf{P}^1, \mathbf{P}^n)_{\mathbf{H}}(S) \to \mathcal{P}^1[d(n+1)](S)/(\mathcal{S}_d)^{n+1}$$

($\mathcal{S}_d$ is the symmetric group) because each set $\{\rho_{i1}, ..., \rho_{id}\}$ is *unordered!*. Moreover, not every pointed curve can occur. Let:

$$U_{\text{(n+1)}-\text{balanced}} \subset \mathbf{P}^1[d(n+1)]$$

be the (open!) subscheme of pointed curves with the property that each line bundle:

$$\mathcal{O}_C(p_{i1} + ... + p_{id})$$

restricts to the *same* line bundle on every component $C_0 \subset C$. Evidently this open subset is invariant under the action of $(\mathcal{S}_d)^{n+1}$.

Lemma: (see [7]) The action of $(\mathcal{S}_d)^{n+1}$ lifts to a $(\mathbf{C}^*)^n$-bundle:

$$\begin{array}{ccc} (\mathbf{C}^*)^{n+1} & \to & U_{\mathbf{H}} \\ & & \downarrow \\ & & U_{\text{(n+1)}-\text{balanced}} \end{array}$$

such that the equivalence relation $U_{\mathbf{H}}/(\mathcal{S}_d)^{n+1} \times U_{\mathbf{H}}$ represents the functor $\overline{\mathcal{M}ap}_d(\mathbf{P}^1, \mathbf{P}^n)_{\mathbf{H}}$, i.e. each $\overline{\mathcal{M}ap}_d(\mathbf{P}^1, \mathbf{P}^n)_{\mathbf{H}}$ is a Deligne-Mumford stack.

Moreover, these form an "open cover" of $\overline{\mathcal{M}ap}_d(\mathbf{P}^1, \mathbf{P}^n)$ itself, and there is an étale equivalence relation:

$$R \Rightarrow \coprod U_{\mathbf{H}}$$

representing the functor $\overline{\mathcal{M}ap}_d(\mathbf{P}^1, \mathbf{P}^n)$ (and finitely many $U_{\mathbf{H}}$ will suffice.) (R consists of $\coprod_{\mathbf{H}}((\mathcal{S}_d)^{n+1} \times U_{\mathbf{H}})$ and "patching data" $\coprod_{\mathbf{H},\mathbf{H}'}((\mathcal{S}_d)^{n+1} \times U_{\mathbf{H},\mathbf{H}'})$.)

Variations: There are several important variations to consider:

Pointed Maps: The functor

$$\overline{\mathcal{M}ap}_{d,m}(\mathbf{P}^1, \mathbf{P}^n)(S) = \left\{ \begin{array}{ccccc} \mathbf{P}^1 \times S & \stackrel{f_S}{\leftarrow} & \mathcal{C} & \stackrel{e_S}{\to} & \mathbf{P}^n \\ & \searrow & \pi \downarrow\Uparrow \rho_i & & \\ & & S & & \end{array} \right\}$$

adds in m sections (with the usual conditions on the sections). There are "forgetful" transformations (this is a theorem!):

$$\overline{\mathcal{M}ap}_{d,m+1}(\mathbf{P}^1, \mathbf{P}^m) \to \overline{\mathcal{M}ap}_{d,m}(\mathbf{P}^1, \mathbf{P}^n)$$

so that $\overline{\mathcal{M}ap}_{d,m+1}(\mathbf{P}^1, \mathbf{P}^m) = \mathcal{C}$ is the universal curve over $\overline{\mathcal{M}ap}_{d,m}(\mathbf{P}^1, \mathbf{P}^m)$.

Their étale equivalence relations are constructed just as $R \Rightarrow U_\beta$ was constructed above, but carrying m additional points in $\mathbf{P}^1[d(n+1)+m]$.

Stable Maps: These are the Kontsevich-Manin functors:

$$\overline{\mathcal{M}}_{0,m}(\mathbf{P}^n, d)(S) = \left\{ \text{families of genus 0 maps} \qquad \begin{array}{ccc} \mathcal{C} & \stackrel{e_S}{\to} & \mathbf{P}^n \\ \pi \downarrow \Uparrow \rho_i & & \\ S & & \end{array} \right\}$$

built out of the spaces $\overline{M}_{0,d(n+1)+m}$ and these generalize to higher genus:

$$\overline{\mathcal{M}}_{g,m}(\mathbf{P}^n, d)(S) = \left\{ \text{families of genus } g \text{ maps} \qquad \begin{array}{ccc} \mathcal{C} & \stackrel{e_S}{\to} & \mathbf{P}^n \\ \pi \downarrow \Uparrow \rho_i & & \\ S & & \end{array} \right\}$$

with a little more care (e.g. since $\overline{\mathcal{M}}_{g,m}$ is itself a Deligne-Mumford stack). Also in this case, the stack is not, in general, smooth.

Maps to other targets: If $X \subset \mathbf{P}^n$, let $I(X) = \langle F_1, ..., F_m \rangle$ be its ideal, where $F_i \in H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(l_i))$. Then each F_i determines a section

$$\overline{F}_i \in H^0(\overline{\mathcal{M}ap}_d(\mathbf{P}^1, \mathbf{P}^n), \pi_* e^* \mathcal{O}_{\mathbf{P}^n}(l_i))$$

(i.e. a vector bundle with descent data on the equivalence relation $R \Rightarrow U$) and the zero schemes give equivalence relations that patch to construct:

$$\overline{\mathcal{M}ap}_d(\mathbf{P}^1, X) \text{ (and } \overline{\mathcal{M}ap}_{d,m}(\mathbf{P}^1, X), \overline{\mathcal{M}}_{0,m}(X, d), \text{ etc.)}$$

Notice that the homology class $\beta = e_*[C_s] \in H_2(X, \mathbf{Z})$ is constant over a connected base scheme, so it follows that we obtain a decomposition:

$$\overline{\mathcal{M}ap}_d(\mathbf{P}^1, X) = \coprod_{\deg(\beta)=d} \overline{\mathcal{M}ap}_\beta(\mathbf{P}^1, X)$$

as a disjoint union (again, this makes sense at the equivalence class level). And of course the other moduli spaces have a similar decomposition.

In particular, if we consider the pair of maps (e, f) defining a family in $\overline{\mathcal{M}ap}_\beta(\mathbf{P}^1, X)$ to be a single map $(e, f) : C \to \mathbf{P}^1 \times X$ of "bidegree" $(1, \beta)$, then we can identify:

$$\overline{\mathcal{M}ap}_\beta(\mathbf{P}^1, X) = \overline{\mathcal{M}}_{0,0}(\mathbf{P}^1 \times X, (\beta, 1))$$

and so all the "map" stacks are Kontsevich-Manin stacks of stable maps.

Deformation Theory: With the functor of points comes the possibility of analyzing the "local properties" of the stacks $\overline{\mathcal{M}ap}_\beta(\mathbf{P}^1, X)$ (and the others) by means of deformation theory. It is evident from the construction that:

$\overline{\mathcal{M}}_{0,0}(\mathbf{P}^n, d)$ is smooth, of dimension $d(n+1)-3+n$, since it is patched out of $(\mathbf{C}^*)^n$ bundles over open subsets of $\overline{M}_{0,d(n+1)}$ (modulo étale equivalence). But what about the others?

We will define the Zariski tangent space at a "closed point" $(C; f) \in \overline{\mathcal{M}}_{g,0}(X, \beta)(\mathrm{Spec}(\mathbf{C}))$ to be the space of extensions to families:

$$(C_\epsilon; f_\epsilon) \in \overline{\mathcal{M}}_{g,0}(X, \beta)(\mathrm{Spec}(\mathbf{C}[\epsilon]))$$

and this turns out to sit in an exact sequence:

$$0 \to H^0(C, TC) \to H^0(C, f^*TX) \to (\text{tangent space}) \to$$

$$\to H^1(C, TC) \to H^1(C, f^*TX) \to (\text{obstruction space}) \to 0$$

where the obstruction space measures the obstructions to the infinitessimal lifting that would give the smoothness of the stack. Notice in particular that if $H^1(C, f^*TX) = 0$ for all maps, then the stack is smooth, of dimension:

$$(*) \quad \chi(C, f^*TX) - \chi(C, TC) = \dim(X) + \int_\beta c_1(TX) - (3 - 3g)$$

and there is one important case where this holds, namely:

When X is homogeneous and $g = 0$ then the $\overline{\mathcal{M}}_{0,0}(X, \beta)$ are all smooth, of dimension $(*)$ above (and it has been shown that they are also irreducible).

Note: When $g > 0$, then even the stacks $\overline{\mathcal{M}}_{g,0}(\mathbf{P}^n, d)$ are not smooth.

Final Remarks: Analogous to the blow-down of the configuration space:

$$\mathbf{P}^1[n] \to \mathbf{P}^n$$

there is a "blow-down" to the naive space of rational maps from §2:

$$\Phi : \overline{\mathcal{M}ap}_d(\mathbf{P}^1, \mathbf{P}^n) \to \mathbf{P}^n_d$$

This is *literally* the blow-down of configuration spaces on the level of equivalence relations. It is interesting to note that if we blow down $R \Rightarrow U$ we get an equivalence relation describing $\mathbf{P}^n_d$, but not an étale one.

Also, the stratification of $\mathbf{P}^1[n]$ has the natural analogue:

$$\gamma_{d_0,..,d_\nu} : \overline{\mathcal{M}ap}_{d_0,\nu}(\mathbf{P}^1, \mathbf{P}^n) \times_X \prod_{i=1}^{\nu} \overline{\mathcal{M}}_{0,1}(\mathbf{P}^n, d_i) \to \overline{\mathcal{M}ap}_d(\mathbf{P}^1, \mathbf{P}^n)$$

stratifying the map space into maps of "comb"-like curves.

4 Enumerative Geometry

I'm going to focus here on the stacks:

$$\overline{\mathcal{M}}_{0,0}(X, \beta)$$

where X is a homogeneous space, though you should be aware that much of what I will say can be extended to arbitrary targets and genus by means of a "virtual fundamental class."

There is, in this case, a well-defined *Chow ring* (over $\mathbf{Q}$) with the nice properties of Fulton-Macpherson's intersection theory:

$$A^*_{\mathbf{Q}}(\overline{\mathcal{M}}_{0,0}(X, \beta))$$

with which to compute intersection numbers. In this Chow ring, we want to study "tautological" classes, which are derived from the universal family:

$$\begin{array}{ccc} \mathcal{C} & \stackrel{e}{\to} & X \\ \pi \downarrow & & \\ \overline{\mathcal{M}}_{0,0}(X, \beta) & & \end{array}$$

These come in three flavors. There are the "enumerative" classes derived from X, which I will name (following the introduction):

- $\sigma_c := \pi_* e^* c \in A^{k-1}_{\mathbf{Q}}(\overline{\mathcal{M}}_{0,0}(X, \beta))$ for $c \in A^k(X)$

(the expected class of maps that meet α)

• $\tau_E := c_{\text{top}}(\pi_* e^* E)$ for a "positive" vector bundle E

(the expected class of maps to the zero scheme of a section of E)

There is a very useful class on $\mathcal{C}$ itself, namely:

• $\psi = c_1(\omega_\pi)$ for ω_π the relative dualizing (cotangent) line bundle

which we use to define classes $\pi_* \psi^k \in A^{k-1}_{\mathbf{Q}}(\overline{\mathcal{M}}_{0,0}(X, \beta))$

And finally, there are push-forwards of the classes above under various "gluing" maps, like the comb-stratification maps:

$$\gamma_{\beta_0,\ldots,\beta_\nu} : \overline{\mathcal{M}ap}_{\beta_0,\nu}(\mathbf{P}^1, X) \times_X \prod_{i=1}^{\nu} \overline{\mathcal{M}}_{0,1}(\mathbf{P}^n, \beta_i) \to \overline{\mathcal{M}ap}_{\sum \beta_i}(\mathbf{P}^1, X)$$

So following the introduction, we wish to find a method for computing intersection numbers of the form:

$$\int_{\overline{\mathcal{M}}_{0,0}(X,\beta)} \sigma_{c_1} \cup \ldots \cup \sigma_{c_n} =: \langle c_1, \ldots, c_n \rangle_\beta$$

(the Gromov-Witten invariants of X) assuming, of course, that:

$$(c_1 - 1) + (c_2 - 1) + \ldots + (c_n - 1) = \dim(\overline{\mathcal{M}}_{0,0}(X, \beta))$$

or, more generally, we wish to compute:

$$\int_{\overline{\mathcal{M}}_{0,0}(X,\beta)} \sigma_{c_1} \cup \ldots \cup \sigma_{c_n} \cup \tau_E =: \langle c_1, \ldots, c_n \rangle_\beta^E$$

assuming that $(c_1 - 1) + \ldots + (c_n - n) + (\int_\beta c_1(E) + \text{rk}(E)) = \dim(\overline{\mathcal{M}}_{0,0}(X, \beta))$.

Examples: If $p_i \in A^2(\mathbf{P}^2)$ are each (the same) class of a point, then:

$$\int_{\overline{\mathcal{M}}_{0,0}(\mathbf{P}^2,d)} \sigma_{p_1} \cup \ldots \cup \sigma_{p_{3d-1}} =: N_d$$

is the # of rational curves of degree d through $3d - 1$ general points of $\mathbf{P}^2$.

(Here, $\dim(\overline{\mathcal{M}}_{0,0}(\mathbf{P}^2, d)) = 3d - 1$.) Another example of some interest is:

$$\int_{\overline{\mathcal{M}}_{0,0}(\mathbf{P}^4,d)} \tau_{\mathcal{O}(5)} =: n_d$$

the expected numbers of degree d maps to a quintic hypersurface $S \subset \mathbf{P}^4$.

(Here, $\dim(\overline{\mathcal{M}}_{0,0}(\mathbf{P}^4, d)) = 5d + 1$.)

But unlike the Grassmannian case, we need to make these computations *without complete knowledge of* $A^*_{\mathbf{Q}}(\overline{\mathcal{M}}_{0,0}(\mathbf{P}^n, d))$, basically because these rings are too complicated! There are now several methods for doing this. The first, exploited by Kontsevich-Manin, is to induct down on the number of σ's using:

The WDVV Equations: Assume for simplicity that $A^*_{\mathbf{Q}}(X) = H^*(X, \mathbf{Q})$ and choose a basis $\{e_i\}$, with intersection pairing:

$$g_{ij} = \int_X e_i \cup e_j$$

Then:
(**n=4 version**) For any 4 classes $c_1, c_2, c_3, c_4 \in H^*(X, \mathbf{Q})$ and $\beta \in H_2(X, \mathbf{Z})$:

$$\sum_{\beta_1+\beta_2=\beta} \left(\int_{\overline{\mathcal{M}}_{0,0}(X,\beta_1)} \sigma_{c_1}\sigma_{c_2}\sigma_{e_i} \right) g^{ij} \left(\int_{\overline{\mathcal{M}}_{0,0}(X,\beta_2)} \sigma_{e_j}\sigma_{c_1}\sigma_{c_2} \right) =$$

$$\sum_{\beta_1+\beta_2=\beta} \left(\int_{\overline{\mathcal{M}}_{0,0}(X,\beta_1)} \sigma_{c_1}\sigma_{c_3}\sigma_{e_i} \right) g^{ij} \left(\int_{\overline{\mathcal{M}}_{0,0}(X,\beta_1)} \sigma_{e_j}\sigma_{c_2}\sigma_{c_4} \right)$$

(with the convention, entirely consistent, that:

$$\int_{\overline{\mathcal{M}}_{0,0}(X,0)} \sigma_{c_1}\sigma_{c_2}\sigma_{c_3} = \int_X c_1 \cup c_2 \cup c_3)$$

Corollary (Witten): Choose $t_1, ..., t_m \in H_2(X, \mathbf{Z})$ independent over $\mathbf{Q}$ with the property that each curve class $\beta = f_*[C]$ is a non-negative (integer) linear combination of the t_i. Let $q_i = e^{t_i}$, and if $\beta = \sum d_i t_i$, let $q^\beta = \prod q_i^{d_i}$. Then there is an associative, commutative "quantum" product $*$ with 1 on the $\mathbf{Q}[[q_1, ..., q_m]]$-module:

$$QH^*(X) := H^*(X)[[q_1, ..., q_m]]$$

uniquely defined by the "structure constants":

$$\int_X (c * c') \cup c'' = \sum_\beta \left(\int_{\overline{\mathcal{M}}_{0,0}(X,\beta)} \sigma_c \sigma_{c'} \sigma_{c''} \right) q^\beta$$

(general version) For any classes $c_1, ..., c_4, c'_1, ..., c'_m \in H^*(X, \mathbf{Q})$ and β:

$$\sum_{S\subseteq\{1,...,m\}} \sum_{\beta_1+\beta_2=\beta} \left(\int_{\overline{\mathcal{M}}_{0,0}(X,\beta_1)} \sigma_{c_1}\sigma_{c_2}\sigma_{e_i} \prod_{s\in S} \sigma_{c'_s} \right) g^{ij} \left(\int_{\overline{\mathcal{M}}_{0,0}(X,\beta_2)} \sigma_{e_j}\sigma_{c_3}\sigma_{c_4} \prod_{s\notin S} \sigma_{c'_s} \right)$$

$$=$$

$$\sum_{S\subseteq\{1,...,m\}}\sum_{\beta_1+\beta_2=\beta}\left(\int_{\overline{\mathcal{M}}_{0,0}(X,\beta_1)}\sigma_{c_1}\sigma_{c_3}\sigma_{e_i}\prod_{s\in S}\sigma_{c'_s}\right)g^{ij}\left(\int_{\overline{\mathcal{M}}_{0,0}(X,\beta_1)}\sigma_{e_j}\sigma_{c_2}\sigma_{c_4}\prod_{s\notin S}\sigma_{c'_s}\right)$$

Corollary [11]: Let $\{t_i\}$ be the dual basis to $\{e_i\}$. There is an associative, commutative "big quantum" product $\bullet$ with 1 on the $\mathbf{Q}[[t_1,...,t_n]]$-module:

$$BQH^*(X) := H^*(X)[[t_1,...,t_n]]$$

uniquely defined by the "structure constants":

$$\int_X (c\bullet c')\cup c'' = \sum_\beta\sum_{m\geq 0}\sum_{m_1+...+m_n=m}\left(\int_{\overline{\mathcal{M}}_{0,0}(X,\beta)}\sigma_c\sigma_{c'}\sigma_{c''}\prod_{i=1}^n\sigma_{e_i}^{m_i}\right)\prod_{i=1}^n\frac{t_i^{m_i}}{m_i!}$$

Remark: $BQH^*(X)$ specializes to $QH^*(X)$ (setting some of the t's to zero) and $QH^*(X)$ specializes to $H^*(X)$ (setting all of the q's to zero). There is much more to this story, starting with the "potential" function whose third partials give the structure constants above when $c,c',c''=e_i,e_j,e_k$. But I want to focus on the application of WDVV, noticed by Kontsevich-Manin, to the computation of Gromov-Witten invariants. But first:

Proof of WDVV (Sketch): Using the projection formula, we reinterpret Gromov-Witten invariants as intersection numbers on *pointed* map stacks:

$$\int_{\overline{\mathcal{M}}_{0,0}(X,\beta)}\sigma_{c_1}\cdots\sigma_{c_n} = \int_{\overline{\mathcal{M}}_{0,n}(X,\beta)}e_1^*c_1\cdots e_n^*c_n$$

where $e_i = e\circ\rho_i$ are determined by the universal family:

$$\begin{array}{ccc} \mathcal{C} & \stackrel{e}{\rightarrow} & X \\ \pi\downarrow\Uparrow\rho_i & & \\ \overline{\mathcal{M}}_{0,n}(X,\beta) & & \end{array}$$

Next, for $m\geq 0$, there is a "cross-ratio" $cr:\overline{\mathcal{M}}_{0,4+m}(X,\beta)\to\overline{M}_{0,4}=\mathbf{P}^1$ forgetting the map e and all but the first four points. Let $0,\infty\in\overline{M}_{0,4}=\mathbf{P}^1$ correspond to curves with two components whose marked points are grouped as (respectively) $\{1,2\}\cup\{3,4\}$ and $\{1,3\}\cup\{2,4\}$ on the two components. Then the left side of WDVV (general version) is:

$$\int_{\overline{\mathcal{M}}_{0,4+m}(X,\beta)}\prod_{i=1}^4 e_i^*c_i\cup\prod_{j=1}^m e_{j+4}^*c'_j\cup cr^*[0]$$

where $[0] \in A^1(\mathbf{P}^1)$ is the class of $\{0\}$ and the right side is:

$$\int_{\overline{\mathcal{M}}_{0,4+m}(X,\beta)} \prod_{i=1}^{4} e_i^* c_i \cup \prod_{j=1}^{m} e_{j+4}^* c_j' \cup cr^*[\infty]$$

and $[0] = [\infty] \in A^1(\mathbf{P}^1)$ gives WDVV.

Application to Enumerative Geometry: First of all, notice that:

$\sigma_{[X]} = 0$ (since $[X] \in A^0(X)$) and

$\sigma_{[D]} = \left(\int_\beta [D]\right) [\overline{M}_{0,0}(X,\beta)] \in A^0(\overline{M}_{0,0}(X,\beta))$ when D is a divisor class.

Thus

$$\int_{\overline{M}_{0,0}(X,\beta)} \sigma_{c_1} \cdots \sigma_{c_n} \sigma_{[D]} = \left(\int_\beta [D]\right) \int_{\overline{M}_{0,0}(X,\beta)} \sigma_{c_1} \cdots \sigma_{c_n}$$

so we can eliminate any divisor classes from the $c_1, ..., c_n$. The Kontsevich-Manin reconstruction theorem allows us to eliminate divisor classes from within the c_i's. The key observation is the following. If we plug:

$$c_1, c_2, c_3, [D], c_1', ..., c_m'$$

into the general version of WDVV, we get:

$$\int_{\overline{M}_{0,0}(X,\beta)} \sigma_{c_1}\sigma_{c_2}\sigma_{[D]\cup c_3} \prod_{i=1}^{m} c_i' + \int_{\overline{M}_{0,0}(X,\beta)} \sigma_{c_1\cup c_2}\sigma_{[D]}\sigma_{c_3} \prod_{i=1}^{m} c_i' + \{\text{lower } \beta \text{ terms}\} =$$

$$\int_{\overline{M}_{0,0}(X,\beta)} \sigma_{c_1}\sigma_{c_3}\sigma_{[D]\cup c_2} \prod_{i=1}^{m} c_i' + \int_{\overline{M}_{0,0}(X,\beta)} \sigma_{c_1\cup c_3}\sigma_{[D]}\sigma_{c_2} \prod_{i=1}^{m} c_i' + \{\text{lower } \beta \text{ terms}\}$$

(from which we can eliminate the $\sigma_{[D]}$'s).

Thus we can inductively express the first term (the one involving $\sigma_{[D]\cup c_3}$) in terms of Gromov-Witten invariants involving either lower β, fewer c's, and one term (the first on the second line) with the same β and same number of c's, but with $[D]$ eliminated from $\sigma_{[D]\cup c_3}$ (and moved over to $c_{[D]\cup c_2}$). This gives the following corollary by induction:

Kontsevich-Manin Reconstruction: All the (σ) invariants of X can be recursively reconstructed from "2-point" invariants:

$$\int_{\overline{M}_{0,0}(X,\beta)} \sigma_{c_1}\sigma_{c_2}$$

and invariants of classes $c_1, ..., c_n$ that are "primitive" (i.e. not obtainable from divisors cupped with other classes). In particular, if $H^*(X)$ is generated by divisor classes, then the 2-point invariants determine all the others.

Remark: This is particularly remarkable when $X = \mathbf{P}^n$, since in that case:

$$\int_{\overline{M}_{0,0}(\mathbf{P}^n,1)} \sigma_{p_1}\sigma_{p_2} = 1$$

(the line through 2 points!) is the only 2-point invariant (dimension count!)

So reconstruction even gives a new way of performing Schubert calculus:

$$\int_{G(2,n+1)} \sigma_{m_1-1}\sigma_{m_2-1}\cdots\sigma_{m_a-1}$$

on the Grassmannian of lines (by recursing from the 1 line through 2 points) and this method, unlike the ordinary Schubert calculus, generalizes to give recursions for the counts of rational curves of all degrees on all manifolds!

Worked Example: $X = \mathbf{P}^2$.

For each $d > 1$, let $c_1 = c_2 = [p], c_3 = c_4 = [H]$ and $c'_1 = ... = c'_{3d-4} = [p]$. Then WDVV (general version) gives the following recursive formula for N_d:

$$N_d + \sum_{\substack{d_1+d_2=d \\ d_i>0}} \binom{3d-4}{3d_1-3}(d_1N_{d_1})(d_2^3N_{d_2}) = \sum_{\substack{d_1+d_2=d \\ d_i>0}} \binom{3d-4}{3d_1-2}(d_1^2N_{d_1})(d_2^2N_{d_2})$$

This is very satisfactory for σ classes, but how do we work in the τ's? The next development was inspired by mirror symmetry, and requires a new tool, namely the localization theorem of Atiyah-Bott.

5 Equivariant Techniques

The WDVV relations and reconstruction corollary are already surprisingly effective ways to compute Gromov-Witten invariants, but this next technique, introduced by Givental following hints from mirror symmetry, is magical. The idea here is to use the localization theorem of Atiyah-Bott, but in a very clever way.

Quick Review of Equivariant Cohomology: If $\mathbf{C}^*$ acts holomorphically on a compact complex manifold X (the theory is more general than this!) there is an *equivariant cohomology* ring:

$$H^*_{\mathbf{C}^*}(X, \mathbf{Q})$$

that "enriches" the ordinary cohomology ring $H^*(X, \mathbf{Q})$. The equivariant cohomology ring $H^*_{\mathbf{C}^*}(X, \mathbf{Q})$ is an algebra over the polynomial ring $\mathbf{Q}[\lambda]$ (the equivariant cohomology ring of a point with trivial action of $\mathbf{C}^*$), and more generally it is functorial in the usual sense for $\mathbf{C}^*$-equivariant maps $f : X \to Y$. (i.e. there are equivariant pull-backs and proper push-forwards)

Examples: (a) A trivial action of $\mathbf{C}^*$ on X gives:

$$H^*_{\mathbf{C}^*}(X, \mathbf{Q}) = H^*(X, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}[\lambda]$$

(b) A free action of $\mathbf{C}^*$ on X gives:

$$H^*_{\mathbf{C}^*}(X, \mathbf{Q}) = H^*(X/\mathbf{C}^*, \mathbf{Q})$$

(as a $\mathbf{Q} = \mathbf{Q}[\lambda]/\langle\lambda\rangle$-module)

Borel's definition gives equivariant cohomology as (ordinary) cohomology of the twisted product of X with the universal principal bundle:

$$X_{\mathbf{C}^*} := X \times_{\mathbf{C}^*} E\mathbf{C}^* \to B\mathbf{C}^*;\ B\mathbf{C}^* = \mathbf{CP}^\infty, E\mathbf{C}^* = \mathbf{C}^{\infty+1} - \{0\}$$

and $H^*(BG) = \mathbf{Q}[\lambda]$ gives the $\mathbf{Q}[\lambda]$ structure.

(In the algebraic setting, we should think of the equivariant Chow ring

$$A^*_{\mathbf{C}^*}(X)$$

as the ordinary Chow ring of the (smooth!) equivalence relation $X/(\mathbf{C}^* \times X)$).

Equivariant cohomology specializes to (a subring of) cohomology:

$$H^*_{\mathbf{C}^*}(X, \mathbf{Q}) \stackrel{\lambda=0}{\to} H^*(X, \mathbf{Q})$$

and any $\mathbf{C}^*$-linearized vector bundle E over X has equivariant Chern classes:

$$c_i^{\mathbf{C}^*}(E) \in H^{2i}_{\mathbf{C}^*}(X, \mathbf{Q})$$

that specialize to the ordinary Chern classes $c_i(E) \in H^{2i}(X, \mathbf{Q})$.

In particular, each connected component $i : F \hookrightarrow X$ of the locus of fixed points (which is necessarily an embedded submanifold) has a canonically linearized normal bundle (for the trivial action on F) hence:

$$\epsilon_{\mathbf{C}^*}(F) := c_{\text{top}}^{\mathbf{C}^*}(N_{F/X})$$

the equivariant "Euler class" of F in X.

Localization Theorem: An equivariant class $c^{\mathbf{C}^*} \in H^*_{\mathbf{C}^*}(X, \mathbf{Q})$ is uniquely determined (modulo torsion) by its restriction to the fixed components by the "excess intersection" formula:

$$c^{\mathbf{C}^*} \equiv \sum_{F \subset X} i_* \left(\frac{c^{\mathbf{C}^*}|_F}{\epsilon_{\mathbf{C}^*}(F)} \right) \text{ (mod torsion)}$$

($\epsilon_{\mathbf{C}^*}(F)$ is invertible in the localization $H^*_{\mathbf{C}^*}(F, \mathbf{Q})_\lambda = H^*(F) \otimes \mathbf{Q}[\lambda, \lambda^{-1}]$ and $i_* : H^*_{\mathbf{C}^*}(F, \mathbf{Q})_\lambda \to H^*_{\mathbf{C}^*}(X, \mathbf{Q})_\lambda$ is the extended proper push-forward.)

There are two important and useful corollaries to this:

Bott Residue Theorem: If $p(c)$ is a polynomial of degree $\dim(X)$ in the Chern classes of linearized vector bundles on X, then $p(c)$ can be integrated over fixed loci:

$$\int_X p(c) = \int_X p(c^{\mathbf{C}^*}) = \sum_{F \subset X} \int_F \frac{p(c^{\mathbf{C}^*})|_F}{\epsilon_{\mathbf{C}^*}(F)}$$

and if the fixed loci are *points*, then $\int_F$ is just evaluation at the points.

Relative Localization: If $f : X \to X'$ is a $\mathbf{C}^*$-equivariant map, then for each $c^{\mathbf{C}^*} \in H^*_{\mathbf{C}^*}(X, \mathbf{Q})$ and fixed locus $i' : F' \hookrightarrow X'$,

$$\sum_{F \subset f^{-1}(F')} (f|_F)_* \left(\frac{c^{\mathbf{C}^*}|_F}{\epsilon_{\mathbf{C}^*}(F)} \right) = \frac{(f_* c^{\mathbf{C}^*})|_{F'}}{\epsilon_{\mathbf{C}^*}(F')}$$

where $(f|_F)_* : H^*_{\mathbf{C}^*}(F, \mathbf{Q})_\lambda \to H^*_{\mathbf{C}^*}(F', \mathbf{Q})_\lambda$ is the extended push-forward.

Two New Ways to do Schubert Calculus (of Lines):

(i) Using the Bott Residue Theorem. Consider the action of $\mathbf{C}^*$ on $(\mathbf{C}^n)^\vee$ (the dual space to $\mathbf{C}^n$):

$$z \times (x_0, x_1, ..., x_n) \mapsto (x_0, zx_1, z^2 x_2, ..., z^n x_n)$$

The induced action of $\mathbf{C}^*$ on $\mathbf{CP}^n$ has isolated fixed points $e_0, ..., e_n \in \mathbf{CP}^n$ (the coordinate points of $\mathbf{CP}^n$) and the induced action on $G(2, n+1)$ has isolated fixed points $e_{ij} := \overline{e_i e_j} \in G(2, n+1)$ (for $i < j$).

The equivariant Euler classes are then:

$$\epsilon_{\mathbf{C}^*}(e_{ij}) = \frac{(-1)^{i+j} i! j! (n-i)!(n-j)!}{(j-i)^2} \lambda^{2n-2}$$

as one checks from the $\mathrm{Hom}(W, \mathbf{C}^{n+1}/W)$ description of the tangent space. Next, if $U^\vee$ is the dual of the universal sub-bundle, one computes:

$$c_1^{\mathbf{C}^*}(U^\vee)|_{e_{ij}} = (i+j)\lambda \text{ and } c_2^{\mathbf{C}^*}(U^\vee)|_{e_{ij}} = (ij)\lambda^2$$

and then for any symmetric $p(q_1, q_2)$ of degree $2n-2$ in the Chern roots,

$$\int_{G(2,n+1)} p(q_1, q_2) = \sum_{i<j} (-1)^{i+j} \frac{p(i,j)(j-i)^2}{i!j!(n-i)!(n-j)!}$$

(note the similarity with the Vafa-Intriligator residue formula of §1).

(ii) Using relative localization. Consider the action of $\mathbf{C}^*$ on $(\mathbf{C}^2)^\vee$:

$$z \times (x:y) \mapsto (x:zy)$$

and the induced action on $\mathrm{Map}_1(\mathbf{P}^1, \mathbf{P}^n)$. Then the map of compactifications:

$$\Phi : \overline{\mathrm{Map}}_1(\mathbf{P}^1, \mathbf{P}^n) \to \mathbf{P}_1^n = \mathbf{P}(\mathrm{Hom}(\mathbf{C}^2, \mathbf{C}^{n+1}))$$

is $\mathbf{C}^*$-equivariant, and we will apply relative localization to it. There are two fixed loci for the action of $\mathbf{C}^*$ on $\mathbf{P}_1^n$:

$$F_0' = \{(0:1)\} \times \mathbf{P}^n \subset \mathbf{P}_1^n$$

$$F_\infty' = \{(1:0)\} \times \mathbf{P}^n \subset \mathbf{P}_1^n$$

The equivariant Euler classes of these are, respectively:

$$\epsilon_{\mathbf{C}^*}(F_0') = (H+\lambda)^{n+1} \text{ and } \epsilon_{\mathbf{C}^*}(F_\infty') = (H-\lambda)^{n+1}$$

Their preimages happen to be also fixed loci. They are:

$$F_0 := \Phi^{-1}(F_0') = \{(0:1)\} \times \mathcal{C} \text{ and } F_\infty := \Phi^{-1}(F_\infty') = \{(1:0)\} \times \mathcal{C}$$

where $\mathcal{C} = Fl(1,2,n+1)$ is the universal line over the Grassmannian. Their equivariant Euler classes are described in terms of the ψ classes (see §4). I.e.

$$\epsilon_{\mathbf{C}^*}(F_0) = \lambda(\lambda - \psi) \text{ and } \epsilon_{\mathbf{C}^*}(F_\infty) = (-\lambda)(-\lambda - \psi)$$

We need one more piece of data before we invoke relative localization. Namely, recall that the universal bundle on $G(2,n+1)$ is $U^\vee = \pi_* e^* \mathcal{O}_{\mathbf{P}^n}(1)$ from the universal curve:

$$\begin{array}{ccc} \mathcal{C} & \stackrel{e}{\to} & \mathbf{P}^n \\ \pi \downarrow & & \\ G(2,n+1) & & \end{array} \quad \text{and if we consider} \quad \begin{array}{ccc} \mathcal{C} & \stackrel{e}{\to} & \mathbf{P}^n \\ \pi \downarrow & & \\ \overline{\mathrm{Map}}_1(\mathbf{P}^1, \mathbf{P}^n) & & \end{array}$$

then the same construction gives a *linearized* vector bundle on $\overline{\mathrm{Map}}_1(\mathbf{P}^1, \mathbf{P}^n)$:

$$U^{\vee}_{\mathbf{C}^*} := \pi_* e^* \mathcal{O}_{\mathbf{P}^n}(1)$$

that restricts to U on both fixed loci. The linearization however is trivial only over F_0. Over F_∞ the action is multiplication by z on the fibers, giving:

$$c_1^{\mathbf{C}^*}(U^{\vee}_{\mathbf{C}^*})|_{F_0} = \pi^* c_1(U^{\vee}), c_2^{\mathbf{C}^*}(U^{\vee}_{\mathbf{C}^*})|_{F_0} = \pi^* c_2(U^{\vee})$$

$$c_1^{\mathbf{C}^*}(U^{\vee}_{\mathbf{C}^*})|_{F_\infty} = \pi^* c_1(U^{\vee}) - 2\lambda, c_2^{\mathbf{C}^*}(U^{\vee}_{\mathbf{C}^*})|_{F_\infty} = \pi^* c_2(U^{\vee}) - \lambda \pi^* c_1(U^{\vee}) + \lambda^2$$

It is better to think in terms of Chern roots. If $q_i^{\mathbf{C}^*}$ are the roots of $U^{\vee}_{\mathbf{C}^*}$ and q_i are the roots of $U^{\vee}$, then they are related as follows:

$$q_i{}^{\mathbf{C}^*}|_{F_0} = \pi^* q_i \text{ and } q_i{}^{\mathbf{C}^*}|_{F_\infty} = \pi^* q_i - \lambda$$

These are the ingredients, which are a bit harder to assemble than in (i), but the payoff is also greater. Forget about F_∞ and concentrate on:

$$\begin{array}{ccc} \overline{\mathrm{Map}}_1(\mathbf{P}^1, \mathbf{P}^n) & \stackrel{\Phi}{\to} & \mathbf{P}^n_1 \\ \cup & & \cup \\ F_0 = \{(0:1)\} \times \mathcal{C} & \stackrel{e}{\to} & F'_0 = \{(0:1)\} \times \mathbf{P}^n \end{array}$$

(Recall that $\Phi|_{F_0} = e$ is the same as the evaluation map!)

Then relative localization applied to any polynomial $p(c_1^{\mathbf{C}^*}, c_2^{\mathbf{C}^*})$ in the equivariant Chern classes of $U^{\vee}_{\mathbf{C}^*}$ says:

$$e_* \left(\frac{\pi^* p(c_1, c_2)}{\lambda(\lambda - \psi)} \right) = \Phi|_{F_0 *} \left(\frac{p(c_1^{\mathbf{C}^*}, c_2^{\mathbf{C}^*})|_{F_0}}{\epsilon_{\mathbf{C}^*}(F_0)} \right) =$$

$$= \left(\frac{\Phi_* p(c_1^{\mathbf{C}^*}, c_2^{\mathbf{C}^*})|_{F'_0}}{\epsilon_{\mathbf{C}^*}(F'_0)} \right) = \left(\frac{\Phi_* p(c_1^{\mathbf{C}^*}, c_2^{\mathbf{C}^*})|_{F'_0}}{(H + \lambda)^{n+1}} \right)$$

Let's stop for a minute to ponder the meaning of this. The object:

$$\Phi_* p(c_1^{\mathbf{C}^*}, c_2^{\mathbf{C}^*})|_{F'_0} = P(H, \lambda)$$

will be some polynomial in H and λ, and the two denominators invert as:

$$\frac{1}{\lambda(\lambda - \psi)} = \lambda^{-2} + \lambda^{-3}\psi + \lambda^{-4}\psi^2 + ... \text{ and}$$

$$\frac{1}{(H+\lambda)^{n+1}} = \lambda^{-(n+1)} - \binom{n+1}{1}\lambda^{-(n+2)}H + \binom{n+2}{2}\lambda^{-(n+3)}H^2 - ...$$

so for each class $H^k \in H^*(\mathbf{P}^n, \mathbf{Z})$, the formula says:

$$\int_{\mathcal{C}} e^* H^k \cdot p(c_1, c_2) \cdot (\lambda^{-2} + \lambda^{-3}\psi + \lambda^{-4}\psi^2 + ...) =$$

$$= \int_{\mathbf{P}^n} H^k \cdot P(H, \lambda) \cdot \left(\lambda^{-(n+1)} - \binom{n+1}{1}\lambda^{-(n+2)}H + ...\right)$$

In particular we address our original question!

$$\int_{G(2,n+1)} p(c_1, c_2) = \int_{\mathcal{C}} e^* H \cdot p(c_1, c_2) = \text{coeff of } \lambda^{-2} \text{ in} \int_{\mathbf{P}^n} \frac{H \cdot P(H, \lambda)}{(H+\lambda)^{n+1}}$$

So now we have to figure out what $P(H, \lambda)$ is. It turns out to be better to find a good "approximation" to it. Namely, if we rewrite $p(c_1, c_2)$ as a symmetric polynomial in the Chern roots: $p(c_1, c_2) = p(q_1 + q_2, q_1 q_2) = p'(q_1, q_2)$, then:

Lemma: (See §5)

$$\frac{P(H, \lambda)}{(H+\lambda)^{n+1}} = \frac{p'(H, H+\lambda)}{(H+\lambda)^{n+1}} + a_{-1}(H)\lambda^{-1} + a_0(H)\lambda^0 + ...$$

(we should think of $a_{-1}(H)\lambda^{-1} + a_0(H)\lambda^0 + ...$ as an "error term")

The Upshot: If $p'(q_1, q_2)$ is a symmetric polynomial in the Chern roots, then:

$$\int_{G(2,n+1)} p'(q_1, q_2) = \text{coeff of } \lambda^{-2} \text{ in } \int_{\mathbf{P}^n} \frac{Hp'(H, H+\lambda)}{(H+\lambda)^{n+1}}$$

Examples: (a) The lines on a cubic surface are counted this way as:

$$\int_{G(2,4)} 3q_1(2q_1 + q_2)(q_1 + 2q_2)3q_2 =$$

$$= \text{coeff of } \lambda^{-2} \text{ in} \int_{\mathbf{P}^3} \frac{3H^2(3H+\lambda)(3H+2\lambda)(3H+3\lambda)}{(H+\lambda)^4} = 27$$

(b) The degree of the Plücker embedding is computed this way as:

$$\int_{G(2,n+1)} (q_1 + q_2)^{2n-2} = \text{coeff of } \lambda^{-2} \text{ in} \int_{\mathbf{P}^n} \frac{H(2H+\lambda)^{2n-2}}{(H+\lambda)^{n+1}} =$$

$$= \text{coeff of } \lambda^{-2} \text{ in} \int_{\mathbf{P}^n} \frac{H^{n-1}}{(H+\lambda)}\binom{2n-2}{n-2} + \frac{H^n}{(H+\lambda)^2}\binom{2n-2}{n-1} =$$

$$= \binom{2n-2}{n-1} - \binom{2n-2}{n-2}$$

Remarks: (i) Kontsevich generalized the technique of (i) to compute Gromov-Witten invariants on $\overline{\mathcal{M}}_{0,0}(\mathbf{P}^n, d)$ by the Bott residue theorem. His approach was successful *in theory* in the sense that he characterized the fixed loci and their equivariant Euler classes. The problem is that it seems impossible to make any sense of the answer given in this way! There are so many of these fixed loci (which are not points in general!) that it is very difficult to see how to simplify the sum over them. And without simplifications, the sum is impossible to carry out in all but a few cases.

(ii) By contrast, the technique of (ii) will generalize to make some of the computations relevant to mirror symmetry and to give computable answers that are compatible with the physics predictions. One can already see virtue in this technique by noticing that it readily generalizes to the relative setting. If V is a vector bundle over a base B, then it gives a Porteous-type formula:

$$\int_{G(2,V)} p'(q_1, q_2) = \text{coeff of } \lambda^{-2} \text{ in} \int_{\mathbf{P}(V)} \frac{Hp'(H, H+\lambda)}{\prod_{i=1}^{\mathrm{rk}(V)}(H + \pi^*\rho_i + \lambda)}$$

where q_i are the Chern roots of the universal bundle $U^\vee$ on $G(2, V)$ and ρ_i are the Chern roots of V on B. This idea was generalized to Grassmann bundles of any rank by Jian Kong in his PhD thesis.

6 Higher Degree Curves and J-Functions

First, we want to generalize the technique (ii) of the previous section to maps of degree d to $\mathbf{P}^n$. An equivariant intersection theory with localization is available for smooth Deligne-Mumford stacks, so we may pretend we are in the compact complex manifold case.

There are $d+1$ fixed loci for the action of $\mathbf{C}^*$ on $\mathbf{P}^n_d$:

$$F'_{k0+(d-k)\infty} := \{(a_0 x^k y^{d-k} : ... : a_n x^k y^{d-k})\} \subset \mathbf{P}^n_d$$

and their equivariant Euler classes are not hard to compute. We will need only one of them, namely:

$$\epsilon_{\mathbf{C}^*}(F'_{d0}) = \prod_{k=1}^{d} (H + k\lambda)^{n+1}$$

We follow the same game-plan as in §4 and apply relative localization to:

$$\begin{array}{ccc} \overline{\mathcal{M}ap}_d(\mathbf{P}^1,\mathbf{P}^n) & \stackrel{\Phi}{\to} & \mathbf{P}^n_d \\ \cup & & \cup \\ F_{d0}=\{(0:1)\}\times\mathcal{C} & \stackrel{e}{\to} & F'_{d0}=\mathbf{P}^n \\ \pi\downarrow & & \\ \overline{\mathcal{M}}_{0,0}(\mathbf{P}^n,d) & & \end{array}$$

and work with the Chern classes of the "universal bundles" $U_k^\vee := \pi_* e^*\mathcal{O}_{\mathbf{P}^n}(k)$ and their equivariant counterparts. Unlike the $d=1$ case these bundles are not equal to $\mathrm{Sym}^k U_1^\vee$, and we will need to treat them each separately.

The equivariant Euler class of F_{d0} is still $\lambda(\lambda-\psi)$ and relative localization now gives the following. If $p(c)$ is a polynomial in Chern classes of the $U_k^\vee$ bundles, let $p(c^{\mathbf{C}^*})$ be the same polynomial in the equivariant Chern classes of the corresponding *linearized* bundles $\pi_* e^*\mathcal{O}_{\mathbf{P}^n}(k)$ on $\overline{\mathcal{M}ap}_d(\mathbf{P}^1,\mathbf{P}^n)$. Then:

$$e_*\left(\frac{\pi^* p(c)}{\lambda(\lambda-\psi)}\right)=\frac{P(H,\lambda)}{\prod_{k=1}^d (H+k\lambda)^{n+1}}$$

where

$$P(H,\lambda)=\Phi_* p(c^{\mathbf{C}^*})|_{F'_{d0}}$$

And then, as before, we would be able to compute:

$$\int_{\overline{\mathcal{M}}_{0,0}(\mathbf{P}^n,d)} p(c) = \text{coeff of } \lambda^{-2} \text{ in } \frac{1}{d}\int_{\mathbf{P}^n}\frac{H\cdot P(H,\lambda)}{\prod_{k=1}^d (H+k\lambda)^{n+1}}$$

if we could figure out $P(H,\lambda)$, or a good enough approximation (see §5). (Note that here the projection formula requires us to divide by d.)

This turns out to be harder to do in degree $d>1$. In fact, no one has been able to figure out what to do except for the classes:

(a) $\tau_{\mathcal{O}_{\mathbf{P}^n}(l)} = c_{dl+1}(U_l^\vee)$ and

(b) $\widehat{\tau}_{\mathcal{O}_{\mathbf{P}^n}(-l)} = c_{dl-1}(R^1\pi_* e^*\mathcal{O}_{\mathbf{P}^n}(-l))$

and products of such classes. Notice that each has an equivariant counterpart that restricts to the ordinary class on F_{d0}.

I should mention one case, which is already interesting, namely $p(c)=1$:

$$e_*\left(\frac{1}{\lambda(\lambda-\psi)}\right)=\frac{1}{\prod_{k=1}^d (H+k\lambda)^{n+1}}$$

(This is gives the J-function of projective space. See below.)

To see what happens in general, we need to recall the "comb" strata:

$$\gamma_{\vec{d}} = \gamma_{d_0,..,d_\nu} : \overline{\mathcal{M}ap}_{d_0,\nu}(\mathbf{P}^1, \mathbf{P}^n) \times_X \prod_{i=1}^{\nu} \overline{\mathcal{M}}_{0,1}(\mathbf{P}^n, d_i) \to \overline{\mathcal{M}ap}_d(\mathbf{P}^1, \mathbf{P}^n)$$

from §2. These (including the case $\nu = 0$) map under Φ as follows:

$$\begin{array}{ccc} \overline{\mathcal{M}ap}_d(\mathbf{P}^1, \mathbf{P}^n) & \stackrel{\Phi}{\to} & \mathbf{P}^n_d \\ \uparrow \gamma_{\vec{d}} & & \uparrow \gamma'_{\vec{d}} \\ \overline{\mathcal{M}ap}_{d_0,\nu}(\mathbf{P}^1, \mathbf{P}^n) \times_X \prod_{i=1}^{\nu} \overline{\mathcal{M}}_{0,1}(\mathbf{P}^n, d_i) & \stackrel{\Phi_{\vec{d}}}{\to} & \mathbf{P}^n_{d_0} \times (\mathbf{P}^1)^\nu \end{array}$$

where $\gamma'_{\vec{d}}$ is the map:

$$(P_0(x, y) : ... : P_n(x, y)) \times ((a_1 : b_1), ..., (a_\nu : b_\nu)) \mapsto (QP_0 : ... : QP_n)$$

where $Q = Q_1^{d_1} \cdots Q_\nu^{d_\nu}$ and each $Q_i = b_i x - a_i y$ (i.e. $Q_i(a_i : b_i) = 0$). The point is that the classes τ and $\widehat{\tau}$ and their products behave well with respect to this. From the point of view developed here, the key is the following:

Theorem 1: (a) The equivariant Chern class $\tau^{\mathbf{C}^*}_{\mathcal{O}_{\mathbf{P}^n}(l)}$ decomposes as:

$$\tau^{\mathbf{C}^*}_{\mathcal{O}_{\mathbf{P}^n}(l)} = \sum_{\vec{d}} \frac{1}{\nu!} \gamma_{\vec{d}*} \left(c^{\mathbf{C}^*}_{\vec{d}} \cup \Phi^*_{\vec{d}} \prod_{k=0}^{d_0 l} (lH + k\lambda) \right)$$

for some collection of classes $c^{\mathbf{C}^*}_{\vec{d}}$, where $H = c_1^{\mathbf{C}^*}(\mathcal{O}_{\mathbf{P}^n_{d_0}}(1)) \in H^2_{\mathbf{C}^*}(\mathbf{P}^n_{d_0}, \mathbf{Q})$ for the given linearization. Similarly,

(b) The equivariant Chern class $\widehat{\tau}^{\mathbf{C}^*}_{\mathcal{O}_{\mathbf{P}^n}(-l)}$ decomposes as:

$$\widehat{\tau}^{\mathbf{C}^*}_{\mathcal{O}_{\mathbf{P}^n}(-l)} = \sum_{\vec{d}} \frac{1}{\nu!} \gamma_{\vec{d}*} \left(b^{\mathbf{C}^*}_{\vec{d}} \cup \Phi^*_{\vec{d}} \prod_{i=1}^{\nu} \xi_i \prod_{k=1}^{d_0 l - 1} (-lH - k\lambda) \right)$$

for some classes $b^{\mathbf{C}^*}_{\vec{d}}$, where $\xi_i = c_1^{\mathbf{C}^*}(T_{\mathbf{P}^1})$ is the first equivariant Chern class of the tangent bundle pulled back from the ith projection of $(\mathbf{P}^1)^\nu$.

And more generally, if

$$\tau = \prod_{i=1}^{m} \tau_{\mathcal{O}_{\mathbf{P}^n}(l_i)} \prod_{j=1}^{m'} \widehat{\tau}_{\mathcal{O}_{\mathbf{P}^n}(-l'_j)}$$

then the corresponding $\tau^{\mathbf{C}^*}$ decomposes according to the same pattern:

$$\sum_{\vec{d}} \frac{1}{d!}\gamma_{\vec{d}*}\left(a_{\vec{d}}^{\mathbf{C}^*} \cup \Phi_{\vec{d}}^* \prod_i \prod_{k=0}^{d_0 l_i}(l_i H + k\lambda) \prod_j \prod_{i=1}^{\nu} \xi_i \prod_{k=1}^{d_0 l'_j - 1}(-l'_j H - k\lambda)\right)$$

The excess intersection formula now gives the following:

Corollary: For the equivariant Chern classes $\tau_{\mathcal{O}_{\mathbf{P}^n}(l)}$:

$$\frac{P(H,\lambda)}{\prod_{k=1}^{d}(H+\lambda)^{n+1}} = \sum_{\vec{d}} \frac{\left(\Phi_{\vec{d}*} c_{\vec{d}}^{\mathbf{C}^*}|_{F'_{d0}}\right)\prod_{k=0}^{d_0 l}(lH+k\lambda)}{\lambda^{\nu}\prod_{k=1}^{d_0}(H+k\lambda)^{n+1}}$$

and similarly (but even better...see below) for the classes $\widehat{\tau}_{\mathcal{O}_{\mathbf{P}^n}(-l)}^{\mathbf{C}^*}$:

$$\frac{P(H,\lambda)}{\prod_{k=1}^{d}(H+\lambda)^{n+1}} = \sum_{\vec{d}} \frac{\left(\Phi_{\vec{d}*} b_{\vec{d}}^{\mathbf{C}^*}|_{F'_{d0}}\right)\prod_{k=1}^{d_0 l-1}(-lH-k\lambda)}{\prod_{k=1}^{d_0}(H+k\lambda)^{n+1}}$$

(note the missing λ^{ν} in the denominator!) and similarly for the general τ.

Part of the Punchline: This already puts severe restrictions on the $c_{\vec{d}}^{\mathbf{C}^*}$ (or rather on their contributions $\left(\Phi_{\vec{d}*} c_{\vec{d}}^{\mathbf{C}^*}|_{F'_{d0}}\right)$ to $P(H,\lambda)$), and even more severe restrictions on the $b_{\vec{d}}^{\mathbf{C}^*}$ and $a_{\vec{d}}^{\mathbf{C}^*}$. The point is that if l is small, then $\left(\Phi_{\vec{d}*} c_{\vec{d}}^{\mathbf{C}^*}|_{F'_{d0}}\right) = 0$ because it has negative degree!

Namely, one computes from the Corollary above:

$$\begin{aligned}\deg\left(\Phi_{\vec{d}*} c_{\vec{d}}^{\mathbf{C}^*}|_{F'_{d0}}\right) &= dl + 1 - d_0 l + 1 + d_0(n+1) - d(n+1) + \nu \\ &= (d-d_0)(l-n-1) + \nu\end{aligned}$$

and since $\nu \le d - d_0$, and necessarily $\deg\left(\Phi_{\vec{d}*} c_{\vec{d}}^{\mathbf{C}^*}|_{F'_{d0}}\right) \ge 0$, we get:

Corollary: If $l < n$, then only one term survives in the previous corollary:

$$\frac{P(H,\lambda)}{\prod_{k=1}^{d}(H+\lambda)^{n+1}} = \frac{\prod_{k=0}^{dl}(lH+k\lambda)}{\prod_{k=1}^{d}(H+k\lambda)^{n+1}}$$

so that (just as in the case of the Grassmannian of lines):

$$e_*\left(\frac{\pi^*\tau_{\mathcal{O}_{\mathbf{P}^n}(l)}}{\lambda(\lambda-\psi)}\right) = \frac{\prod_{k=0}^{dl}(lH+k\lambda)}{\prod_{k=1}^{d}(H+k\lambda)^{n+1}}$$

In the general case, we compute:

$$\begin{aligned}\deg\left(\Phi_{\vec{d}*}a^{\mathbf{C}^*}_{\vec{d}}|_{F'_{d0}}\right) &= \textstyle\sum_{i=1}^{m}(dl_i+1)+\sum_{j=1}^{m'}(dl'_j-1)-\sum_{i=1}^{m}(d_0l_i+1) \\ &\quad -\textstyle\sum_{j=1}^{m'}(d_0l'_j-1)+d_0(n+1)-d(n+1)+\nu-m'\nu \\ &= (d-d_0)(\textstyle\sum l_i+\sum l'_j-n-1)+(1-m')\nu\end{aligned}$$

and we conclude that in this case:

Corollary: If $\sum l_i + \sum l'_j < n$ or $\sum l_i + \sum l'_j = n$ and $m' \geq 1$ or finally $\sum l_i + \sum l'_j = n+1$ and $m' > 1$, then:

$$e_*\left(\frac{\pi^*\tau}{\lambda(\lambda-\psi)}\right) = \frac{\prod_{i=1}^{m}\prod_{k=0}^{dl}(lH+k\lambda)\prod_{j=1}^{m'}\prod_{k=1}^{dl-1}(-lH-k\lambda)}{\prod_{k=1}^{d}(H+k\lambda)^{n+1}}$$

Application: If one computes (as we will do) the intersection number:

$$n_d := \int_{\overline{\mathcal{M}}_{0,0}(\mathbf{P}^4,d)} \tau_{\mathcal{O}_{\mathbf{P}^4}(5)}$$

then one obtains a rational (not integral) number of maps of degree d to a rational curve in a quintic threefold when $d > 1$. One reason for this is that multiple covers of a $\mathbf{P}^1$ (of lower degree) count for rational numbers. The specific count for a degree d map to a smooth curve of normal bundle $\mathcal{O}_{\mathbf{P}^1}(-1)\oplus\mathcal{O}_{\mathbf{P}^1}(-1)$ is given by the following τ integral:

Aspinwall-Morrison Number:

$$\int_{\overline{\mathcal{M}}_{0,0}(\mathbf{P}^1(d)}\widehat{\tau}^2_{\mathcal{O}_{\mathbf{P}^1}(-1)} = \text{coeff of } \lambda^2 \text{ in}\frac{1}{d}\int_{\mathbf{P}^1}\frac{(\prod_{k=1}^{d-1}(-H-k\lambda))^2}{\prod_{k=1}^{d}(H+k\lambda)^2} = \frac{1}{d^3}$$

and this is valid because $m' = 2$ and $l_1 + l_2 = 2$.

But we really want to compute n_d above, and for this we notice:

Corollary: If $\sum l_i = n+1$ and $m' = 0$, then:

$$\deg\left(\Phi_{\vec{d}*}a^{\mathbf{C}^*}_{\vec{d}}|_{F'_{d0}}\right) = \nu$$

Theorem 2: In this case (i.e. $\sum l_i = n+1$ and $m' = 0$) define:

$$a_dH + b_d\lambda := \Phi_{0,d_*}a^{\mathbf{C}^*}_{0,d}|_{F'_{d0}}$$

Then:

(a) $\Phi_{(d_0,d_1)_*} a^{\mathbf{C}^*}_{(d_0,d_1)}|_{F'_{(d_0+d_1)0}} = a_{d_1}(H + d_0\lambda) + b_{d_1}\lambda$ for "simple" $\vec{d} = (d_0, d_1)$, and more generally:

(b) $\Phi_{\vec{d}*} a^{\mathbf{C}^*}_{\vec{d}}|_{F'_{d0}} = \prod_{i=1}^{\nu} \left(a_{d_i}(H + (\sum_{k=0}^{i} d_k)\lambda) + b_{d_i}\lambda \right)$

The Rest of the Punchline: When $\sum l_i = n+1$ and $m' = 0$, that is, when τ measures the expected class of rational curves of degree d in a Calabi-Yau complete intersection in $\mathbf{P}^n$, then Theorem 2 recursively determines all the pairs (a_d, b_d), hence it gives a simple (and very computable!) algorithm for counting the degree d maps of rational curves to such a Calabi-Yau manifold.

Application: To compute the numbers n_d above for the quintic threefold:

$$(1)\quad e_*\left(\frac{\pi^*\tau_{\mathcal{O}_{\mathbf{P}^4}(5)}}{\lambda(\lambda - \psi)}\right) = \frac{\prod_{k=0}^{5}(5H + k\lambda)}{(H + \lambda)^5} + \frac{(a_1 H + b_1\lambda)}{\lambda}$$

$$\Rightarrow n_1 = 2875 \text{ and } a_1 = -770, b_1 = -120$$

$$(2)\quad e_*\left(\frac{\pi^*\tau_{\mathcal{O}_{\mathbf{P}^4}(5)}}{\lambda(\lambda - \psi)}\right) = \frac{\prod_{k=0}^{10}(5H + k\lambda)}{\prod_{k=1}^{2}(H + k\lambda)^5} + \frac{(a_1(H + \lambda) + b_1\lambda)\prod_{k=0}^{5}(5H + k\lambda)}{\lambda(H + \lambda)^5}$$

$$+ \frac{(a_1 H + b_1\lambda)(a_1(H + \lambda) + b_1\lambda)}{\lambda^2} + \frac{(a_2 H + b_2\lambda)}{\lambda}$$

$$\Rightarrow n_2 = 609250 + \frac{2875}{8} \text{ and } a_2 = -421375, b_2 = -60000$$

(Recall the Aspinwall-Morrison correction!) etc.

General Remarks: These computations are motivated by mirror symmetry, which, among other things, relates Gromov-Witten invariants of a Calabi-Yau manifold to the Hodge theory of the family of mirror Calabi-Yau manifolds. From this point of view, the "correct" way of packaging the enumerative data of rational curves on a projective manifold X is with:

$$J^X_\beta := e_*\left(\frac{1}{\lambda(\lambda - \psi)}\right) \in \lambda^{-2} H^*(X)[\lambda^{-1}]$$

which (as in the $\mathbf{P}^n$ case) is the push-forward of the equivariant Euler class:

$$\begin{array}{ccc} \overline{\mathcal{M}ap}_\beta(\mathbf{P}^1, \mathbf{P}^n) & & \\ \cup & & \\ \{(0:1)\} \times \mathcal{C} & \stackrel{e}{\to} & X \\ \pi \downarrow & & \\ \overline{\mathcal{M}}_{0,0}(X, \beta) & & \end{array}$$

but *there is no way to fill in with a smooth* X_β *analogue of* $\mathbf{P}^n_d$ *when* $X \neq \mathbf{P}^n$.

Nevertheless, this appears to be an important invariant, not just for Calabi-Yau manifolds. Precisely, the J-function is the generating function:

$$J^X(q) := 1 + \sum_{\beta \neq 0} J^X_\beta q^\beta$$

and there is an entire $\mathcal{D}$-module theory of Gromov-Witten invariants that I have not mentioned, which motivates the following:

Principle: The J-function solves interesting differential equations

One example of this comes from mirror symmetry, but another is:

Proposition: (Dijkgraaf, Givental) Recall the small quantum cohomology and in particular the interpretation $q^\beta = e^{\beta_1 t_1 + \ldots + \beta_m t_m}$ where the t_i generate the curve classes, and let H_i be the dual divisor classes in $H^2(X)$. If:

$$\mathcal{D} = \mathcal{D}(\lambda, \lambda \frac{\partial}{\partial t_i}, e^{t_i})$$

is any polynomial operator that satisfies: $\mathcal{D}(e^{\frac{t_1 H_1 + \ldots + t_m H_m}{\lambda}} J(e^t)) = 0$, then $\mathcal{D}(0, H_i, q_i) = 0$ is a relation in small quantum cohomology.

Example: The J-function of $\mathbf{P}^n$ is:

$$J^{\mathbf{P}^n}(e^t) = 1 + \sum_{d>0} \frac{e^{dt}}{\prod_{k=1}^{n+1} (H + k\lambda)^{n+1}}$$

and one can easily check that:

$$\mathcal{D} = \left(\lambda \frac{\partial}{\partial t}\right)^{n+1} - e^t$$

satisfies $\mathcal{D}(e^{\frac{tH}{\lambda}} J(e^t)) = 0$, hence that $H^{n+1} - q$ is a relation in $\mathrm{QH}^*(\mathbf{P}^n)$.

This is no surprise since this relation is easy to see by more direct methods. But Givental's generalization to Fano complete intersections $X \subset \mathbf{P}^n$ gives a relation that is very hard to see any other way.

One other remark I should make about these is the following:

Theorem: (Lee-Pandharipande [13], Bertram-Kley [6]) Suppose the coefficients of the J-function of X lie in a subalgebra $R \subseteq H^*(X)$ satisfying the following:

- R is generated as an algebra by classes of divisors
- the intersection pairing $\int_X c \cup c'$ is non-degenerate on R.

Then as in the Kontsevich-Manin theorem, every Gromov-Witten invariant of rational curves can be reconstructed from the J-function.

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